§ Conformal Perturbation Theory.

Let us consider a QFT,

i.e. an RG flow from one CFT to another.

$$\mathrm{CFT}_1 \longrightarrow \mathrm{CFT}_2$$

In general, the description of the theory (e.g. choice of field variables) can be quite different between high energy scales (near UV fixed point) and low energy scales (near IR fixed point).

However, if the two fixed points are "close" in some way, then we may be able to describe the entire flow using a single description.

Today, we discuss such an attempt using perturbation theory at the UV fixed point.

Suppose we have an action $S_{CFT}(\varphi) = \int d^dx\, \mathcal{L}_{CFT}(\varphi, \partial_\mu\varphi, \dots)$ that describes a CFT, and let's perturb it by relevant or marginal operators, $\mathcal{L}_{CFT} \to \mathcal{L}_{CFT} + \sum_i g_i \phi_i(x)$, or

$$S_{CFT}(\varphi) \longrightarrow S_{CFT}(\varphi) + \int d^dx \sum_i g_i \phi_i(x).$$

Correlation functions of the perturbed system are given by

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\varphi\, e^{-S_{CFT}(\varphi) - \int d^dx \sum_i g_i \phi_i(x)}\, \mathcal{O}(\varphi).$$

Perturbation theory is to expand $e^{-\int d^dx \sum_i g_i \phi_i(x)}$ as $\sum_{m=0}^{\infty} \frac{1}{m!}\left(-\int d^dx \sum_i g_i \phi_i(x)\right)^m$ and exchange the order of the integral $\int \mathcal{D}\phi$ and the sum $\sum_{m=0}^{\infty}$. i.e,

$$\begin{aligned}\langle \mathcal{O} \rangle^{\text{perturb}} = \langle \mathcal{O} \rangle_{CFT} &- \int d^dx \sum_i g_i \langle \mathcal{O}\, \phi_i(x) \rangle_{CFT} \\ &+ \frac{1}{2}(-1)^2 \int d^dx_1 d^dx_2 \sum_{i,j} g_i g_j \langle \mathcal{O}\, \phi_i(x_1)\, \phi_j(x_2) \rangle_{CFT} \\ &+ \frac{1}{3!}(-1)^3 \int d^dx_1 d^dx_2 d^dx_3 \sum_{i,j,k} g_i g_j g_k \langle \mathcal{O}\, \phi_i(x_1) \phi_j(x_2) \phi_k(x_3) \rangle_{CFT} \\ &+ \cdots\end{aligned}$$

where $\langle \cdots \rangle_{CFT}$ is the CFT correlation function.

This formal expression has some problems. In addition to the issue of summability of $\sum_{m=0}^{\infty}$, there is also a problem at each order: the integral

$$\int \langle \mathcal{O}\, \phi_{i_1}(x_1) \cdots \phi_{i_m}(x_m) \rangle_{CFT}\, d^d x_1 \cdots d^d x_m$$

may be divergent from the singularities as $x_i \to x_j$, and also, if $\mathcal{O} = \mathcal{O}_1(y_1) \cdots \mathcal{O}_s(y_s)$, as $x_i \to y_a$'s.

For example, let's look at $x_i \to x_j$: If we have the OPE

$$\phi_i(x_i)\, \phi_j(x_j) = \sum_k \frac{C_{ijk}}{|x_i - x_j|^{d_i + d_j - d_k}}\, \phi_k(x_j)$$

then the integral $\int \frac{d^d x_i}{|x_i - x_j|^{d_i + d_j - d_k}}$ is divergent if

$$\boxed{d_i + d_j - d_k \geq d} \qquad (*)$$

which occurs in general. This requires us to introduce a short distance cut-off of the integrals, i.e. we need a <u>regularization</u> in the perturbed CFT.

In the standard QFT course/textbook, we go to the momentum space via Fourier transform and then cut-off the momentum integral from above, e.g. $\int_{|k|\leq\Lambda} d^d k \cdots$ or $\int d^d k \, e^{-k^2/\Lambda^2} \cdots$, etc.

Instead, here we introduce a short-distance cut-off directly in the position space, e.g., a hard-core cut-off at distance a : <u>remove $|x_i - x_j| < a$, $|x_i - y_a| < a$ from the integrals.</u>

Then, each term in the perturbative expansion is finite, and depends on the cut-off a.
Of course it is divergent as $a \to 0$, but we give a-dependence to the couplings g_i's so that the correlation function is finite at each order in the g_i-expansion. This is the <u>renormalization</u>.

Note that the divergence criterion (*) can be written as

$$d_k \leq d_i + d_j - d \underset{\substack{\uparrow \\ \text{if } \phi_i, \phi_j \text{ are relevant/marginal.}}}{\leq} d.$$

Thus $\phi_i \times \phi_j \longrightarrow \phi_k$ is a "divergent channel" only if ϕ_k is a relevant or marginal operator.

