Correlation functions of Critical Ising Model

$C=\frac{1}{2}$ minimal unitary CFT has 1 $(\Delta=\tilde{\Delta}=0)$
ε $(\Delta=\tilde{\Delta}=\frac{1}{2})$
σ $(\Delta=\tilde{\Delta}=\frac{1}{16})$ as the primaries.

We shall compute 4-point functions of ε and σ.

By the fusion rule, only non-zero 4pt functions are

$$\langle\sigma\sigma\sigma\sigma\rangle,\ \langle\varepsilon\varepsilon\varepsilon\varepsilon\rangle,\ \langle\sigma\sigma\varepsilon\varepsilon\rangle.$$

As we will see, they can be determined completely by

- Projective Ward identity
- The differential eqn for $\Delta=\Delta_{1,2}$ or $\Delta_{2,1}$
- Single valuedness
- normalization condition $\mathcal{O}_i(x)\mathcal{O}_i(y)\sim\dfrac{1}{(x-y)^{2\Delta_i}(\bar{x}-\bar{y})^{2\tilde{\Delta}_i}}$

The result allows us to determine C_{ijk}.

We shall also compute correlation functions involving the disorder operator μ as well as fermions ψ_z.
[The result will not be single valued.]

$\underline{\langle \sigma\sigma\sigma\sigma \rangle}$

By projective Ward identity, we have

$$\langle \sigma(1)\sigma(2)\sigma(3)\sigma(4)\rangle = \frac{1}{z_{12}^{\frac{1}{8}} z_{34}^{\frac{1}{8}} \bar{z}_{12}^{\frac{1}{8}} \bar{z}_{34}^{\frac{1}{8}}} F(x,\bar{x})$$

for $x = \dfrac{z_{14} z_{32}}{z_{12} z_{34}}$, $z_{ij} = z_i - z_j$.

The differential equation for $\Delta_\sigma = \Delta_{1,2}$

$$\left[\frac{\partial^2}{\partial z_4^2} - \underbrace{\frac{2}{3}(2\Delta_\sigma + 1)}_{= \frac{3}{4}} \sum_{i=1}^{3} \left(\frac{\Delta_\sigma = \frac{1}{16}}{(z_4 - z_i)^2} + \frac{1}{z_4 - z_i}\frac{\partial}{\partial z_i} \right) \right] \langle \sigma \ldots \sigma \rangle = 0$$

leads to (Hint: after applying $\frac{\partial^2}{\partial z_4^2}$, $\frac{\partial}{\partial z_i}$, send $\begin{matrix} z_1 \to 0 \\ z_2 \to 1 \\ z_3 \to \infty \end{matrix}$):

$$\left\{ \frac{d^2}{dx^2} + \frac{3}{4}\left(\frac{1}{x} + \frac{1}{x-1}\right)\frac{d}{dx} - \frac{3}{4}\left[\frac{\frac{1}{16}}{x^2} + \frac{\frac{1}{16}}{(x-1)^2} + \frac{\frac{1}{8}}{x} - \frac{\frac{1}{8}}{x-1} \right] \right\} F = 0$$

Since we expect $\langle \sigma(\infty)\sigma(1)\sigma(x)\sigma(0)\rangle \sim \dfrac{1}{x^{\frac{1}{8}} \bar{x}^{\frac{1}{8}}}$ as $x \to 0$

$\sim \dfrac{1}{(1-x)^{\frac{1}{8}}(1-\bar{x})^{\frac{1}{8}}}$ as $x \to 1$

We put $F(x,\bar{x}) = x^{-\frac{1}{8}}(1-x)^{\frac{1}{8}}\bar{x}^{-\frac{1}{8}}(1-\bar{x})^{\frac{1}{8}} u(x,\bar{x})$. Then the eqn is

$$\left\{ x(1-x)\frac{d^2}{dx^2} + \frac{1}{2}(1-2x)\frac{d}{dx} + \frac{1}{16} \right\} u = 0$$

With the change of variables $x = \sin^2\theta$, this is

$$\left(\frac{d^2}{d\theta^2} + \frac{1}{4}\right) u = 0$$

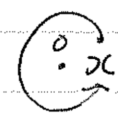
which is solved by $u = \cos\left(\frac{\theta}{2}\right),\ \sin\left(\frac{\theta}{2}\right)$.

