§ C-Theorem

Set-up

Riemannian metric

A QFT formulated on a d-dimensional (Σ, g)

Need to specify :

regularization with UV cut-off Λ_0

[e.g. $\Lambda_0 = \frac{1}{a}$, a = lattice spacing of a lattice that approximates Σ]

renormalization : cut-off dependent coupling constants $\lambda_0 = \lambda(\Lambda_0) = (\lambda^1(\Lambda_0), \cdots, \lambda^n(\Lambda_0))$

Alternatively, with a choice of renormalization condition, the theory depends on

- renormalization point Λ
- renormalized coupling constant $\lambda = (\lambda^1, \cdots, \lambda^n)$

Statement of Renormalization Group flow :

The content of the theory does not changes as Λ_0 is varied if the bare coupling λ_0 varies according to $\lambda_0 = \lambda(\Lambda_0)$.

$\overset{\text{or}}{\Longleftrightarrow}$ The content of the theory does not change as $(\Lambda, \lambda) \longrightarrow (e^{-t}\Lambda, \lambda(-t))$

a certain function of t.

The β-function is defined by

$$\frac{d}{dt}\lambda^i(t)\Big|_{t=0} =: \beta^i(\lambda).$$

- Here, we have assumed that the couplings $\lambda=(\lambda^1,\cdots,\lambda^n)$ have <u>zero canonical dimension.</u>
- We will also choose a set of variables $\phi=(\phi^1,\cdots,\phi^m)$ which have zero canonical dimension.

e.g. Kinetic term is of the form for a scalar

$$\int\sqrt{g}\,d^dx\;\Lambda^{d-2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\times K$$

here K is one of the coupling constants.
We decide <u>not to</u> normalize the field so that it has always the canonical kinetic term $\int(\partial_\mu\phi)^2\sqrt{g}\,d^dx$.
Therefore the field variables does not flow, but the coefficients K's flow.

Let us put the RG flow statement into an equation. For this, let's introduce the notation

$$d\mu(\phi)_{g,\Lambda,\lambda} = \mathcal{D}_{g,\Lambda,\lambda}\phi\; e^{-S_{g,\Lambda,\lambda}(\phi)}\Big/Z$$

Then, <u>RG flow</u> amounts to

$$d\mu(\phi)_{g,\Lambda,\lambda} = d\mu(\phi)_{g,\, e^{-t}\Lambda,\, \lambda(-t)} . \tag{1}$$

By the <u>dimensional analysis</u>, we have

$$d\mu(\phi)_{g,\Lambda,\lambda} = d\mu(\phi)_{e^{2t}g,\, e^{-t}\Lambda,\, \lambda} . \tag{2}$$

We also require <u>diffeomorphism invariance</u>

$$d\mu(\phi)_{g,\Lambda,\lambda} = d\mu(f^{*}\phi)_{f^{*}g,\, \Lambda,\, \lambda} \tag{3}$$

for $f: \Sigma \to \Sigma$ a diffeomorphism.

We will derive some consequences of these facts/requirements, on correlation functions

$$\langle \mathcal{O}(x_1, \cdots, x_s) \rangle_{g,\Lambda,\lambda} = \int d\mu(\phi)_{g,\Lambda,\lambda}\, \mathcal{O}^{x_1,\cdots,x_s}_{g,\Lambda,\lambda}(\phi)$$

We will be particularly interested in $(\Sigma, g) = (\mathbb{R}^d, \sum_{\mu=1}^{d} (dx^\mu)^2)$ which has scale transformation (dilatation)

$$f_t : (x^1, x^2, \cdots, x^d) \mapsto (e^t x^1, e^t x^2, \cdots, e^t x^d)$$

which does $f_t^* g = e^{2t} g$.