Namely, the divergence can be cancelled by a suitable a-dependence of g_k. We do not have to turn on irrelevant operators. This is the renormalizability.

Now, let us study the regularized correlation function

$$\langle \mathcal{O} \rangle_a^{\text{perturb}} = \langle \mathcal{O} \rangle_{\text{CFT}} - \int_a d^dx \sum_i g_i(a) \langle \mathcal{O}\, \phi_i(x) \rangle_{\text{CFT}}$$

$$+ \frac{1}{2}(-1)^2 \int_a d^dx_1\, d^dx_2 \sum_{i,j} g_i(a) g_j(a) \langle \mathcal{O}\, \phi_i(x_1)\, \phi_j(x_2) \rangle_{\text{CFT}}$$

$$+ \cdots$$

where $\int_a$ is the regularized integral where $|x_i - x_j| < a$ and $|x_i - y_a| < a$ are removed.

The idea is to find $g_i(a)$ s.t. this $\langle \mathcal{O} \rangle_a^{pert}$ has a limit as $a \to 0$, or equivalently, is independent of a for small a's.

Let us see how it changes as $\underline{a \to e^t a}$, if we did not have a-dependence of g_i's. Let's focus on

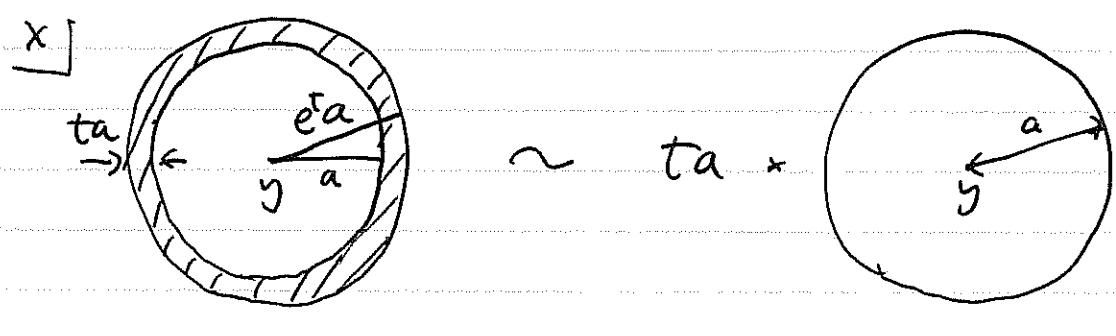
$$\cdots \int_{|x-y| \geq a} \sum_i g_i \phi_i(x) \sum_j g_j \phi_j(y)\, d^dx\, d^dy \cdots$$

The change is

$$\int_{a \leq |x-y| \leq e^t a} \sum_{i,j} g_i g_j\, \phi_i(x) \phi_j(y)\, d^dx\, d^dy$$

$$\overset{a \text{ small}}{\approx} \int_{a \leq |x-y| \leq e^t a} \sum_{i,j,k} g_i g_j \frac{C_{ij}^k\, \phi_k(y)}{|x-y|^{d_i+d_j-d_k}}\, d^dx\, d^dy .$$

For small t, the x-integral can be approximated by ta times the integral on the sphere $\{x \mid |x-y| = a\}$:



The change is thus

$$ta \cdot a^{d-1} \mathrm{Vol}(S^{d-1}) \int \sum_{i,j,k} \frac{C_{ij}^{k}\, g_i g_j}{a^{d_i+d_j-d_k}} \phi_k(y)\, d^d y$$

where $\mathrm{Vol}(S^{d-1})$ is the volume of the $(d-1)$-dimensional unit sphere ($\mathrm{Vol}(S^1) = 2\pi$, $\mathrm{Vol}(S^2) = 4\pi$, $\dots$).

From such two-body collisions, the whole expansion

$$\left[e^{-\int d^d x \sum_i g_i \phi_i(x)} \right]_a = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_a \left(d^d x \sum_\ell g_\ell \phi_\ell(x) \right)^m$$

changes by

$$t\, \mathrm{Vol}(S^{d-1}) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \binom{m}{2} \int_a \left(d^d x' \sum_\ell g_\ell \phi_\ell(x') \right)^{m-2} \sum_{i,j,k} \frac{C_{ij}^{k}\, g_i g_j\, a^d}{a^{d_i+d_j-d_k}} \phi_k(y)\, d^d y .$$

