Correlation functions of Critical Ising Model

$c = \frac{1}{2}$ minimal unitary CFT has $1 \ (\Delta = \tilde{\Delta} = 0)$
$\varepsilon \ (\Delta = \tilde{\Delta} = \frac{1}{2})$
$\sigma \ (\Delta = \tilde{\Delta} = \frac{1}{16})$ as the primaries.

We shall compute <u>4-point functions</u> of ε and σ.

By the fusion rule, Only non-zero 4pt functions are

$$\langle \sigma\sigma\sigma\sigma \rangle, \ \langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle, \ \langle \sigma\sigma\varepsilon\varepsilon \rangle.$$

As we will see, they can be determined completely by

- Projective Ward identity
- The differential eqn for $\Delta = \Delta_{1,2}$ or $\Delta_{2,1}$
- Single valued ness
- normalization condition $\mathcal{O}_i(x)\mathcal{O}_i(y) \sim \dfrac{1}{(x-y)^{2\Delta_i}(\bar{x}-\bar{y})^{2\tilde{\Delta}_i}}$

The result allows us to determine C_{ijk}.

We shall also compute correlation functions involving the disorder operator μ as well as fermions ψ_z.
[The result will <u>not</u> be single valued.]

$\langle \sigma\sigma\sigma\sigma \rangle$

By projective Ward identity, we have

$$\langle \sigma(1)\sigma(2)\sigma(3)\sigma(4)\rangle = \frac{1}{z_{12}^{\frac{1}{8}} z_{34}^{\frac{1}{8}} \bar{z}_{12}^{\frac{1}{8}} \bar{z}_{34}^{\frac{1}{8}}} F(x,\bar{x})$$

for $x = \dfrac{z_{14} z_{32}}{z_{12} z_{34}}$, $z_{ij} = z_i - z_j$.

The differential equation for $\Delta_\sigma = \Delta_{1,2}$

$$\left[\frac{\partial^2}{\partial z_4^2} - \underbrace{\frac{2}{3}(2\Delta_\sigma + 1)}_{= \frac{3}{4}} \sum_{i=1}^{3} \left(\frac{\Delta_\sigma = \frac{1}{16}}{(z_4 - z_i)^2} + \frac{1}{z_4 - z_i}\frac{\partial}{\partial z_i} \right) \right] \langle \sigma \cdots \sigma \rangle = 0$$

leads to (Hint: after applying $\frac{\partial^2}{\partial z_4^2}$, $\frac{\partial}{\partial z_i}$, send $z_1 \to 0$, $z_2 \to 1$, $z_3 \to \infty$):

$$\left\{ \frac{d^2}{dx^2} + \frac{3}{4}\left(\frac{1}{x} + \frac{1}{x-1}\right)\frac{d}{dx} - \frac{3}{4}\left[\frac{\frac{1}{16}}{x^2} + \frac{\frac{1}{16}}{(x-1)^2} + \frac{\frac{1}{8}}{x} - \frac{\frac{1}{8}}{x-1} \right] \right\} F = 0$$

Since we expect $\langle \sigma(\infty)\sigma(1)\sigma(x)\sigma(0)\rangle \sim \dfrac{1}{x^{\frac{1}{8}} \bar{x}^{\frac{1}{8}}}$ as $x \to 0$

$$\sim \frac{1}{(1-x)^{\frac{1}{8}}(1-\bar{x})^{\frac{1}{8}}} \quad \text{as } x \to 1$$

We put $F(x,\bar{x}) = x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}} \bar{x}^{-\frac{1}{8}}(1-\bar{x})^{-\frac{1}{8}} u(x,\bar{x})$. Then the eqn is

$$\left\{ x(1-x)\frac{d^2}{dx^2} + \frac{1}{2}(1-2x)\frac{d}{dx} + \frac{1}{16} \right\} u = 0$$

With the change of variables $x = \sin^2\theta$, this is

$$\left(\frac{d^2}{d\theta^2} + \frac{1}{4}\right) u = 0$$

which is solved by $u = \cos\left(\frac{\theta}{2}\right),\ \sin\left(\frac{\theta}{2}\right)$.

