For the primary with $\Delta = \Delta_{r,s}$, the fusion rule is

$$[\phi_{\Delta_{r,s}}] \times [\phi_{\Delta(\alpha)}] \subset \sum_{\substack{k=-(r-1) \\ k+r:\,\mathrm{odd}}}^{r-1} \sum_{\substack{l=-(s-1) \\ l+s:\,\mathrm{odd}}}^{s-1} [\phi_{\Delta(\alpha + k\alpha_+ + l\alpha_-)}]$$

Since $\Delta_{r,s} = \Delta(r\alpha_+ + s\alpha_-)$, degenerate primaries are closed under OPE

$$[\phi_{\Delta_{r_1,s_1}}] \times [\phi_{\Delta_{r_2,s_2}}] \subset \sum_{\substack{r=r_2-r_1+1 \\ r+r_1+r_2\ \mathrm{odd}}}^{r_2+r_1-1} \sum_{s=s_2-s_1+1}^{s_2+s_1-1} [\phi_{\Delta_{r,s}}]$$

Actually, non-positive r and s drops out :

$$[\phi_{\Delta_{r_1,s_1}}] \times [\phi_{\Delta_{r_2,s_2}}] \subset \sum_{\substack{r=|r_2-r_1|+1 \\ r+r_1+r_2:\,\mathrm{odd}}}^{r_2+r_1-1} \sum_{s=|s_2-s_1|+1}^{s_2+s_1-1} [\phi_{\Delta_{r,s}}]$$

This truncation can be understood as follows :

Consider, e.g. the OPE of $\phi_{\Delta_{2,1}}$ and $\phi_{\Delta_{1,2}}$.

Using the fusion rule for $\phi_{\Delta_{2,1}}$ we find

$$[\phi_{\Delta_{2,1}}] \times [\phi_{\Delta_{1,2}}] \subset [\phi_{\Delta_{0,2}}] + [\phi_{\Delta_{2,2}}]$$

If we use the fusion rule for $\phi_{\Delta_{1,2}}$, we find instead

$$[\phi_{\Delta_{2,1}}] \times [\phi_{\Delta_{1,2}}] \subset [\phi_{\Delta_{2,0}}] + [\phi_{\Delta_{2,2}}]$$

Since $\Delta_{0,2} \neq \Delta_{2,0}$ we see that only $[\phi_{\Delta_{2,2}}]$ can appear on the RHS :

$$[\phi_{\Delta_{2,1}}] \times [\phi_{\Delta_{1,2}}] \subset [\phi_{\Delta_{2,2}}].$$

* * *

For special values of central charge, truncation above is also possible. Suppose

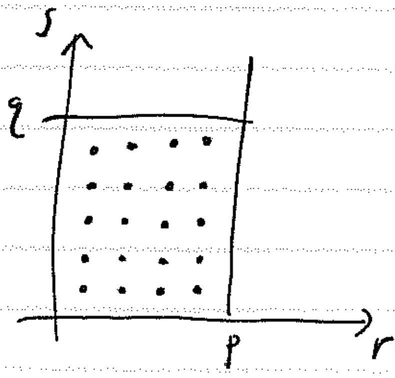
$$-\alpha_-/\alpha_+ = p/q \quad \text{for some coprime integers } p, q.$$

i.e. $p\alpha_+ + q\alpha_- = 0$. This gives $\Delta_{r,s} = \Delta_{r+p,s+q}$

Note also $\Delta_{r,s} = \Delta_{-r,-s}$. In particular $\Delta_{r,s} = \Delta_{p-r,q-s}$.