If we introduce the "dimensionless" couplings λ_i by

$$\boxed{g_i =: a^{d_i - d} \lambda_i},$$

then the change can be written as

$$\left.\begin{aligned} -\frac{t}{2} \mathrm{Vol}(S^{d-1}) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(m-1)!} (m-1) \int_a \left(d^d x' \sum_\ell a^{d_\ell - d} \lambda_\ell \phi_\ell(x') \right)^{m-2} \times \\ \sum_{i,j,k} C_{ij}^{k} \lambda_i \lambda_j\, a^{d_k - d} \phi_k(y)\, d^d y . \end{aligned}\right] \text{—} ☆$$

Now let us give a-dependence to g_i's. i.e.

as $a \to e^t a$, we change g_i as

$$g_i = a^{d_i - d} \lambda_i \longrightarrow (e^t a)^{d_i - d} \lambda_i(-t).$$

The change of $\left[e^{-\int d^d x \sum_i a^{d_i - d} \lambda_i \phi_i(x)} \right]_a$ from this is

$$t \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} m \int \left(d^d x' \sum_\ell a^{d_\ell - d} \lambda_\ell \phi_\ell(x') \right)^{m-1} \times$$

$$\sum_k \underbrace{\frac{d}{dt} \left[(e^t a)^{d_k - d} \lambda_k(-t) \right]_{t=0}}_{(d_k - d) a^{d_k - d} \lambda_k + a^{d_k - d} \underbrace{\frac{d}{dt} \lambda_k(-t) \Big|_0}_{= \dot{\lambda}_k}} \phi_k(y) d^d y$$

This cancels the change ☆ if

$$-\frac{1}{2} \mathrm{Vol}(S^{d-1}) \sum_{i,j} C_{ij}^k \lambda_i \lambda_j + (d_k - d) \lambda_k - \dot{\lambda}_k = 0$$

In this way, we find the β-function $\beta_k = \dot{\lambda}_k$

$$\boxed{\beta_k(\lambda) = (d_k - d) \lambda_k - \frac{1}{2} \mathrm{Vol}(S^{d-1}) \sum_{i,j} C_{ij}^k \lambda_i \lambda_j + O(\lambda^3).}$$

The error $O(\lambda^3)$ comes from the contribution of 3 (or higher) body collisions, Using the property that $C_{ij}^{k} = C_{ijk}$ is symmetric, we find that the β-function is the gradient

$$\boxed{\begin{aligned} \beta_k(\lambda) &= \frac{\partial}{\partial \lambda_k} \tilde{C}(\lambda) + O(\lambda^3), \\ \tilde{C}(\lambda) &= \frac{1}{2} \sum_i (d_i - d) \lambda_i^2 - \frac{1}{6} \mathrm{Vol}(S^{d-1}) \sum_{i,j,k} C_{ijk} \lambda_i \lambda_j \lambda_k \end{aligned}}$$

That is, <u>RG flow is a gradient flow</u>, at least to order $O(\lambda^2)$ [Use a metric $G_{ij} = \delta_{ij} + O(\lambda^2)$].

Let us now specialize to $\underline{d=2}$. We have

$$\beta^i(\lambda) = \frac{\partial}{\partial \lambda^i} \tilde{C}(\lambda) + O(\lambda^3)$$

for $\tilde{C}(\lambda) = \frac{1}{2}\sum_i (d_i - 2)(\lambda^i)^2 - \frac{\pi}{3}\sum_{i,j,k} C_{ijk}\lambda^i\lambda^j\lambda^k$.

On the other hand, we have $\sum_i \beta^i \frac{\partial C}{\partial \lambda^i} = 6\pi^2 \sum_{ij} G_{ij}\beta^i\beta^j$

where C is the C-function and G_{ij} is Zamolodchikov metric.

If we <u>assume</u>

$$\beta^i(\lambda) = \frac{1}{6\pi^2}\sum_j G^{ij}(\lambda)\frac{\partial}{\partial \lambda^j} C(\lambda)$$

(which is OK for a one-parameter flow), then, for a metric $G_{ij} = \delta_{ij} + O(\lambda^2)$, we must have $C = c_{UV} + \tilde{C}(\lambda)\times 6\pi^2$, i.e.

$$\boxed{C(\lambda) = c_{UV} + 6\pi^2\left\{\frac{1}{2}\sum_i (d_i-2)(\lambda^i)^2 - \frac{\pi}{3}\sum_{i,j,k} C_{ijk}\lambda^i\lambda^j\lambda^k\right\} + O(\lambda^4)}$$

Here c_{UV} is the central charge of the U.V. CFT.

Suppose we have a one parameter flow generated by some relevant operator $\mathcal{O}$, $d_{\mathcal{O}} < 2$.