Since we have the same equation for $\bar{x} = \sin^2\bar{\theta}$ dependence we have

$$U = U_{11}\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\bar\theta}{2}\right) + U_{12}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\bar\theta}{2}\right) + U_{21}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\bar\theta}{2}\right) + U_{22}\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\bar\theta}{2}\right)$$

We still have to determine U_{ij}.

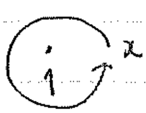
First, we use the fact that $\langle \sigma(\infty)\sigma(1)\sigma(x)\sigma(0)\rangle$ is a single valued function of x.

Note that $x \sim \theta^2$ as $x \sim 0$.  $\Longleftrightarrow$ 

i.e. $\theta \to -\theta$

invariance under $\theta \to -\theta$: $\underline{U_{12} = U_{21} = 0}$

Also $\sin\Big(\underbrace{\frac{\pi}{2} + \epsilon}_{\theta}\Big) = \cos\epsilon = -\left(1 - \frac{\epsilon^2}{2} + \cdots\right)$ $\therefore$ $x = 1 - \epsilon^2 + \cdots$

 $\Longleftrightarrow$ $\epsilon \to -\epsilon$ $\Longleftrightarrow$ $\theta \to \pi - \theta$: $\cos\left(\frac{\theta}{2}\right) \to \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \sin\frac{\theta}{2}$

$\sin\left(\frac{\theta}{2}\right) \to \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos\frac{\theta}{2}$.

invariance under this : $\underline{U_{11} = U_{22}}$.

Finally we require that

$$\langle \sigma(1)\sigma(2)\sigma(3)\sigma(4)\rangle = \frac{1}{z_{12}^{\frac{1}{8}} z_{34}^{\frac{1}{8}} \bar{z}_{12}^{\frac{1}{8}} \bar{z}_{34}^{\frac{1}{8}}} F$$

$$= \frac{1}{z_{14}^{\frac{1}{8}} z_{32}^{\frac{1}{8}} \bar{z}_{14}^{\frac{1}{8}} \bar{z}_{32}^{\frac{1}{8}}} U(x,\bar{x})$$

must approach $\sim \left\langle \frac{1}{z_{14}^{\frac{1}{8}} \bar{z}_{14}^{\frac{1}{8}}} \sigma(2)\sigma(3) \right\rangle = \frac{1}{z_{14}^{\frac{1}{8}} \bar{z}_{14}^{\frac{1}{8}} z_{32}^{\frac{1}{8}} \bar{z}_{32}^{\frac{1}{8}}}$

as $z_1 \to z_4$. i.e. $U(x,\bar{x}) \to 1$ as $x \to 0$.

This determines $U_{11} = U_{22} = \underline{1}$

Finally we find

$$\boxed{\begin{aligned}\langle \sigma(\infty)\sigma(1)\sigma(x)\sigma(0)\rangle &= F(x,\bar{x}) \\ &= \left| x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}} \cos\left(\frac{\theta}{2}\right)\right|^2 + \left| x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}} \sin\left(\frac{\theta}{2}\right)\right|^2\end{aligned}}$$

for $x = \sin^2\theta$

From the general consideration, we have

$$\langle \sigma(\infty)\sigma(1)\sigma(x)\sigma(0)\rangle = \sum_p C_{\sigma\sigma}^{p} C_{p\sigma\sigma} \left| \mathcal{F}_{\frac{1}{16}\frac{1}{16}}^{\frac{1}{16}\frac{1}{16}}(\Delta_p|x)\right|^2$$

for $\mathcal{F}_{\Delta\Delta'}^{\Delta''\Delta'''}(\Delta_p|x) = x^{\Delta_p-\Delta-\Delta'} \sum_{\{k\}} \beta_{\Delta\Delta'}^{\Delta_p\{k\}} \alpha_{\Delta''\Delta'''}^{\Delta_p\{k\}} x^{|k|}$

$$= x^{\Delta_p-\Delta-\Delta'} \sum_{\substack{\{k\}\\ \{k'\}}} \beta_{\Delta\Delta'}^{\Delta_p\{k\}} M_{\Delta_p}^{\{k\}\{k'\}} \beta_{\Delta''\Delta'''}^{\Delta_p\{k'\}} x^{|k|}$$