$$\langle \mathcal{O}(x_1,\cdots,x_s)\rangle_{g,\Lambda,\lambda} = \int d\mu(\phi)_{g,\Lambda,\lambda}\, \mathcal{O}^{x_1,\cdots,x_s}_{g,\Lambda,\lambda}(\phi)$$

$$\overset{(1)}{=} \int d\mu(\phi)_{g,e^t\Lambda,\lambda(t)}\, \mathcal{O}^{x_1,\cdots,x_s}_{g,\Lambda,\lambda}(\phi)$$

$$\overset{(2)}{=} \int d\mu(\phi)_{\underbrace{e^{2t}g}_{=f_t^* g},\Lambda,\lambda(t)}\, \mathcal{O}^{x_1,\cdots,x_s}_{\underbrace{g}_{=e^{-2t}f_t^* g},\Lambda,\lambda}(\phi)$$

denote $\phi = f_t^*\phi'$

$$= \int d\mu(f_t^*\phi')_{f_t^* g,\Lambda,\lambda(t)}\, \mathcal{O}^{x_1,\cdots,x_s}_{e^{-2t}f_t^* g,\Lambda,\lambda}(f_t^*\phi')$$

$$\overset{(3)}{=} \int d\mu(\phi')_{g,\Lambda,\lambda(t)}\, \mathcal{O}^{f_t x_1,\cdots,f_t x_s}_{e^{-2t}g,\Lambda,\underbrace{\lambda}_{=\lambda(t-t)}}(\phi')$$

At this point, we define

$$\boxed{(\hat{\Gamma}\mathcal{O})^{x_1,\cdots,x_s}_{g,\Lambda,\lambda}(\phi) := \frac{d}{dt}\mathcal{O}^{x_1,\cdots,x_s}_{e^{-2t}g,\Lambda,\lambda(-t)}(\phi)\Big|_{t=0}.}$$

Then we have

$$0 = \frac{d}{dt}\langle \mathcal{O}(f_t x_1,\cdots,f_t x_s)\rangle_{g,\Lambda,\lambda(t)}\Big|_{t=0} + \langle \hat{\Gamma}\mathcal{O}(x_1,\cdots,x_s)\rangle_{g,\Lambda,\lambda}$$

i.e.

$$\boxed{\sum_{a=1}^{s} x_a^\mu \frac{\partial}{\partial x_a^\mu} \langle \mathcal{O}(x_1,\cdots,x_s)\rangle_{g,\Lambda,\lambda} + \langle \hat{\Gamma}\mathcal{O}(x_1,\cdots,x_s)\rangle_{g,\Lambda,\lambda} + \sum_{i=1}^{n} \beta^i(\lambda)\frac{\partial}{\partial\lambda^i}\langle \mathcal{O}(x_1,\cdots,x_s)\rangle_{g,\Lambda,\lambda} = 0}$$

This is the (Callan-Symanzik type) RG equation for correlation functions.

Next, we derive some other consequences of (1), (2), (3).
(1) and (2) yield:

$$d\mu(\varphi)_{g,\Lambda,\lambda} = d\mu(\varphi)_{ge^{-2t},\Lambda,\lambda(-t)}. \qquad (4).$$

Take the metric variation of this equation:
$g \to g+\delta g$

energy-momentum tensor.

$$\delta(\text{LHS}) = d\mu(\varphi)_{g,\Lambda,\lambda}\frac{1}{4\pi}\int\sqrt{g}\,d^dx\,\delta g^{\mu\nu}\left(T_{\mu\nu}\right)_{g,\Lambda,\lambda}$$

$$\delta(\text{RHS}) = \underbrace{d\mu(\varphi)_{e^{-2t}g,\Lambda,\lambda(-t)}}_{\substack{\| \text{(4) again} \\ d\mu(\varphi)_{g,\Lambda,\lambda}}}\frac{1}{4\pi}\int\sqrt{e^{-2t}g}\,d^dx\,e^{2t}\,\delta g^{\mu\nu}\left(T_{\mu\nu}\right)_{e^{-2t}g,\Lambda,\lambda(-t)}$$

Thus, we find

$$(T_{\mu\nu})_{g,\Lambda,\lambda} = e^{-dt} e^{2t} (T_{\mu\nu})_{e^{-2t}g,\Lambda,\lambda(-t)}.$$

Taking $\frac{d}{dt}\Big|_{t=0}$, we have $0 = (-d+2) T_{\mu\nu} + \hat{\Gamma} T_{\mu\nu}$, i.e.

$$\boxed{\hat{\Gamma} T_{\mu\nu} = (d-2) T_{\mu\nu}}$$

i.e. $T_{\mu\nu}$ always have dimension $(d-2)$. This is of course from the fact that $\int \sqrt{g}\, d^d x\, \delta g^{\mu\nu} T_{\mu\nu}$ is dimensionless.