Since we have the same equation for $\bar{x} = \sin^2\bar{\theta}$ dependence we have

$$U = U_{11}\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\bar\theta}{2}\right) + U_{12}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\bar\theta}{2}\right) + U_{21}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\bar\theta}{2}\right) + U_{22}\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\bar\theta}{2}\right)$$

We still have to determine U_{ij}.

First, we use the fact that $\langle\sigma(\infty)\sigma(1)\sigma(x)\sigma(0)\rangle$ is a single valued function of x.

Note that $x \sim \theta^2$ as $x \sim 0$. $\quad 0\ x \iff 0\ \theta$

i.e. $\theta \to -\theta$

invariance under $\theta \to -\theta$: $\underline{U_{12} = U_{21} = 0}$

Also $\sin\underbrace{\left(\frac{\pi}{2} + \epsilon\right)}_{\theta} = \cancel{\mp}\cos\epsilon = -\left(1 - \frac{\epsilon^2}{2} + \cdots\right) \quad \therefore\ x = 1 - \epsilon^2 + \cdots$

$1\ x \iff \epsilon \to -\epsilon \iff \theta \to \pi - \theta$: $\cos\left(\frac{\theta}{2}\right) \to \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \sin\frac{\theta}{2}$

$\sin\left(\frac{\theta}{2}\right) \to \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos\frac{\theta}{2}$.

invariance under this : $\underline{U_{11} = U_{22}}$.

Finally we require that

$$\langle \sigma(1)\sigma(2)\sigma(3)\sigma(4)\rangle = \frac{1}{z_{12}^{\frac{1}{8}}\, z_{34}^{\frac{1}{8}}\, \bar{z}_{12}^{\frac{1}{8}}\, \bar{z}_{34}^{\frac{1}{8}}} F$$

$$= \frac{1}{z_{14}^{\frac{1}{8}}\, z_{32}^{\frac{1}{8}}\, \bar{z}_{14}^{\frac{1}{8}}\, \bar{z}_{32}^{\frac{1}{8}}} U(x,\bar{x})$$

must approach $\sim \left\langle \frac{1}{z_{14}^{\frac{1}{8}}\, \bar{z}_{14}^{\frac{1}{8}}} \sigma(2)\sigma(3) \right\rangle = \frac{1}{z_{14}^{\frac{1}{8}}\, \bar{z}_{14}^{\frac{1}{8}}\, z_{32}^{\frac{1}{8}}\, \bar{z}_{32}^{\frac{1}{8}}}$

as $z_1 \to z_4$. i.e. $U(x,\bar{x}) \to 1$ as $x \to 0$.

This determines $U_{11} = U_{22} = \underline{1}$

Finally we find

$$\boxed{\begin{aligned} \langle \sigma(\infty)\sigma(1)\sigma(x)\sigma(0)\rangle &= F(x,\bar{x}) \\ &= \left| x^{-\frac{1}{8}} (1-x)^{-\frac{1}{8}} \cos\left(\frac{\theta}{2}\right) \right|^2 + \left| x^{-\frac{1}{8}} (1-x)^{-\frac{1}{8}} \sin\left(\frac{\theta}{2}\right) \right|^2 \end{aligned}}$$

for $x = \sin^2\theta$

From the general consideration, we have

$$\langle \sigma(\infty)\,\sigma(1)\,\sigma(x)\,\sigma(0)\rangle = \sum_p C^p_{\sigma\sigma}\, C_{p\sigma\sigma} \left| \mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(\Delta_p|x)\right|^2$$

for $\mathcal{F}^{\Delta''\Delta'''}_{\Delta\Delta'}(\Delta_p|x) = x^{\Delta_p-\Delta-\Delta'} \sum_{\{k\}} \beta^{\Delta_p\{k\}}_{\Delta\Delta'}\, \alpha^{\Delta_p\{k\}}_{\Delta''\Delta'''}\, x^{|k|}$

$$= x^{\Delta_p-\Delta-\Delta'} \sum_{\substack{\{k\}\\ \{k'\}}} \beta^{\Delta_p\{k\}}_{\Delta\Delta'}\, M^{\{k\}\{k'\}}_{\Delta_p}\, \beta^{\Delta_p\{k'\}}_{\Delta''\Delta'''}\, x^{|k|}$$