Note that

$$x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\cos\left(\frac{\theta}{2}\right) = x^{-\frac{1}{8}}\left(1+\frac{1}{8^2}x^2+\cdots\right)$$

$$x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}x^{\frac{1}{2}-\frac{1}{8}}\left(1+\frac{x}{4}+\frac{9}{8^2}x^2+\cdots\right)$$

This implies $x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\cos\left(\frac{\theta}{2}\right) = \mathcal{F}_{\frac{1}{16}\frac{1}{16}}^{\frac{1}{16}\frac{1}{16}}(0|x)$

$$x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}\mathcal{F}_{\frac{1}{16}\frac{1}{16}}^{\frac{1}{16}\frac{1}{16}}\left(\tfrac{1}{2}|x\right)$$

and $\boxed{\langle \sigma(\infty)\sigma(1)\sigma(x)\sigma(0)\rangle = \left|\mathcal{F}_{\frac{1}{16}\frac{1}{16}}^{\frac{1}{16}\frac{1}{16}}(0|x)\right|^2 + \frac{1}{4}\left|\mathcal{F}_{\frac{1}{16}\frac{1}{16}}^{\frac{1}{16}\frac{1}{16}}\left(\tfrac{1}{2}|x\right)\right|^2}$

This shows $\boxed{C_{\sigma\sigma\varepsilon} = \frac{1}{2}}$

In particular, it is non-zero, and hence $[\sigma]\times[\sigma] = [1]+[\varepsilon]$

$[\sigma]\times[\varepsilon] = [\sigma]$.

Check of $F^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(0|x) = x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\cos\left(\frac{\theta}{2}\right)$ (1)

$F^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(\frac{1}{2}|x) = 2x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\sin\left(\frac{\theta}{2}\right)$ (2)

<u>(1)</u>

<u>level 1</u> $\alpha^{0\{1\}}_{\frac{1}{16}\frac{1}{16}} = 0$ (no term of $O(x^1)$)

<u>level 2</u> $\alpha^{0\{2\}}_{\frac{1}{16}\frac{1}{16}} = \frac{1}{16}$, $\alpha^{0\{1,1\}}_{\frac{1}{16}\frac{1}{16}} = 0$, $M^{(2)}_0 = \begin{pmatrix} \frac{c}{2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore \beta^{0\{2\}}_{\frac{1}{16}\frac{1}{16}} = \frac{1}{4}$ ($\beta^{0\{1,1\}}_{\frac{1}{16}\frac{1}{16}}$ is immaterial)

$\therefore F^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(0|x) = x^{0-\frac{1}{16}-\frac{1}{16}}\left(1 + 0\cdot x + \underbrace{\frac{1}{16}\cdot\frac{1}{4}}_{=\frac{1}{8^2}}x^2 + O(x^3)\right)$

<u>matches</u>.

<u>(2)</u>

<u>level 1</u> $\alpha^{\frac{1}{2}\{1\}}_{\frac{1}{16}\frac{1}{16}} = \frac{1}{2}$, $M^{(1)}_{\frac{1}{2}} = \frac{1}{2\Delta_\varepsilon} = 1$ $\therefore \beta^{\frac{1}{2}\{1\}}_{\frac{1}{16}\frac{1}{16}} = \frac{1}{2}$ $\alpha^{(1)}\cdot\beta^{(1)} = \frac{1}{4}$

<u>level 2</u> $\alpha^{\frac{1}{2}\{2\}}_{\frac{1}{16}\frac{1}{16}} = \frac{9}{16}$, $\alpha^{\frac{1}{2}\{1,1\}}_{\frac{1}{16}\frac{1}{16}} = \frac{3}{4}$, $M^{(2)}_{\frac{1}{2}} = \begin{pmatrix} \frac{9}{4} & 3 \\ 3 & 4 \end{pmatrix}$

$\left(\beta^{\frac{1}{2}\{2\}}_{\frac{1}{16}\frac{1}{16}}, \beta^{\frac{1}{2}\{1,1\}}_{\frac{1}{16}\frac{1}{16}}\right) = \left(\frac{1}{4}, 0\right) + c\,(4, -3)$ (c is immaterial)

$\alpha^{(2)}\cdot\beta^{(2)} = \frac{9}{16}\cdot\frac{1}{4} = \frac{9}{8^2}$

$\therefore F^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(\frac{1}{2}|x) = \left(1 + \frac{1}{4}x + \frac{9}{8^2}x^2 + O(x^3)\right)x^{\frac{1}{2}-\frac{1}{16}-\frac{1}{16}}$

<u>matches</u>

$\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$

From the fusion rule $[\varepsilon]\times[\varepsilon]=[1]$, we expect the form

$$\langle \varepsilon(1)\,\varepsilon(2)\,\varepsilon(3)\,\varepsilon(4)\rangle = \left|\text{holomorphic}\right|^2$$

And this is indeed the form expected from the Majorana fermion representation $\varepsilon(x) = i\psi_-(x)\psi_+(x)$.

