Landau-Ginzburg description of unitary minimal CFTs.

Q: Is there an ordinary field theory whose IR fixed point is the unitary minimal CFT?

Zamolodchikov proposes an answer to this question:

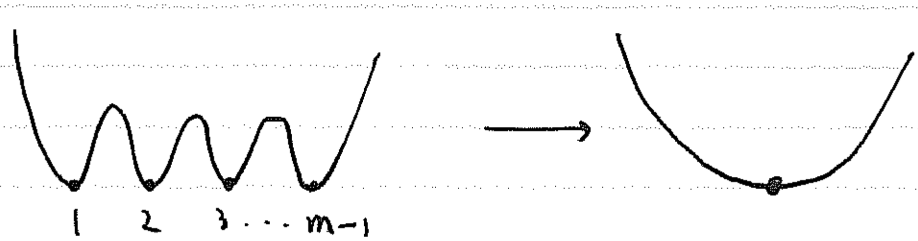
It is the scalar field theory with an even potential

$$U(\varphi) = \sum_{n=1}^{N} g_{2n} \varphi^{2n}$$

where the coupling constants g_{2n} are fine tuned, so that the (1PI) effective potential has a degenerate minimum at $\varphi = 0$, in the form

$$U_{\text{1PI eff}}(\varphi) = g\,\varphi^{2(m-1)} \quad (+\ \text{higher}).$$

Such a potential can be realized as a limit of a potential with separate $(m-1)$ degenerate minima



Thus, the claim is that the m-th minimal CFT describes the multicritical behaviour where $(m-1)$ degenerate minima coalesce.

The main supporting evidence is the spectrum of relevant operators and their operator product algebra.

The equation of motion is $\partial_\mu \partial_\mu \varphi = g \varphi^{2m-3}$.

Since the dimension of $\varphi \neq 1$ must be positive, φ^{2m-3} must be an irrelevant operator.

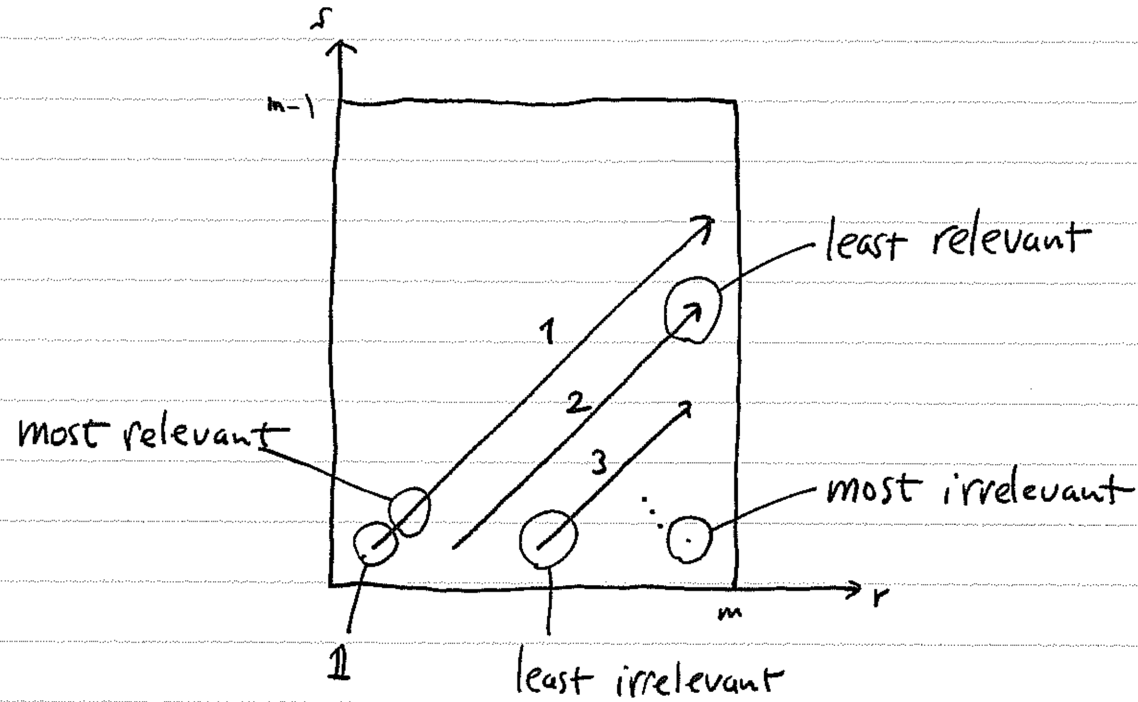
On the other hand, $\varphi^{2(m-2)}$ changes the criticality and cannot be irrelevant nor marginal.