Note that

$$x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\cos\left(\frac{\theta}{2}\right) = x^{-\frac{1}{8}}\left(1+\frac{1}{8^2}x^2+\cdots\right)$$

$$x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}x^{\frac{1}{2}-\frac{1}{8}}\left(1+\frac{x}{4}+\frac{9}{8^2}x^2+\cdots\right)$$

This implies $x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\cos\left(\frac{\theta}{2}\right) = \mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(0|x)$

$$x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}\left(\tfrac{1}{2}\middle|x\right)$$

and
$$\boxed{\langle \sigma(\infty)\,\sigma(1)\,\sigma(x)\,\sigma(0)\rangle = \left|\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(0|x)\right|^2 + \frac{1}{4}\left|\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}\left(\tfrac{1}{2}\middle|x\right)\right|^2}$$

This shows $\boxed{C_{\sigma\sigma\varepsilon} = \frac{1}{2}}$

In particular, it is non-zero, and hence $[\sigma]\times[\sigma] = [1]+[\varepsilon]$

$[\sigma]\times[\varepsilon] = [\sigma]$.

Check of $\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(0|x) = x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\cos\left(\frac{\theta}{2}\right)$ (1)

$\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(\tfrac{1}{2}|x) = 2x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\sin\left(\frac{\theta}{2}\right)$ (2)

(1)

level 1 $\alpha^{0\{1\}}_{\frac{1}{16}\frac{1}{16}} = 0$ (no term of $O(x^1)$)

level 2 $\alpha^{0\{2\}}_{\frac{1}{16}\frac{1}{16}} = \frac{1}{16}$, $\alpha^{0\{1,1\}}_{\frac{1}{16}\frac{1}{16}} = 0$, $M^{(2)}_0 = \begin{pmatrix} \frac{c}{2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore \beta^{0\{2\}}_{\frac{1}{16}\frac{1}{16}} = \frac{1}{4}$ ($\beta^{0\{1,1\}}_{\frac{1}{16}\frac{1}{16}}$ is immaterial)

$\therefore \mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(0|x) = x^{0-\frac{1}{16}-\frac{1}{16}}\left(1 + 0\cdot x + \underbrace{\frac{1}{16}\cdot\frac{1}{4}}_{=\frac{1}{8^2}}x^2 + O(x^3)\right)$

matches.

(2)

level 1 $\alpha^{\frac{1}{2}\{1\}}_{\frac{1}{16}\frac{1}{16}} = \frac{1}{2}$, $M^{(1)}_{\frac{1}{2}} = \frac{1}{2\Delta_\varepsilon} = 1$ $\therefore \beta^{\frac{1}{2}\{1\}}_{\frac{1}{16}\frac{1}{16}} = \frac{1}{2}$ $\alpha^{(1)}\cdot\beta^{(1)} = \frac{1}{4}$

level 2 $\alpha^{\frac{1}{2}\{2\}}_{\frac{1}{16}\frac{1}{16}} = \frac{9}{16}$, $\alpha^{\frac{1}{2}\{1,1\}}_{\frac{1}{16},\frac{1}{16}} = \frac{3}{4}$, $M^{(2)}_{\frac{1}{2}} = \begin{pmatrix} \frac{9}{4} & 3 \\ 3 & 4 \end{pmatrix}$

$\left(\beta^{\frac{1}{2}\{2\}}_{\frac{1}{16}\frac{1}{16}}, \beta^{\frac{1}{2}\{1,1\}}_{\frac{1}{16}\frac{1}{16}}\right) = \left(\frac{1}{4}, 0\right) + c\,(4, -3)$ (c is immaterial)

$\alpha^{(2)}\cdot\beta^{(2)} = \frac{9}{16}\cdot\frac{1}{4} = \frac{9}{8^2}$

$\therefore \mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(\tfrac{1}{2}|x) = \left(1 + \frac{1}{4}x + \frac{9}{8^2}x^2 + O(x^3)\right)x^{\frac{1}{2}-\frac{1}{16}-\frac{1}{16}}$

matches

$\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$

From the fusion rule $[\varepsilon]\times[\varepsilon]=[1]$, we expect the form

$$\langle \varepsilon(1)\,\varepsilon(2)\,\varepsilon(3)\,\varepsilon(4)\rangle = \left|\text{holomorphic}\right|^2$$

And this is indeed the form expected from the Majorana fermion representation $\varepsilon(x) = i\psi_-(x)\psi_+(x)$.