These relation lead to the truncation

$$1 \leq r \leq p-1, \quad 1 \leq s \leq q-1$$



e.g.
$$\begin{array}{ccccccc} \bigl[\phi_{\Delta_{r,q-1}}\bigr] & \times & \bigl[\phi_{\Delta_{1,2}}\bigr] & = & \bigl[\phi_{\Delta_{r,q}}\bigr] & + & \bigl[\phi_{\Delta_{r,q-2}}\bigr] \\ \parallel & & & & \parallel & & \parallel \\ \bigl[\phi_{\Delta_{p-r,1}}\bigr] & \times & \bigl[\phi_{\Delta_{1,2}}\bigr] & = & \xcancel{\bigl[\phi_{\Delta_{p-r,0}}\bigr]} & + & \bigl[\phi_{\Delta_{p-r,2}}\bigr] \\ & & & & \uparrow & & \end{array}$$

by truncation below.

$$\Rightarrow \xcancel{\phi_{\Delta_{r,q}}} \quad \text{(truncation above)}.$$

This suggest existence of a CFT with a finite # of degenerate primaries only (minimal CFT).

- \# of primaries $= \dfrac{(p-1)(q-1)}{2}$ ("$\frac{1}{2}$" from $\Delta_{r,s} = \Delta_{p-r,q-s}$)

$$-\alpha_-/\alpha_+ = p/q \iff (p+q)\sqrt{1-c} = (q-p)\sqrt{25-c}$$

$$\iff \boxed{c = 1 - \frac{6(p-q)^2}{pq}}$$

The case $p=m,\ q=m+1$ is precisely where all the representations $M_{\Delta_{r,s},c}/N$ are unitary!

(unitary minimal model)

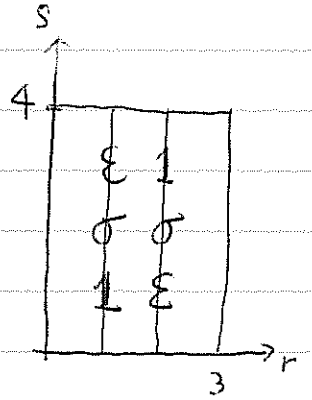
$\underline{(p,q)=(3,4)}$ m=3 unitary minimal model

$$c = 1-\frac{6}{3\cdot 4} = \frac{1}{2}$$

$\Delta_{1,1} = \Delta_{2,3} = 0 \qquad \phi_0 = 1$

$\Delta_{2,1} = \Delta_{1,3} = \frac{1}{2} \qquad \phi_{\frac{1}{2}} = \varepsilon$

$\Delta_{1,2} = \Delta_{2,2} = \frac{1}{16} \qquad \phi_{\frac{1}{16}} = \sigma$



Critical Ising model

$[1]\times[1]=[1], \quad [1]\times[\varepsilon]=[\varepsilon], \quad [1]\times[\sigma]=[\sigma]$

$[\sigma]\times[\varepsilon] = [\phi_{\Delta_{1,2}}]\times[\phi_{\Delta_{2,1}}] \subset [\phi_{\Delta_{2,2}}] = [\sigma]$

$[\sigma]\times[\sigma] = [\phi_{\Delta_{1,2}}]\times[\phi_{\Delta_{1,2}}] \subset [\phi_{\Delta_{1,1}}]+[\phi_{\Delta_{1,3}}] = [1]+[\varepsilon]$

$[\varepsilon]\times[\varepsilon] = [\phi_{\Delta_{2,1}}]\times[\phi_{\Delta_{2,1}}] \subset [\phi_{\Delta_{1,1}}]+[\phi_{\Delta_{3,1}}]$

$\overset{\text{or}}{=} [\phi_{\Delta_{1,3}}]\times[\phi_{\Delta_{1,3}}] \subset [\phi_{\Delta_{1,5}}]+[\phi_{\Delta_{1,3}}]+[\phi_{\Delta_{1,1}}]$

$\left(\overset{\text{or}}{=} [\phi_{\Delta_{2,1}}]\times[\phi_{\Delta_{1,3}}] \subset [\phi_{\Delta_{2,3}}] + \cancel{[\phi_{\Delta_{0,3}}]} \right)$

$\Delta_{3,1} = \frac{5}{3}, \quad \Delta_{1,3} = \frac{1}{2}, \quad \Delta_{1,5} = \frac{5}{2}$ all different

$\therefore$ Only $[\phi_{\underset{\overset{\shortparallel}{\Delta_{2,3}}}{\Delta_{1,1}}}]=[1]$ remains on the RHS. $\therefore [\varepsilon]\times[\varepsilon]\subset[1]$.