Thus, we must have relevant operators

$$1, \varphi, \varphi^2, \varphi^3, \cdots, \varphi^{2(m-2)}, \qquad (2m-3 \text{ of them})$$

with increasing dimensions.

For the m-th unitary minimal CFT, the primaries are ordered with the increasing dimensions as



i.e.

(marginal)

$$0 \longrightarrow 1 \longrightarrow \Delta$$

$$\underbrace{\Delta_{1,1},\ \Delta_{2,2},\ \Delta_{3,3},\ \cdots\ \Delta_{m-1,m-1},\ \Delta_{2,1},\ \Delta_{3,2},\ \cdots\ \Delta_{m-1,m-2}}_{\text{There are } 2m-3 \text{ relevant primaries.}},\ \Delta_{3,1},\ \Delta_{4,2},\ \cdots\ \Delta_{m-1,1}$$

$$\Delta_{2,2} = \frac{3}{4m(m+1)}$$

$$\Delta_{m-1,m-2} = \Delta_{1,3} = \frac{m-1}{m+1}$$

$$\Delta_{3,1} = \frac{m^2+3m+2}{m(m+1)}$$

$$\Delta_{m-1,1} = \frac{(m-1)(m-2)}{4}$$

This suggests the identification (up to constant)

$$\phi_{\Delta_{2,2}} = \varphi \qquad \phi_{\Delta_{2,1}} = :\varphi^{m-1}:$$

$$\phi_{\Delta_{3,3}} = :\varphi^{2}: \qquad \phi_{\Delta_{3,2}} = :\varphi^{m}:$$

$$\vdots \qquad\qquad \vdots$$

$$\phi_{\Delta_{m-1,m-1}} = :\varphi^{m-2}: \qquad \phi_{\underset{\Delta_{1,3}}{\overset{\|}{\Delta_{m-1,m-2}}}} = :\varphi^{2m-4}:$$

Here $:\varphi^n:$ are defined recursively by

$$:\varphi^{n+1}:(0) = \lim_{x\to 0} |x|^{d_1 + d_n - d_{n+1}} \Big\{ \varphi(x) :\varphi^n:(0) - \sum_{i<n} a_i |x|^{d_i - d_1 - d_n} :\varphi^i:(0) \Big\}_{\text{scalar}} .$$

i.e. $|x|^{d_{n+1}-d_1-d_n} :\varphi^{n+1}:(0)$ is the most singular scalar term in the OPE of $\varphi(x):\varphi^n:(0)$ other than the term involving $:\varphi^i:$ with $i<n$.

Let's see ~~wheather~~ whether this identification is consistent with the fusion rule.

$$[\phi_{\Delta_{2,2}}]\times[\phi_{\Delta_{2,2}}] = [\phi_{\Delta_{1,1}}]+[\phi_{\Delta_{1,3}}]+[\phi_{\Delta_{3,1}}]+[\phi_{\Delta_{3,3}}]$$

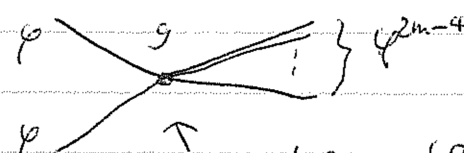
(a term $[\phi_{\Delta_{r,s}}]$ drops if $r \geq m$, $s \geq m+1$. We assume m is sufficiently large for simplicity.)