By Wick contraction, we can easily find

$$\langle \varepsilon(1)\,\varepsilon(2)\,\varepsilon(3)\,\varepsilon(4)\rangle = \left| \frac{1}{z_{12}z_{34}} + \frac{1}{z_{14}z_{23}} - \frac{1}{z_{13}z_{24}} \right|^2$$

In particular

$$\langle \varepsilon(\infty)\,\varepsilon(1)\,\varepsilon(x)\,\varepsilon(0)\rangle = \left| -1 + \frac{1}{x} + \frac{1}{1-x} \right|^2$$

This indeed satisfies the differential eqn

$$\left\{ \frac{\partial^2}{\partial z_4^2} - \underbrace{\frac{2}{3}(2\Delta_\varepsilon + 1)}_{=\frac{4}{3}} \sum_{i=1}^{3} \left(\frac{\Delta_i = \frac{1}{2}}{(z_4 - z_i)^2} + \frac{1}{z_4 - z_i}\frac{\partial}{\partial z_i} \right) \right\} \langle \varepsilon(1) \cdots \varepsilon(4)\rangle = 0$$

or $$\left\{ \frac{d^2}{dx^2} + \frac{4}{3}\left(\frac{1}{x} + \frac{1}{x-1}\right)\frac{d}{dx} - \frac{2}{3}\left[\frac{1}{x^2} + \frac{1}{(x-1)^2} + \frac{2}{x(1-x)}\right] \right\} F = 0$$

for $$\langle \varepsilon(1)\,\varepsilon(2)\,\varepsilon(3)\,\varepsilon(4)\rangle = \frac{1}{z_{12}z_{34}\,\bar z_{12}\bar z_{34}} F(x,\bar x)$$

$$\left(\text{i.e.} \quad \langle \varepsilon(\infty)\,\varepsilon(1)\,\varepsilon(x)\,\varepsilon(0)\rangle = F(x,\bar x) \right)$$

Since $[\varepsilon]\times[\varepsilon]=[1]$, we must have

$$\langle \varepsilon(\infty)\,\varepsilon(1)\,\varepsilon(x)\,\varepsilon(0)\rangle = \underbrace{C^{1}_{\varepsilon\varepsilon}\,C_{1\varepsilon\varepsilon}}_{\substack{\parallel \\ 1 \text{ by normalization.}}}\left|\mathcal{F}^{\frac12\frac12}_{\frac12\frac12}(0|x)\right|^{2}$$

This implies

$$\mathcal{F}^{\frac12\frac12}_{\frac12\frac12}(0|x) = \frac{1}{x} - 1 + \frac{1}{1-x} = \frac{1}{x} + \sum_{n=1}^{\infty} x^{n}$$

<u>Check</u> <u>level 1</u> $\alpha^{0\,\{1\}}_{\frac12\frac12} = \beta^{0\,\{1\}}_{\frac12\frac12} = 0$ $\qquad$ <u>$\alpha^{(1)}\cdot\beta^{(1)} = 0$</u>

<u>level 2</u> $\alpha^{0\,\{2\}}_{\frac12\frac12} = \frac12$, $\alpha^{0\,\{1,1\}}_{\frac12\frac12} = 0$, $M^{(2)}_0 = \begin{pmatrix} \frac14 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore$ one may take $\beta^{0\,\{2\}}_{\frac12\frac12} = 4\cdot\frac12 = 2$ $\quad\therefore$ <u>$\alpha^{(2)}\cdot\beta^{(2)} = 1$</u>

<u>level 3</u> $\alpha^{0\,\{3\}}_{\frac12\frac12} = 1$, $\alpha^{0\,\{2,1\}}_{\frac12\frac12} = \alpha^{0\,\{1,1,1\}}_{\frac12\frac12} = 0$

$$M^{(3)}_0 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad \text{wrt } L_{-3}|0\rangle,\ L_{-2}L_{-1}|0\rangle,\ L_{-1}L_{-1}L_{-1}|0\rangle$$