$\subset$'s are actually $=$'s:

$[\sigma]\times[\sigma]$ and $[\varepsilon]\times[\varepsilon]$ must obviously include $[1]$.

Nontrivial is $[\sigma]\times[\sigma]\supset[\varepsilon]$ and $[\varepsilon]\times[\sigma]\supset[\sigma]$.

These are equivalent to $C_{\sigma\sigma\varepsilon}\neq 0$.

This can be seen by looking at the 4pt fcn $\langle\sigma\sigma\varepsilon\varepsilon\rangle$.

Since $[\sigma]\times[\sigma]\supset[1]$ and $[\varepsilon]\times[\varepsilon]=[1]$, this is non-zero: $\langle\sigma\sigma\varepsilon\varepsilon\rangle\neq 0$. This in turn means that $\varepsilon\sigma$ OPE is non-zero. Thus we <u>must</u> have $[\varepsilon]\times[\sigma]=[\sigma]$.

This shows $C_{\sigma\sigma\varepsilon}\neq 0$ and hence $[\sigma]\times[\sigma]\supset[\varepsilon]$.

In this way, we find that the precise fusion rule is

$$[1]\times[\text{any}] = [\text{any}]$$

$$[\varepsilon]\times[\sigma] = [\sigma]$$

$$[\sigma]\times[\sigma] = [1]+[\varepsilon]$$

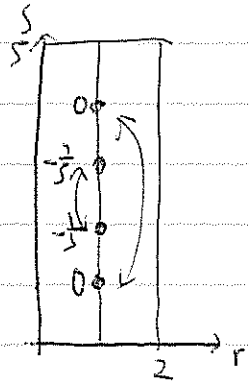
$$[\varepsilon]\times[\varepsilon] = [1].$$

$\underline{(p,q) = (2,5)}$ $\qquad C = 1 - \frac{6 \cdot 3^2}{5 \cdot 2} = -\frac{22}{5}$ $\qquad$ (non unitary)

$$\alpha_+ = \sqrt{\frac{5}{2}} \ , \ \alpha_- = -\sqrt{\frac{2}{5}}$$

$$\Delta_{1,1} = \Delta_{1,4} = 0$$

$$\Delta_{1,2} = \Delta_{1,3} = -\frac{1}{5}$$



$$[\phi_{-\frac{1}{5}}] \times [\phi_{-\frac{1}{5}}] = [\phi_{\Delta_{1,2}}] \times [\phi_{\Delta_{1,2}}] \subseteq [\phi_{\Delta_{1,1}}] + [\phi_{\Delta_{1,3}}]$$

$$\overset{\text{or}}{=} (\phi_{\Delta_{1,3}}] \times [\phi_{\Delta_{1,3}}] \subset [\phi_{\Delta_{1,1}}] + [\phi_{\Delta_{1,3}}] + [\cancel{\phi_{\Delta_{1,5}}}]$$

$$\overset{\text{or}}{=} [\phi_{\Delta_{1,2}}] \times [\phi_{\Delta_{1,3}}] \subset [\phi_{\Delta_{1,2}}] + [\phi_{\Delta_{1,4}}]$$

$$\overset{\text{or}}{=} [\phi_{\Delta_{1,3}}] \times (\phi_{\Delta_{1,2}}] \subset [\phi_{\Delta_{1,4}}) + [\phi_{\Delta_{1,2}}] + [\cancel{\phi_{\Delta_{1,0}}}]$$

The fact is: " $\subset$ is $=$ ".