Next, take $\frac{d}{dt}\Big|_{t=0}$ of the eqn (4) itself. For this, we introduce operators $\phi_1(x), \cdots, \phi_n(x)$ by

$$\frac{\partial}{\partial\lambda^i} d\mu(\phi)_{g,\Lambda,\lambda} = d\mu(\phi)_{g,\Lambda,\lambda}\left(-\int \sqrt{g}\, d^d x \left(\phi_i(x)\right)_{g,\Lambda,\lambda}\right).$$

The idea behind this definition is

$$\text{"} S_{g,\Lambda,\lambda}(\phi) = \int \sqrt{g}\, d^d x \left(\cdots + \sum_i \lambda^i \phi_i(x) \right) \text{"}.$$

$\frac{d}{dt}(4)\Big|_0$ now gives

$$0 = \frac{1}{2\pi} T_{\mu\nu} g^{\mu\nu} + \sum_i \beta^i(\lambda)\, \phi_i \quad , \qquad \text{i.e.}$$

$$\boxed{T^\mu_\mu = -2\pi \sum_i \beta^i(\lambda)\, \phi_i}$$

Finally take $\frac{\partial}{\partial \lambda^i}$ of (4):

$$d\mu(\phi)_{g,\Lambda,\lambda} \left(-\int \sqrt{g}\, d^d x \left(\phi_i(x) \right)_{g,\Lambda,\lambda} \right)$$

$$= d\mu(\phi)_{ge^{-2t},\Lambda,\lambda(-t)} \left(-\int \underbrace{\sqrt{e^{-2t} g}}_{= e^{-dt}\sqrt{g}}\, d^d x \left(\phi_j(x) \right)_{e^{-2t}g,\Lambda,\lambda(-t)} \right) \frac{\partial \lambda^j(-t)}{\partial \lambda^i}$$

Take the $\frac{d}{dt}\Big|_0$ of this eqn:

$$0 = d\,\phi_i - \hat{\Gamma}\phi_i + \sum_j \phi_j \frac{\partial \beta^j(\lambda)}{\partial \lambda^i} \quad , \quad \text{i.e.}$$

$$\boxed{\hat{\Gamma}\phi_i = d\,\phi_i + \sum_j \frac{\partial \beta^j(\lambda)}{\partial \lambda^i}\phi_j}.$$

Note that $\hat{\Gamma} T^\mu_\mu = -2\pi \sum_i \left\{ \cancel{-\sum_j \beta^j \left(\frac{\partial}{\partial \lambda^j} \beta^i\right) \phi_i} + \beta^i \left(d\phi_i + \cancel{\sum_j \frac{\partial \beta^j}{\partial \lambda^i}\phi_j} \right) \right\}$

$$= d\, T^\mu_\mu$$

This is consistent with $\hat{\Gamma} T_{\mu\nu} = (d-2) T_{\mu\nu}$

and $T^\mu_\mu := g^{\mu\nu} T_{\mu\nu}$.

Now, let's consider two-dimensions ($\underline{d=2}$).

We know that $T_{\mu\nu}$ has dimension $d-2=0$, $\boxed{\hat{\Gamma} T_{\mu\nu} = 0}$

For a flat metric g, we choose a complex coordinate z s.t. $g = |dz|^2$. Under the rescaling $g \to e^{-2t} g$, it changes as $z \to e^{-t} z =: z_t$. Then, the components $T_{zz}, T_{z\bar{z}}, T_{\bar{z}\bar{z}}$ of the energy-momentum tensor change as

$$T_{zz} \to T_{z_t z_t} = e^{2t} T_{zz} \quad , \text{ etc.}$$

This shows that they have dimension 2:

$$\boxed{\hat{\Gamma} T_{zz} = 2 T_{zz}, \quad \hat{\Gamma} T_{z\bar{z}} = 2 T_{z\bar{z}}, \quad \hat{\Gamma} T_{\bar{z}\bar{z}} = 2 T_{\bar{z}\bar{z}}}$$