$\Rightarrow \beta^{0\,\{3\}}_{\frac12\frac12} = 1$ $\qquad$ <u>$\alpha^{(3)}\cdot\beta^{(3)} = 1$</u>

<u>level 4</u> $\alpha^{0\,\{4\}}_{\frac12\frac12} = \frac32$, $\alpha^{0\,\{2,2\}}_{\frac12\frac12} = \frac54$, $\alpha^{0\,\{3,1\}}_{\frac12\frac12} = \alpha^{(0)\,\{2,1,1\}}_{\frac12\frac12} = \alpha^{(0)\,\{1111\}}_{\frac12\frac12} = 0$

$$M^{(4)}_0 = \begin{pmatrix} \frac52 & \frac32 & & & \\ \frac32 & \frac{17}{8} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \quad \text{wrt } L_{-4}|0\rangle,\ L_{-2}L_{-2}|0\rangle,\ L_{-3}L_{-1}|0\rangle,\ L_{-2}L_{-1}^{2}|0\rangle,\ L_{-1}^{4}|0\rangle.$$

$\Rightarrow \beta^{0\,\{4\}}_{\frac12\frac12} = \frac37$, $\beta^{0\,\{2,2\}}_{\frac12\frac12} = \frac27$, <u>$\alpha^{(4)}\cdot\beta^{(4)} = \frac32\cdot\frac37 + \frac54\cdot\frac27 = \frac{14}{14} = 1$</u>

$\langle \sigma\sigma\,\varepsilon\varepsilon \rangle$

From $[\varepsilon]\times[\varepsilon]=[1]$, we again expect the form $\left|\text{holomorphic}\right|^2$.

By projective Ward identity

$$\langle \sigma(1)\,\sigma(2)\,\varepsilon(3)\,\varepsilon(4)\rangle = \frac{1}{z_{12}^{2\Delta_\sigma}\, z_{34}^{2\Delta_\varepsilon}\, \bar{z}_{12}^{2\Delta_\sigma}\, \bar{z}_{34}^{2\Delta_\varepsilon}}\, F(x,\bar{x})$$

$$= \frac{F(x,\bar{x})}{|z_{12}|^{\frac{1}{4}}\cdot |z_{34}|^{2}} \qquad \left[\; F(x,\bar{x}) = \langle \varepsilon(\infty)\,\sigma(1)\,\varepsilon(x)\,\sigma(0)\rangle \;\right]$$

Applying the $\Delta=\Delta_{2,1}$ differential eqn for $\varepsilon(z_4)$, we find

$$\left\{ \frac{d^2}{dx^2} + \frac{4}{3}\left(\frac{1}{x}+\frac{1}{x-1}\right)\frac{d}{dx} - \frac{4}{3}\left[\frac{\frac{1}{16}}{x^2} + \frac{\frac{1}{16}}{(x-1)^2} + \frac{\frac{1}{8}}{x} - \frac{\frac{1}{8}}{x-1} \right]\right\} F = 0 .$$

By $[\varepsilon]\times[\sigma]=[\sigma]$, we expect

$$F \sim \frac{C^{\sigma}_{\varepsilon\sigma}}{x^{\frac{1}{2}}\cdot \bar{x}^{\frac{1}{2}}}\,\langle \varepsilon(\infty)\,\sigma(1)\,\sigma(0)\rangle \quad \text{as } x\sim 0$$

$$\sim \frac{C^{\sigma}_{\varepsilon\sigma}}{(x-1)^{\frac{1}{2}}(\bar{x}-1)^{\frac{1}{2}}}\,\langle \varepsilon(\infty)\,\sigma(1)\,\sigma(0)\rangle \quad \text{as } x\sim 1$$

This motivates us to put

$$F = x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,\bar{x}^{-\frac{1}{2}}(1-\bar{x})^{-\frac{1}{2}}\,U(x,\bar{x}).$$

The eqn is then

$$\left\{ 3x(1-x)\frac{d^2}{dx^2} + (1-2x)\frac{d}{dx} + 2 \right\} v = 0$$

Puting $v = x^K +$ higher, we find $3K(K-1)+K=0 \Rightarrow K=0, \frac{2}{3}$

By $F \sim \dfrac{C^{\sigma}_{\varepsilon\sigma}}{x^{\frac{1}{2}}\bar{x}^{\frac{1}{2}}} \langle \varepsilon(\infty)\sigma(1)\sigma(0)\rangle$, we must choose $K=0$.