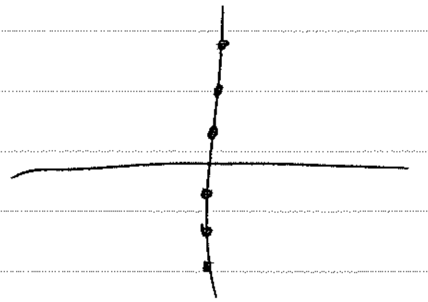
$$\therefore [\phi_{\frac{1}{5}}] \times [\phi_{\frac{1}{5}}] = [\mathbb{1}] + [\phi_{-\frac{1}{5}}].$$

This theory describe the so-called "Lee-Yang edge singularity".

Ising model with external field H

$$Z(T,H) = \sum_{\sigma} e^{\frac{J}{kT}\sum_{\langle ij\rangle}\sigma_i\sigma_j - \frac{\mu H}{kT}\sum_i \sigma_i}$$

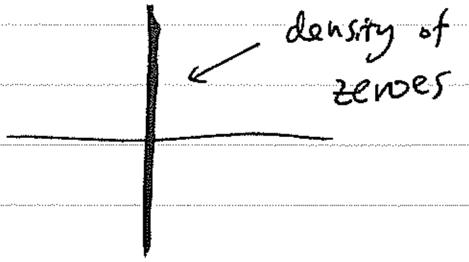
It has zeros at points on pure imaginary H line



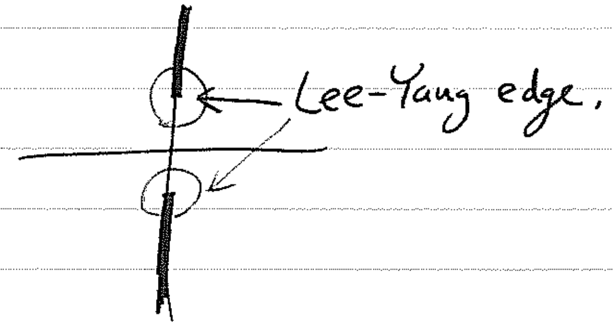
(for finite lattice)

In the thermodynamic limit (∞ lattice)

$T < T_c$



$T > T_c$



The system develops a critical behaviour at the Lee-Yang edge.

$(p, q) = (4, 5)$ $m=4$ unitary minimal model

$$C = 1 - \frac{6}{4 \cdot 5} = \frac{7}{10}$$

$\alpha_+ = \sqrt{\frac{5}{4}}$

$\alpha_- = -\sqrt{\frac{4}{5}}$

$\frac{3}{2}$	$\frac{7}{16}$	0
$\frac{3}{5}$	$\frac{3}{80}$	$\frac{1}{10}$
$\frac{1}{10}$	$\frac{3}{80}$	$\frac{3}{5}$
0	$\frac{7}{16}$	$\frac{3}{2}$

write $\phi_{\frac{1}{10}} = \varepsilon$

$\phi_{\frac{3}{80}} = \sigma$ $\phi_{\frac{3}{5}} = t$

$\phi_0 = 1$ $\phi_{\frac{7}{16}} = \sigma'$ $\phi_{\frac{3}{2}} = \mathbb{G}$

Fusion rule

$[1] \times [\text{any}] = [\text{any}]$

$[\varepsilon] \times [\varepsilon] = [1] + [t]$

$[\varepsilon] \times [t] = [\varepsilon] + [\mathbb{G}]$

$[\varepsilon] \times [\mathbb{G}] = [t]$

$[\varepsilon] \times [\sigma'] = [\sigma]$

$[\varepsilon] \times [\sigma] = [\sigma] + [\sigma']$

$[t] \times [t] = [1] + [t]$

$[t] \times [\mathbb{G}] = [\varepsilon]$

$[t] \times [\sigma'] = [\sigma]$

$[t] \times [\sigma] = [\sigma'] \times [\sigma]$

$[\mathbb{G}] \times [\mathbb{G}] = [1]$

$[\mathbb{G}] \times [\sigma'] = [\sigma']$

$[\mathbb{G}] \times [\sigma] = [\sigma]$

$[\sigma'] \times [\sigma'] = [1] + [\mathbb{G}]$

$[\sigma'] \times [\sigma] = [\varepsilon] + [t]$

$[\sigma] \times [\sigma] = [1] + [\varepsilon] + [t] + [\mathbb{G}]$.