Since $\Delta_{3,3} < \Delta_{1,3}, \Delta_{3,1}$, the most singular term is $[\phi_{\Delta_{3,3}}]$. (other than $[\phi_{\Delta_{1,1}}] = [1]$)

Thus, if $\phi_{\Delta_{2,2}} = \varphi$, then $\phi_{\Delta_{3,3}} = :\varphi^2:$

※ The appearance of $\phi_{\Delta_{1,3}}$ "=" $:\varphi^{2m-4}:$ is also consistent:

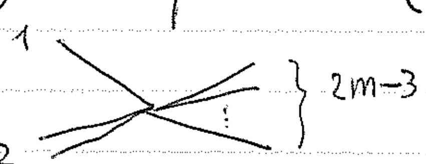
$\langle \varphi\ \varphi\ \varphi^{2m-4} \rangle \neq 0$ by



from $V = g\varphi^{2m-2}$

Next

$$\underset{\substack{\uparrow\\ \varphi}}{[\phi_{\Delta_{2,2}}]}\times\underset{\substack{\uparrow\\ :\varphi^2:}}{[\phi_{\Delta_{3,3}}]} = \underset{\substack{\uparrow\\ \varphi\varphi\varphi}}{[\phi_{\Delta_{2,2}}]}+\underset{\substack{\uparrow\\ :\varphi^3:}}{[\phi_{\Delta_{4,4}}]}+\underset{\substack{\shortparallel\\ \Delta_{m-2,m-3}\\ \uparrow}}{[\phi_{\Delta_{2,4}}]}+\underset{\substack{\uparrow\\ \text{irrelevant}\\ \text{(less singular)}}}{[\phi_{\Delta_{4,2}}]}$$



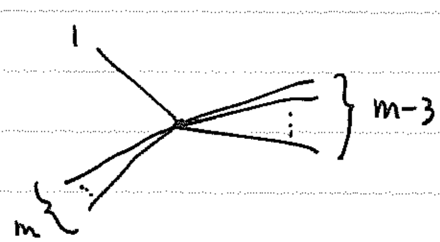
$\Rightarrow \phi_{\Delta_{4,4}} = :\varphi^3:$

$\vdots$

$$\begin{array}{ccccccccccccc}
[\phi_{\Delta_{2,2}}] & \times & [\phi_{\Delta_{m-1,m-1}}] & = & [\phi_{\Delta_{m-2,m-2}}] & + & [\phi_{\Delta_{m-2,m}}] & + & \cancel{[\phi_{\Delta_{m,m}}]} & + & \cancel{[\phi_{\Delta_{m,m-2}}]} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & & & \\
\varphi & & :\varphi^{m-2}: & & \varphi\,:\varphi^{m-2}: & & :\varphi^{m-1}: & & & &
\end{array}$$

$$\begin{array}{ccccccccccccc}
[\phi_{\Delta_{2,2}}] & \times & [\phi_{\Delta_{2,1}}] & = & [\phi_{\Delta_{1,2}}] & + & [\phi_{\Delta_{3,2}}] & + & \cancel{[\phi_{\Delta_{1,0}}]} & + & \cancel{[\phi_{\Delta_{3,0}}]} \\
 & & & & \overset{\shortparallel}{\Delta_{m-1,m-1}} & & & & & & \\
 & & & & \shortparallel & & \uparrow & & & & \\
\uparrow & & \uparrow & & [:\varphi^{m-2}:] & & :\varphi^{m}: & & & & \\
 & & & & \uparrow & & & & & & \\
\varphi & & :\varphi^{m-1}: & & \varphi\,:\varphi^{m-1}: & & & & & &
\end{array}$$