By Wick contraction, we can easily find

$$\langle \varepsilon(1)\,\varepsilon(2)\,\varepsilon(3)\,\varepsilon(4)\rangle = \left|\frac{1}{z_{12}z_{34}} + \frac{1}{z_{14}z_{23}} - \frac{1}{z_{13}z_{24}}\right|^2$$

In particular

$$\langle \varepsilon(\infty)\,\varepsilon(1)\,\varepsilon(x)\,\varepsilon(0)\rangle = \left|-1 + \frac{1}{x} + \frac{1}{1-x}\right|^2$$

This indeed satisfies the differential eqn

$$\left\{\frac{\partial^2}{\partial z_4^2} - \underbrace{\frac{2}{3}(2\Delta_\varepsilon+1)}_{=\frac{4}{3}}\sum_{i=1}^{3}\left(\frac{\overset{\Delta_\varepsilon=\frac{1}{2}}{\Delta_\varepsilon}}{(z_4-z_i)^2} + \frac{1}{z_4-z_i}\frac{\partial}{\partial z_i}\right)\right\}\langle\varepsilon(1)-\varepsilon(4)\rangle = 0$$

or

$$\left\{\frac{d^2}{dx^2} + \frac{4}{3}\left(\frac{1}{x}+\frac{1}{x-1}\right)\frac{d}{dx} - \frac{2}{3}\left[\frac{1}{x^2}+\frac{1}{(x-1)^2}+\frac{2}{x(1-x)}\right]\right\}F = 0$$

for $\langle \varepsilon(1)\,\varepsilon(2)\,\varepsilon(3)\,\varepsilon(4)\rangle = \dfrac{1}{z_{12}z_{34}\,\bar z_{12}\bar z_{34}}F(x,\bar x)$

(i.e. $\langle \varepsilon(\infty)\,\varepsilon(1)\,\varepsilon(x)\,\varepsilon(0)\rangle = F(x,\bar x)$)

Since $[\varepsilon]\times[\varepsilon]=[1]$, we must have

$$\langle \varepsilon(\infty)\,\varepsilon(1)\,\varepsilon(x)\,\varepsilon(0)\rangle = \underbrace{C^{1}_{\varepsilon\varepsilon}\,C_{1\varepsilon\varepsilon}}_{\substack{\parallel \\ 1 \text{ by normalization.}}} \left| \mathcal{F}^{\frac12\frac12}_{\frac12\frac12}(0|x)\right|^2$$

This implies

$$\mathcal{F}^{\frac12\frac12}_{\frac12\frac12}(0|x) = \frac{1}{x} - 1 + \frac{1}{1-x} = \frac{1}{x} + \sum_{n=1}^{\infty} x^n$$

Check

level 1 $\quad \alpha^{0\,\{1\}}_{\frac12\frac12} = \beta^{0\,\{1\}}_{\frac12\frac12} = 0 \qquad \underline{\alpha^{(1)}\cdot\beta^{(1)} = 0}$

level 2 $\quad \alpha^{0\,\{2\}}_{\frac12\frac12} = \frac12, \quad \alpha^{0\,\{1,1\}}_{\frac12\frac12} = 0, \quad M^{(2)}_0 = \begin{pmatrix} \frac14 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore$ one may take $\beta^{0\,\{2\}}_{\frac12\frac12} = 4\cdot\frac12 = 2 \qquad \therefore \underline{\alpha^{(2)}\cdot\beta^{(2)} = 1}$

level 3 $\quad \alpha^{0\,\{3\}}_{\frac12\frac12} = 1, \quad \alpha^{0\,\{2,1\}}_{\frac12\frac12} = \alpha^{0\,\{1,1,1\}}_{\frac12\frac12} = 0$

$$M^{(3)}_0 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad \text{wrt} \quad L_{-3}|0\rangle,\ L_{-2}L_{-1}|0\rangle,\ L_{-1}L_{-1}L_{-1}|0\rangle$$