Remarks

1. It describes "Tricritical Ising model"

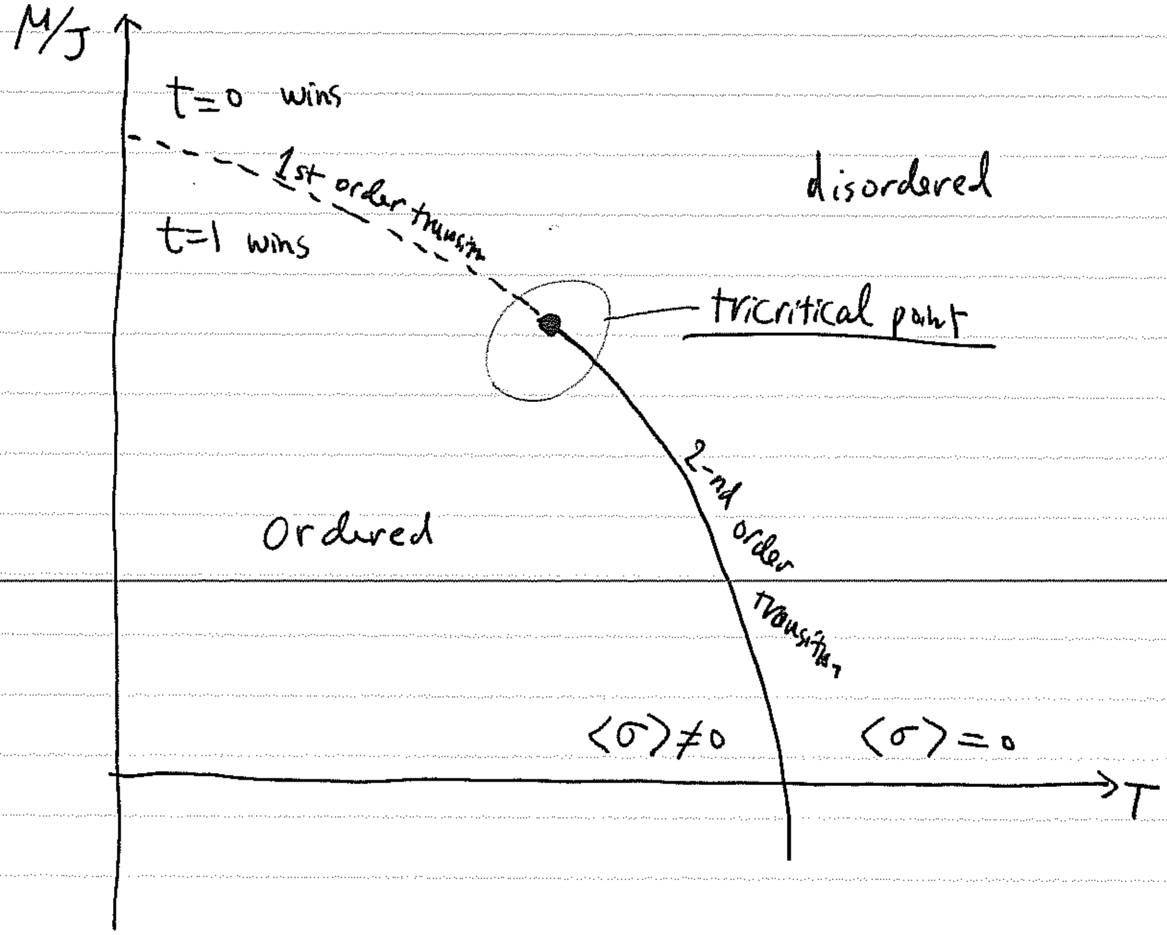
Ising model with vacancies

$$\sigma_i \in \{1,-1\},\quad \underset{\substack{\uparrow \\ \text{vacant}}}{t_i \in \{0}, \underset{\substack{\uparrow \\ \text{present}}}{1\}} \qquad \left(s_i = \sigma_i t_i \in \{1, 0, -1\} \right)$$