Thus, the RG eqn for correlation functions involving $T = T_{zz}, T_{z\bar{z}}, T_{\bar{z}\bar{z}}$ takes the form

$$\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} + 2 \right) \langle T(z)\, \mathcal{O}(x_1, \cdots, x_s) \rangle$$
$$+ \sum_{a=1}^{s} \left(z_a \frac{\partial}{\partial z_a} + \bar{z}_a \frac{\partial}{\partial \bar{z}_a} \right) \langle T(z)\, \mathcal{O}(x_1, \cdots, x_s) \rangle + \langle T(z)\, \hat{\Gamma}\mathcal{O}(x_1, \cdots, x_s) \rangle$$
$$+ \sum_{i=1}^{n} \beta^i(\lambda) \frac{\partial}{\partial \lambda^i} \langle T(z)\, \mathcal{O}(x_1, \cdots, x_s) \rangle = 0.$$

Let us put

$$F = z^4 \langle T_{zz}(z)\, T_{zz}(0) \rangle$$

$$H = z^3 \bar{z} \langle T_{z\bar{z}}(z)\, T_{zz}(0) \rangle$$

$$G = z^2 \bar{z}^2 \langle T_{z\bar{z}}(z)\, T_{z\bar{z}}(0) \rangle$$

By the rotational symmetry of the system, they depend on $z, \bar{z}$ only through $|z|^2$. i.e. they are annihilated by $z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}$:

$$0 = \left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) F = z^4 \left(4 + z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) \langle T_{zz}(z)\, T_{zz}(0) \rangle$$

$$0 = \left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) H = z^3 \bar{z} \left(2 + z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) \langle T_{z\bar{z}}(z)\, T_{zz}(0) \rangle$$

$$0 = \left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) G = z^2 \bar{z}^2 \left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) \langle T_{z\bar{z}}(z)\, T_{z\bar{z}}(0) \rangle .$$

Let us now compute $\dot{F} = \left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right) F$, etc :

$$\dot{F} = z^4 \left(4 + z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right) \langle T_{zz}(z)\, T_{zz}(0) \rangle$$

$$\dot{G} = z^3 \bar{z} \left(4 + z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right) \langle T_{z\bar{z}}(z)\, T_{zz}(0) \rangle$$

$$\dot{H} = z^2 \bar{z}^2 \left(4 + z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right) \langle T_{z\bar{z}}(z)\, T_{z\bar{z}}(0) \rangle$$

Using the relation from rotational invariance, we can write them as

$$\dot{F} = z^4\left(2\bar{z}\frac{\partial}{\partial\bar{z}}\right)\langle T_{zz}(z)\,T_{zz}(0)\rangle$$

$$\dot{H} = z^3\bar{z}\left(6 + 2z\frac{\partial}{\partial z}\right)\langle T_{z\bar{z}}(z)\,T_{zz}(0)\rangle$$

$$\overset{\text{or}}{=} z^3\bar{z}\left(2 + 2\bar{z}\frac{\partial}{\partial\bar{z}}\right)\langle T_{z\bar{z}}(z)\,T_{zz}(0)\rangle$$

$$\dot{G} = z^2\bar{z}^2\left(4 + 2z\frac{\partial}{\partial z}\right)\langle T_{z\bar{z}}(z)\,T_{z\bar{z}}(0)\rangle$$

Using the conservation equation $\partial_{\bar{z}}T_{zz} + \partial_z T_{z\bar{z}} = 0$, we may write $\dot{F}$ as $\dot{F} = -z^4\left(2\bar{z}\frac{\partial}{\partial z}\right)\langle T_{z\bar{z}}(z)\,T_{zz}(0)\rangle$. Then we find

$$\boxed{\dot{F} + \dot{H} = 6H}$$

Using the translation symmetry, we may write $\underbrace{\frac{\partial}{\partial\bar{z}}}_{\text{for } \dot{H}}\langle T_{z\bar{z}}(z)\,T_{zz}(0)\rangle$ $= -\frac{\partial}{\partial\bar{w}}\langle T_{z\bar{z}}(z)\,T_{zz}(w)\rangle\Big|_{w=0}$. We can then again use the conservation equation to write it $= \frac{\partial}{\partial w}\langle T_{z\bar{z}}(z)\,T_{z\bar{z}}(w)\rangle\Big|_{w=0}$ $= -\frac{\partial}{\partial z}\langle T_{z\bar{z}}(z)\,T_{z\bar{z}}(0)\rangle$ where translation symmetry is used again.