It turns out that the solution is very simple:

$$v = 1-2x \quad !$$

$$\therefore \quad F = \left| x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}(1-2x) \right|^2 \times \text{const.}$$

To determine the constant, we use the requirement

$$\langle \sigma(1)\sigma(2)\varepsilon(3)\varepsilon(4)\rangle \xrightarrow{z_1\to z_2} \frac{1}{|z_{12}|^{\frac{1}{4}}}\langle \varepsilon(3)\varepsilon(4)\rangle = \frac{1}{|z_{12}|^{\frac{1}{4}}|z_{34}|^2}$$

$$\langle \sigma(1)\sigma(2)\varepsilon(3)\varepsilon(4)\rangle = \left| \frac{1-2x}{z_{12}^{\frac{1}{8}}\, z_{34}\, x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} \right|^2 \times \text{const}$$

$$= \left| \frac{1}{z_{12}^{\frac{1}{8}}\cdot z_{34}} \cdot \frac{z_{12}z_{34} - 2z_{14}z_{32}}{(z_{14}z_{13})^{\frac{1}{2}}(z_{32}z_{24})^{\frac{1}{2}}} \right|^2 \times \text{const}$$

$$\xrightarrow{z_1\to z_2} \frac{4\cdot\text{const}}{\left| z_{12}^{\frac{1}{8}}\cdot z_{34}\right|^2} \qquad \therefore \quad \text{const} = \frac{1}{4}$$

$$\therefore\ \boxed{\langle \varepsilon(\infty)\,\sigma(1)\,\varepsilon(x)\,\sigma(0)\rangle = \frac{1}{4}\left|\frac{1-2x}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\right|^2}$$

From the general consideration, we must have

$$\langle \varepsilon(\infty)\,\sigma(1)\,\varepsilon(x)\,\sigma(0)\rangle = \underbrace{C^{\sigma}_{\varepsilon\sigma}\,C_{\sigma\varepsilon\sigma}}_{\left(\frac{1}{2}\right)^2=\frac{1}{4}}\left|\mathcal{F}^{\frac{1}{16}\ \frac{1}{2}}_{\frac{1}{2}\ \frac{1}{16}}\left(\tfrac{1}{16}\middle|x\right)\right|^2$$

This implies $\mathcal{F}^{\frac{1}{16}\ \frac{1}{2}}_{\frac{1}{2}\ \frac{1}{16}}\left(\tfrac{1}{16}\middle|x\right) = x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}(1-2x)$

$$= x^{-\frac{1}{2}}\left(1-\frac{3}{2}x-\frac{5}{8}x^2+\cdots\right).$$

<u>Check</u> : The power $x^{-\frac{1}{2}} = x^{\frac{1}{16}-\frac{1}{2}-\frac{1}{16}}$ is OK.

<u>level 1</u> $\alpha^{\frac{1}{16}\{1\}}_{\frac{1}{2}\frac{1}{16}} = \frac{1}{2}$ $\quad M^{(1)}_{\frac{1}{16}} = \frac{1}{8}$ $\quad\therefore\ \beta^{\frac{1}{16}\{1\}}_{\frac{1}{2}\frac{1}{16}} = 8\cdot\frac{1}{2} = 4$

$\alpha^{\frac{1}{16}\{1\}}_{\frac{1}{16}\frac{1}{2}} = -\frac{3}{8}$ $\quad\therefore\ \alpha^{\frac{1}{16}\{1\}}_{\frac{1}{16}\frac{1}{2}}\,\beta^{\frac{1}{16}\{1\}}_{\frac{1}{2}\frac{1}{16}} = -\frac{3}{8}\cdot 4 = \underline{-\frac{3}{2}}$ OK.