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j t_i t_j - H \sum_i \sigma_i t_i + \mu \sum_i t_i$$

$$\left(= -J \sum_{\langle i,j \rangle} s_i s_j - H \sum_i s_i + \mu \sum_i s_i^2 \right)$$

Phase diagram at $H=0$:



2. It has a supersymmetry!

It is a GSO projected model of an $\mathcal{N}=(1,1)$ superconformal field theory

$$\phi_{\frac{3}{2}} = \mathbb{G} \quad \text{is acturally} \quad \underset{\uparrow}{G}\cdot\underset{\uparrow}{\overline{G}}\,.$$

supercurrents (right/left)

The symmetry algebra is larger

$$\mathrm{Vir} \oplus G \;=\; \text{Super Virasoro} \qquad \times 2 \ \swarrow \text{right/left}$$

$$[0]_{Vir} \overset{\overset{G}{\frown}}{\oplus} \left[\tfrac{3}{2}\right]_{Vir} = [0]_{NS}$$

$$\left[\tfrac{1}{10}\right]_{Vir} \overset{\overset{G}{\frown}}{\oplus} \left[\tfrac{3}{5}\right]_{Vir} = \left[\tfrac{1}{10}\right]_{NS}$$

$$\left[\tfrac{3}{80}\right]_{Vir} = \left[\tfrac{3}{80}\right]_{R}$$

$$\left[\tfrac{7}{16}\right]_{Vir} = \left[\tfrac{7}{16}\right]_{R}$$

(as consistent with the fusion rule)

It is the first nontrivial one in the series of unitary minimal models with superconformal symmety

$$C = \frac{3}{2}\left[1 - \frac{8}{m(m+2)}\right] \qquad m = \underset{\substack{\uparrow \\ \text{trivial}}}{2},\ \underset{\Uparrow}{3},\ 4, 5, \cdots$$

3. The operator algebra plays a certain important rôle in compactification of string theory with minimal supersymmetry in 2+1 dimensions.
(e.g. Compactification of a 7-dimensional manifold
with G_2 - holonomy.)

The 3d spacetime supersymmetry requires and is guaranteed by the existence of the algebra <u>inside</u> the CFT (with $c = \frac{3}{2} \cdot 7$) describing the internal dimension.

<u>Cf.</u> 4d spacetime supersymmetry

$\Leftrightarrow$ $\mathcal{N} = (2,2)$ superconformal symmetry
+ "spectral flow operators"

2d spacetime supersymmetry

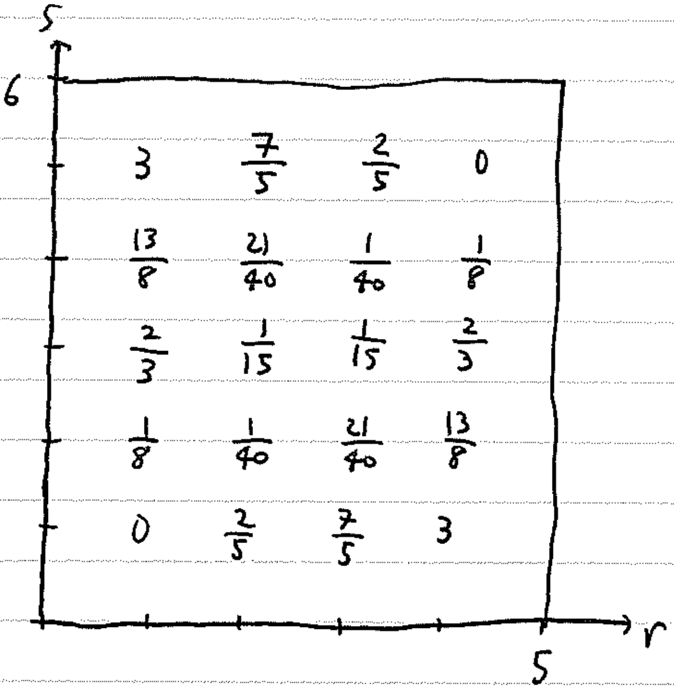
$\Leftrightarrow$ $\exists$ 2d Ising model algebra