$\Rightarrow \beta^{0\,\{3\}}_{\frac12,\frac12} = 1 \qquad \underline{\alpha^{(3)}\cdot\beta^{(3)} = 1}$

level 4 $\quad \alpha^{0\{4\}}_{\frac12\frac12} = \frac32, \quad \alpha^{0\{2,2\}}_{\frac12\frac12} = \frac54, \quad \alpha^{0\{3,1\}}_{\frac12\frac12} = \alpha^{(0)\{2,1,1\}}_{\frac12\frac12} = \alpha^{(0)\{1111\}}_{\frac12\frac12} = 0$

$$M^{(4)}_0 = \begin{pmatrix} \frac52 & \frac32 & & & \\ \frac32 & \frac{17}{8} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \quad \text{wrt} \quad L_{-4}|0\rangle,\ L_{-2}L_{-2}|0\rangle,\ L_{-3}L_{-1}|0\rangle,\ L_{-2}L_{-1}^2|0\rangle,\ L_{-1}^4|0\rangle.$$

$\Rightarrow \beta^{0\{4\}}_{\frac12\frac12} = \frac37, \quad \beta^{0\{2,2\}}_{\frac12\frac12} = \frac27, \quad \underline{\alpha^{(4)}\cdot\beta^{(4)} = \frac{3}{2}\cdot\frac{3}{7} + \frac{5}{4}\cdot\frac{2}{7} = \frac{14}{14} = 1}$

$\langle \sigma\sigma\varepsilon\varepsilon \rangle$: Homework

① Find the expression for $\langle \sigma(1)\sigma(2)\varepsilon(3)\varepsilon(4) \rangle$

in the form $f(z_{12}, z_{34}, \bar{z}_{12}, \bar{z}_{34}) \cdot F(x, \bar{x})$

using projective Ward identity.

② Write down the differential eqn for $\underbrace{\langle \varepsilon(\infty)\sigma(1)\varepsilon(x)\sigma(0) \rangle}_{=F(x,\bar{x})}$.

③ Solve the equation.

Hint Using the behaviour as $x \to 0$ & $x \to 1$,

it is convenient to write

$$F(x, \bar{x}) = x^{\star}(1-x)^{\star'}\bar{x}^{\star}(1-\bar{x})^{\star'}\, U(x, \bar{x})$$

for some $\star$, $\star'$.

Then write the eqn for $U(x, \bar{x})$.

To find the solution, assume the form $U = x^{K} + \text{higher}$

and write the eqn for K.

Choose the "right" K, found by the $x \to 0$ condition.

④ Find the correct normalization of the solution

⑤ → continued

⑤ : Find the expression for

$$\mathcal{F}^{\frac{1}{16}\,\frac{1}{2}}_{\frac{1}{2}\,\frac{1}{16}}\left(\tfrac{1}{16}\,\middle|\,x\right) \quad \text{and} \quad \mathcal{F}^{\frac{1}{16}\,\frac{1}{16}}_{\frac{1}{2}\,\frac{1}{2}}\left(0\,\middle|\,x\right)$$

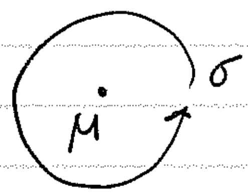
⑥ If you have enough Guts, check the expression in ⑤ for the first few terms by computing $\alpha^{\Delta''\{k\}}_{\Delta\Delta'}$, $M^{(\cdot\cdot)}_{\Delta''}$, $\beta^{\Delta''\{k\}}_{\Delta\Delta'}$.

$\langle \sigma\sigma\mu\mu \rangle$

$\mu =$ disorder operator. (σ in the dual system).

Since $T = T_c$ is the self dual point, μ must have the same conformal weights as σ:

$$\Delta_\mu = \tilde{\Delta}_\mu = \frac{1}{16}.$$

We also know that  produces a sign (-1).

Since $\Delta_\mu = \tilde{\Delta}_\mu = \frac{1}{16}$, the four point function

$G = \langle \mu(\infty)\,\sigma(1)\,\sigma(x)\,\mu(0) \rangle$ must obey the same differential eqn as $\langle \sigma(\infty)\sigma(1)\sigma(x)\sigma(0)\rangle$.