<u>level 2</u> $\alpha^{\frac{1}{16}\{2\}}_{\frac{1}{2}\frac{1}{16}} = 1$, $\alpha^{\frac{1}{16}\{1,1\}}_{\frac{1}{2},\frac{1}{16}} = \frac{3}{4}$ $\quad M^{(2)}_{\frac{1}{16}} = \begin{pmatrix}\frac{1}{2} & \frac{3}{8}\\ \frac{3}{8} & \frac{9}{2\cdot 16}\end{pmatrix}$

$$\begin{pmatrix}1\\ \frac{3}{4}\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{3}{8}\\ \frac{3}{8} & \frac{9}{2\cdot 16}\end{pmatrix}\underbrace{\begin{pmatrix}2+\frac{3}{4}c\\ -c\end{pmatrix}}_{=\ \beta^{\frac{1}{16}(2)}_{\frac{1}{2},\frac{1}{16}}}\qquad \forall c$$

$$\alpha^{\frac{1}{16}\{2\}}_{\frac{1}{16}\frac{1}{2}} = -\frac{5}{16}, \quad \alpha^{\frac{1}{16}\{1,1\}}_{\frac{1}{16}\frac{1}{2}} = -\frac{5\cdot 3}{8^2}$$

$$\alpha^{\frac{1}{16}(2)}_{\frac{1}{16}\frac{1}{2}} \cdot \beta^{\frac{1}{16}(2)}_{\frac{1}{2}\frac{1}{16}} = -\frac{5}{16}\cdot 2 = -\frac{5}{8} \quad \text{OK.}$$

We may also consider $z_1 \to \infty$, $z_2 \to 1$, $z_3 \to x$, $z_4 \to 0$

$$\Rightarrow \langle \sigma(\infty)\sigma(1)\,\varepsilon(x)\,\varepsilon(0)\rangle = \frac{1}{4}\left|\frac{1}{x}\cdot\frac{x-2(x-\cancel{1})}{(1-x)^{\frac{1}{2}}}\right|^2$$

$$= \left| x^{-1}(1-x)^{-\frac{1}{2}}\left(1-\frac{x}{2}\right)\right|^2$$

This implies

$$\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{2}\frac{1}{2}}(0|x) = x^{-1}(1-x)^{-\frac{1}{2}}\left(1-\frac{x}{2}\right)$$

$$= x^{-1}\left(1+\frac{1}{8}x^2+\cdots\right)$$

<u>Check</u> The power $x^{0-\frac{1}{2}-\frac{1}{2}} = x^{-1}$ is OK.

<u>level 1</u> $\alpha^{0\{1\}}_{\frac{1}{16}\frac{1}{16}} = 0 = \alpha^{0\{1\}}_{\frac{1}{2}\frac{1}{2}}$

<u>level 2</u> $\alpha^{0\{2\}}_{\frac{1}{16}\frac{1}{16}} = \frac{1}{16}$, $\alpha^{0\{2\}}_{\frac{1}{2}\frac{1}{2}} = \frac{1}{2}$ $\quad M^{(2)} = \begin{pmatrix}\frac{1}{4} & 0\\ 0 & 0\end{pmatrix}$ $\therefore \alpha^{0(2)}_{\frac{1}{16}\frac{1}{16}}\cdot\beta^{0(2)}_{\frac{1}{2}\frac{1}{2}} = \frac{1}{16}\cdot 4\cdot\frac{1}{2} = \frac{1}{8}$

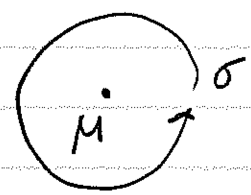
OK.

$\langle \sigma\sigma\mu\mu \rangle$

μ = disorder operator. (σ in the dual system).

Since $T = T_c$ is the self dual point, μ must have the same conformal weights as σ:

$$\Delta_\mu = \tilde{\Delta}_\mu = \frac{1}{16}.$$

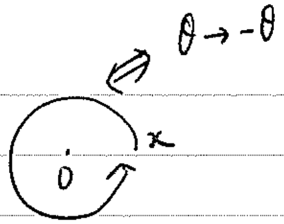
We also know that  produces a sign (-1).

Since $\Delta_\mu = \tilde{\Delta}_\mu = \frac{1}{16}$, the four point function $G = \langle \mu(\infty)\, \sigma(1)\, \sigma(x)\, \mu(0) \rangle$ must obey the same differential eqn as $\langle \sigma(\infty)\sigma(1)\sigma(x)\sigma(0) \rangle$.