$$[\varepsilon] \times [\varepsilon] = [1], \quad [\sigma] \times [\sigma] = [1] + [\varepsilon]$$
$$[\sigma] \times [\varepsilon] = [\sigma]$$

$\underline{(p,q)=(5,6)}$ $m=5$ unitary minimal model

$$c = 1 - \frac{6}{5\cdot 6} = \frac{4}{5}$$

$\alpha_+ = \sqrt{\frac{6}{5}}$, $\alpha_- = -\sqrt{\frac{5}{6}}$



3	$\frac{7}{5}$	$\frac{2}{5}$	0
$\frac{13}{8}$	$\frac{21}{40}$	$\frac{1}{40}$	$\frac{1}{8}$
$\frac{2}{3}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{3}$
$\frac{1}{8}$	$\frac{1}{40}$	$\frac{21}{40}$	$\frac{13}{8}$
0	$\frac{2}{5}$	$\frac{7}{5}$	3

☆ $\phi_{3,0}$ generates a new symmetry algebra (W-algebra)

☆ ∃ a unitary CFT that includes operators

$$\phi_{\Delta,\Delta} \quad \text{with } \Delta = 0, \frac{2}{5}, \frac{7}{5}, 3, \overset{\text{2 copies}}{\frac{1}{15}}, \overset{\text{2 copies}}{\frac{2}{3}}$$

& $\phi_{0,3}$, $\phi_{3,0}$, $\phi_{\frac{2}{5},\frac{7}{5}}$, $\phi_{\frac{7}{5},\frac{2}{5}}$ <u>only</u>.

(Note $\frac{1}{8}, \frac{1}{40}, \frac{21}{40}, \frac{13}{8}$ are missing!)

[This is a feature that exists in all higher $m=5,6,7,\cdots$.]

The "Strange" CFT

It describes the critical behaviour of the "3-state Potts model"

$$\mathcal{H} = -J \sum_{\langle ij \rangle} (\sigma_i \bar{\sigma}_j + \bar{\sigma}_i \sigma_j) \qquad \sigma_i \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$$

$$\phi_{\frac{2}{5},\frac{2}{5}} = \varepsilon \ , \quad \phi^{i=1,2}_{\frac{1}{15},\frac{1}{15}} = \sigma, \bar{\sigma} .$$

- It is different from the CFT with operators $\phi_{\Delta,\Delta}$ for all $\Delta = 0, 2/5, 7/5, 3, 1/8, 1/40, 21/40, 13/8, 1/15, 2/3$

← "diagonal modular invariant"

$$Z = \sum_{\Delta = \text{all}} |\chi_\Delta(\tau)|^2$$

- $$Z_{\text{Crit 3 Potts}} = |\chi_0 + \chi_3|^2 + |\chi_{\frac{2}{5}} + \chi_{\frac{7}{5}}|^2 + 2|\chi_{\frac{1}{15}}|^2 + 2|\chi_{\frac{2}{3}}|^2$$

$$= \sum_{\Delta,\tilde{\Delta}} N_{\Delta,\tilde{\Delta}} \chi_\Delta(\tau) \chi_{\tilde{\Delta}}(\bar{\tau})$$

$N_{\Delta,\tilde{\Delta}} \neq \delta_{\Delta,\tilde{\Delta}}$ "off-diagonal modular invariant".

<u>NB</u> Each of these two theories has its own Fusion rule.