This shows

$$\boxed{\dot{H} + \dot{G} = 2H + 4G}$$

If we put $C = 2F - 4H - 6G$, we have

$$\boxed{\begin{aligned}\dot{C} &= 2\dot{F} - 4\dot{H} - 6\dot{G} \\ &= 2(6\cancel{H} - \cancel{\dot{H}}) - \cancel{4\dot{H}} - 6(\cancel{2H} + 4G - \cancel{\dot{H}}) = -24G.\end{aligned}}$$

Now, we use the RG Eqn for $T, T' = T_{zz}, T_{z\bar{z}}, T_{\bar{z}\bar{z}}$:

$$\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} + 2\right)\langle T(z) T'(w)\rangle + \left(w\frac{\partial}{\partial w} + \bar{w}\frac{\partial}{\partial \bar{w}} + 2\right)\langle T(z) T'(w)\rangle$$

$$+ \sum_{i=1}^{n} \beta^i(\lambda)\frac{\partial}{\partial \lambda^i}\langle T(z) T'(w)\rangle = 0$$

$$\xrightarrow{w=0} \quad \left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} + 4\right)\langle T(z) T'(0)\rangle = -\sum_{i=1}^{n} \beta^i(\lambda)\frac{\partial}{\partial \lambda^i}\langle T(z) T'(w)\rangle$$

Using this, $\dot{C} = -24G$ can be written as

$$-\sum_{i=1}^{n} \beta^i(\lambda)\frac{\partial}{\partial \lambda^i} C = -24G \qquad \forall z.$$

As a convention, let us set $z = 1$ and put $C(\lambda) = C\big|_{z=1}$.

Then, the above equation can be written as

$$\boxed{\frac{d}{dt} C(\lambda(-t))\Big|_{t=0} = -24 \langle T_{z\bar{z}}(1) T_{z\bar{z}}(0)\rangle.}$$

Since $\langle T_{z\bar{z}}(z)\, T_{z\bar{z}}(0)\rangle \geq 0$ and $= 0$ iff $T_{z\bar{z}} = 0$ (i.e. CFT point),
in a unitary QFT, this equation means

- $C(\lambda)$ decreases under the RG flow (toward longer distance scales = lower energy scales).

- $C(\lambda)$ is stationary, $\frac{d}{dt} C(\lambda(-t))\Big|_0 = 0$, at and only at CFT points.

Furthermore, at a CFT point λ_*, $H = G = 0$ and
$F\big|_{z=1} = 2\langle T_{zz}(1)\, T_{zz}(0)\rangle$,

- $C(\lambda_*)$ = the central charge of the CFT.

In particular, if there is an RG flow from one CFT to another, there is an inequality

- $C_{UV} > C_{IR}$.

These statements are <u>Zamolodchikov's C-theorem</u>.
The function $C(\lambda)$ that interpolates the central charges is called <u>the C-function</u>

Using the expression $T_{z\bar{z}} = \frac{1}{4}T_{11} + \frac{1}{4}T_{22} = \frac{1}{4}T^{\mu}{}_{\mu} = \frac{1}{4}\left(-2\pi \sum_i \beta^i \phi_i\right)$

we can write

$$G\Big|_{z=1} = \left(\frac{\pi}{2}\right)^2 \left\langle \sum_i \beta^i \phi_i(1) \sum_j \beta^j \phi_j(0) \right\rangle = \left(\frac{\pi}{2}\right)^2 \sum_{i,j} G_{ij} \beta^i \beta^j$$

where
$$\boxed{G_{ij} := \langle \phi_i(1)\, \phi_j(0) \rangle}.$$

G_{ij} defines a metric in the space of coupling constants (called Zamolodchikov metric).

Then, the eqn $-\beta^i \frac{\partial}{\partial \lambda^i} C = -24 G$ can be written as

$$\boxed{-\sum_{i=1}^{n} \beta^i \frac{\partial}{\partial \lambda^i} C = -6\pi^2 \sum_{i,j=1}^{n} G_{ij}\, \beta^i \beta^j}$$

In the case of a one-parameter flow, we have

$$-\beta^{\lambda} \frac{dC}{d\lambda} = -6\pi^2 G_{\lambda\lambda} \beta^{\lambda} \beta^{\lambda}.$$

Since $\beta^{\lambda} \neq 0$ during the flow, this means

$$\boxed{\beta^{\lambda} = \frac{1}{6\pi^2} G^{\lambda\lambda} \frac{dC}{d\lambda}}.$$

i.e. The RG flow is a gradient flow for the C-function.