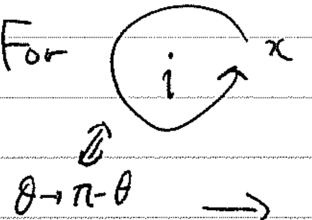
In particular it can be written as

$$G = \left| x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}} \right|^2 \Big\{ g_{11} \cos\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\bar\theta}{2}\right) + g_{12} \cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\bar\theta}{2}\right) + g_{21} \sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\bar\theta}{2}\right) + g_{22} \sin\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\bar\theta}{2}\right) \Big\}$$

The difference is the monodromy:

For 

σ circles μ, we must have (-1) sign.

For 

σ circles σ, the result must be invariant.

$$\Longrightarrow \quad g_{11} = g_{22} = 0 \quad \& \quad g_{12} = g_{21} =: g$$

$$\therefore\ G = g\left|x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\right|^2 \sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\bar\theta}{2}\right) + g\left|x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\right| \cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\bar\theta}{2}\right).$$

As $x \to 1$ we must have $G \to \frac{1}{|x-1|^{\frac{1}{4}}}\langle \mu(\infty)\mu(0)\rangle \sim \frac{1}{|x-1|^{\frac{1}{4}}}$

$$x \sim 1 \Leftrightarrow \theta \sim \frac{\pi}{2} \qquad \therefore\ \sin\frac{\theta}{2} \sim \frac{1}{\sqrt{2}} \sim \cos\frac{\theta}{2}$$

$$\therefore\ \boxed{g = \mathbb{1}}$$

$$\therefore\ \langle \mu(\infty)\,\sigma(1)\,\sigma(x)\,\mu(0)\rangle$$

$$= \mathbb{1}\left|x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\right|^2\left(\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\bar\theta}{2}\right) + \cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\bar\theta}{2}\right)\right)$$

$$= \mathbb{1}x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\sin\frac{\theta}{2}\cdot\overline{x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\cos\frac{\theta}{2}} + x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\cos\frac{\theta}{2}\cdot\overline{\mathbb{1}x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\sin\frac{\theta}{2}}$$

$$= \underbrace{\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}}_{\frac{1}{2}}\left(\tfrac{1}{2}\middle|x\right)\cdot\overline{\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(0|x)} + \underbrace{\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}}_{\frac{1}{2}}(0|x)\,\overline{\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}\left(\tfrac{1}{2}\middle|x\right)}$$

This shows that the OPE $\sigma(x)\mu(0)$ includes two primaries, one with $(\Delta,\bar{\Delta})=(\frac{1}{2},0)$ another with $(0,\frac{1}{2})$.

The obvious candidates are the fermions ψ_- and ψ_+.

Moreover we find that the structure constants are

$$C_{\sigma\mu}^{\psi_-} = C_{\sigma\mu}^{\psi_+} = \frac{1}{\sqrt{2}}$$

Thus, the precise OPE is

$$\sigma(x)\mu(0) = \frac{1}{\sqrt{2}}\underbrace{x^{\frac{3}{8}}\bar{x}^{-\frac{1}{8}}}_{= x^{\frac{1}{2}}/|x|^{\frac{1}{4}}}(\psi_-(0)+\cdots) + \frac{1}{\sqrt{2}}\underbrace{x^{-\frac{1}{8}}\bar{x}^{\frac{3}{8}}}_{= \bar{x}^{\frac{1}{2}}/|x|^{\frac{1}{4}}}(\psi_+(0)+\cdots)$$

This also implies

$$\psi_-(z)\sigma(0) = \frac{1/\sqrt{2}}{z^{\frac{1}{2}}}(\mu(0)+\cdots), \quad \psi_+(z)\sigma(0) = \frac{1/\sqrt{2}}{\bar{z}^{\frac{1}{2}}}(\mu(0)+\cdots)$$

$$\psi_-(z)\mu(0) = \frac{1/\sqrt{2}}{z^{\frac{1}{2}}}(\sigma(0)+\cdots), \quad \psi_+(z)\mu(0) = \frac{1/\sqrt{2}}{\bar{z}^{\frac{1}{2}}}(\sigma(0)+\cdots).$$