In particular it can be written as

$$G = \left| x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}} \right|^2 \Big\{ g_{11} \cos\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\bar\theta}{2}\right) + g_{12} \cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\bar\theta}{2}\right) + g_{21} \sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\bar\theta}{2}\right) + g_{22} \sin\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\bar\theta}{2}\right) \Big\}$$

The difference is the monodromy:

$\theta \to -\theta$

For $\underset{0}{\circlearrowleft}\, x$ $\quad\sigma$ circles μ, we must have (-1) sign.

For $\underset{i}{\circlearrowleft}\, x$ $\quad\sigma$ circles σ, the result must be invariant.

$\theta \to \pi - \theta$ $\quad\Longrightarrow\quad g_{11} = g_{22} = 0 \quad \& \quad g_{12} = g_{21} =: g$

$$\therefore\ G = g\left|x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\right|^2 \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\bar{\theta}}{2}\right) + g\left|x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\right| \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\bar{\theta}}{2}\right).$$

As $x \to 1$ we must have $G \to \dfrac{1}{|x-1|^{\frac{1}{4}}}\langle \mu(\infty)\mu(0)\rangle \sim \dfrac{1}{|x-1|^{\frac{1}{4}}}$

$$x \sim 1 \Leftrightarrow \theta \sim \frac{\pi}{2} \quad \therefore\ \sin\frac{\theta}{2} \sim \frac{1}{\sqrt{2}} \sim \cos\frac{\theta}{2}$$

$$\therefore\ \boxed{g = 1}$$

$\therefore \langle \mu(\infty)\,\sigma(1)\,\sigma(x)\,\mu(0)\rangle$

$$= 1\left|x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\right|^2\left(\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\bar{\theta}}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\bar{\theta}}{2}\right)\right)$$

$$= 1\,x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\sin\frac{\theta}{2}\cdot\overline{x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\cos\frac{\theta}{2}} + x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\cos\frac{\theta}{2}\cdot\overline{1\,x^{-\frac{1}{8}}(1-x)^{-\frac{1}{8}}\sin\frac{\theta}{2}}$$

$$= \underset{\frac{1}{2}}{\wedge}\,\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}\left(\tfrac{1}{2}\middle|x\right)\cdot\overline{\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(0|x)} + \underset{\frac{1}{2}}{\wedge}\,\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}(0|x)\,\overline{\mathcal{F}^{\frac{1}{16}\frac{1}{16}}_{\frac{1}{16}\frac{1}{16}}\left(\tfrac{1}{2}\middle|x\right)}$$

This shows that the OPE $\sigma(x)\mu(0)$ includes two primaries, one with $(\Delta, \bar{\Delta}) = (\frac{1}{2}, 0)$ another with $(0, \frac{1}{2})$.

The obvious candidates are the fermions ψ_- and ψ_+.

Moreover we find that the structure constants are

$$C_{\sigma\mu}^{\psi_-} = C_{\sigma\mu}^{\psi_+} = \frac{1}{\sqrt{2}}$$

Thus, the precise OPE is

$$\sigma(x)\mu(0) = \frac{1}{\sqrt{2}} \underbrace{x^{\frac{3}{8}} \bar{x}^{-\frac{1}{8}}}_{= x^{\frac{1}{2}}/|x|^{\frac{1}{4}}} (\psi_-(0) + \cdots) + \frac{1}{\sqrt{2}} \underbrace{x^{-\frac{1}{8}} \bar{x}^{\frac{3}{8}}}_{= \bar{x}^{\frac{1}{2}}/|x|^{\frac{1}{4}}} (\psi_+(0) + \cdots)$$

This also implies

$$\psi_-(z)\sigma(0) = \frac{1}{z^{\frac{1}{2}}}(\mu(0) + \cdots), \quad \psi_+(z)\sigma(0) = \frac{1}{\bar{z}^{\frac{1}{2}}}(\mu(0) + \cdots)$$

$$\psi_-(z)\mu(0) = \frac{1}{z^{\frac{1}{2}}}(\sigma(0) + \cdots), \quad \psi_+(z)\mu(0) = \frac{1}{\bar{z}^{\frac{1}{2}}}(\sigma(0) + \cdots).$$