$$\begin{array}{ccccccccccccccc}
[\phi_{\Delta_{2,2}}] & \times & [\phi_{\Delta_{3,2}}] & = & [\phi_{\Delta_{2,1}}] & + & [\phi_{\Delta_{2,3}}] & + & [\phi_{\Delta_{4,3}}] & + & [\phi_{\Delta_{4,1}}] \\
 & & & & & & \overset{\shortparallel}{\Delta_{m-2,m-2}} & & & & \uparrow \\
\uparrow & & \uparrow & & \uparrow & & \shortparallel & & \uparrow & & \text{irrelevant} \\
\varphi & & :\varphi^{m}: & & \varphi\,:\varphi^{m}: & & [:\varphi^{m-3}:] & & :\varphi^{m+1}: & & \text{(less singular)} \\
 & & & & & & \uparrow & & & &
\end{array}$$

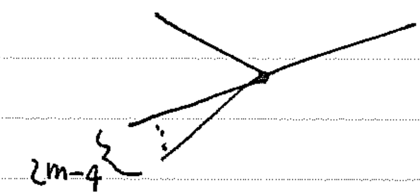


$\vdots$

$$\underset{\varphi}{\underset{\uparrow}{[\phi_{\Delta_{2,2}}]}} \times \underset{:\varphi^{2m-4}:}{\underset{\uparrow}{[\phi_{\Delta_{m-1,m-2}}]}} = \underset{\varphi\,:\varphi^{2m-4}:}{\underset{\uparrow}{\overset{[:\varphi^{2m-5}:]}{\overset{\parallel}{[\phi_{\Delta_{m-2,m-3}}]}}}} + \underset{[\varphi]}{\underset{\parallel}{[\phi_{\Delta_{m-2,m-1}}]}} + \cancel{[\phi_{\Delta_{m,m-1}}]} + \cancel{[\phi_{\Delta_{m,m-2}}]}$$

$\Delta_{m-2,m-1} = \Delta_{2,2}$

$\uparrow$



The most singular scalar term in $\varphi(x) :\varphi^{2m-4}:(0)$

Other than $|x|^{d_{2m-5}-d_1-d_{2m-4}} :\varphi^{2m-5}:(0)$

$$+ g\,|x|^{d_1-d_1-d_{2m-4}}\,\varphi(0)$$

is $g\,|x|^{d_1-2-d_1-d_{2m-4}}\,L_{-1}\tilde{L}_{-1}\varphi(0) \times \#$

$$\Longrightarrow \quad :\varphi^{2m-3}: \sim L_{-1}\tilde{L}_{-1}\varphi = \partial_\mu\partial_\mu\varphi$$

— Equation of motion!

<u>Everything is consistent</u>

This is good, But the LG description does not allow us to compute detailed information, such as the central charge c and the dimensions (conformal weights) of the operators.

However, it gives us a qualitative and intuitive picture of the unitary minimal CFT.

It is also useful in understanding the RG flows induced by perturbations of the CFT by relevant operators. [e.g. Perturbation by φ^{2m-4} shifts the potential as $g\varphi^{2(m-1)} \longrightarrow g\varphi^{2(m-1)} + \epsilon\varphi^{2(m-2)}$, and we expect that it flows to $(m-1)$-th CFT. (This will be confirmed by a more precise argument later.).]

With an extended supersymmetry, (i.e. $\mathcal{N}=(2,2)$ SUSY), LG description is also powerful quantitatively.

The supercharges $Q_\pm, \bar{Q}_\pm$ allow us to study the "chiral sector" which is protected from renormalizations.

We can compute exactly the "chiral ring" (OPE relations of operators in the chiral sector). This includes the superconformal algebra $T(z), G(z), \bar{G}(z), J(z)$ [in the $\bar{Q}_+$-chiral sector]. By computing $T(z)T(0) \sim \frac{c/2}{(z-0)^4} + \cdots$ we can compute the central charge.

Also, "U(1) R-symmetry" and conformal symmetry are related by supersymmetry, and from this one can compute the conformal weights of operators in the chiral sector just by looking at their classical "U(1) R-charges".