The fixed point equation is

$$\beta = (d_{\mathcal{O}} - 2)\lambda - \pi C_{\mathcal{OOO}} \lambda^2 + O(\lambda^3) = 0$$

If we ignore the $O(\lambda^3)$ terms, the solution is

$$\lambda = 0 \quad \text{and} \quad -\frac{(2-d_{\mathcal{O}})}{\pi C_{\mathcal{OOO}}} .$$

The latter is OK if $\varepsilon = (2-d_{\mathcal{O}})/C_{\mathcal{OOO}}^2 \ll 1$. Then, we find another fixed point at

$$\lambda_* = -\frac{(2-d_{\mathcal{O}})}{\pi C_{\mathcal{OOO}}} + O(\varepsilon^2).$$

The new central charge is

$$\begin{aligned}
C_{IR} &= C_{UV} + 6\pi^2 \left(\frac{1}{2}(d_{\mathcal{O}}-2)\lambda_*^2 - \frac{\pi}{3} C_{\mathcal{OOO}} \lambda_*^3 \right) + O(\lambda_*^4) \\
&= C_{UV} + 6\pi^2 \left(-\frac{1}{2} \frac{(2-d_{\mathcal{O}})}{(\pi C_{\mathcal{OOO}})^2} + \frac{\pi}{3} C_{\mathcal{OOO}} \left(\frac{2-d_{\mathcal{O}}}{\pi C_{\mathcal{OOO}}} \right)^3 \right) + O(\varepsilon^4) \\
&= C_{UV} - \frac{(2-d_{\mathcal{O}})^3}{C_{\mathcal{OOO}}^2} + O(\varepsilon)
\end{aligned}$$

Flow of the dimension of $\mathcal{O}$

Recall $\quad \hat{\Gamma}\,\mathcal{O} = \left(2 + \frac{\partial \beta}{\partial \lambda}\right)\mathcal{O}$

i.e. $\quad d_{\mathcal{O}}(\lambda) = 2 + \frac{\partial \beta}{\partial \lambda} = d_{\mathcal{O}}^{UV} - 2\pi C_{\mathcal{OOO}}\,\lambda + O(\lambda^2)$

- $d_{\mathcal{O}}^{UV}$ is the dimension of $\mathcal{O}$ at the U.V. CFT.

- $$d_{\mathcal{O}}^{IR} = d_{\mathcal{O}}^{UV} - 2\pi C_{\mathcal{OOO}}\,\lambda_* + O(\lambda_*^2)$$
$$= d_{\mathcal{O}}^{UV} + 2(2 - d_{\mathcal{O}}^{UV}) + O(\varepsilon^2)$$
$$= 2 + (2 - d_{\mathcal{O}}^{UV}) + O(\varepsilon^2) \qquad > 2 \quad \text{slightly irrelevant.}$$

Let us apply this to the unitary minimal model $\left(c = 1 - \frac{6}{m(m+1)}\right)$.

We consider perturbation by the least relevant operator $\mathcal{O}$, so that the coefficient of the β-function $|d_{\mathcal{O}} - 2|$ is minimized.

linear term in

The least relevant operator is $\mathcal{O} = \phi_{\Delta_{1,3}}$ with

$$d_{\mathcal{O}} = 2\Delta_{1,3} = 2\frac{m-1}{m+1} = 2(1-\epsilon), \quad \boxed{\epsilon = \frac{2}{m+1}.}$$

Fusion rule is

$$[\phi_{\Delta_{1,3}}] \times [\phi_{\Delta_{1,3}}] = \underbrace{[\phi_{\Delta_{1,1}}]}_{\text{identity}} + \underbrace{[\phi_{\Delta_{1,3}}]}_{= \mathcal{O}} + \underbrace{[\phi_{\Delta_{1,5}}]}_{\text{irrelevant}}$$

Thus, the only divergent channel is $\mathcal{O}$ itself!

<u>We have a one-parameter flow.</u>

It is known that $\boxed{C_{\mathcal{O}\mathcal{O}\mathcal{O}} = \frac{4}{\sqrt{3}} + O(\epsilon)}$

thus we have $|2-d| = 2\epsilon \ll C_{\mathcal{O}\mathcal{O}\mathcal{O}}^2 = \frac{16}{3} + O(\epsilon)$

Applying the general one-parameter flow story, we find another fixed point at

$$\lambda_* = -\frac{2\epsilon}{\pi\frac{4}{\sqrt{3}}(1+O(\epsilon))} + O(\epsilon^2) = -\frac{\sqrt{3}}{2\pi}\epsilon + O(\epsilon^2).$$

The new central charge is

$$C_{IR} = C_{UV} - \frac{(2\epsilon)^3}{\left(\frac{4}{\sqrt{3}}+O(\epsilon)\right)^2} = C_{UV} - \frac{3}{2}\epsilon^3 + O(\epsilon^4)$$

$$= 1 - \frac{6}{m(m+1)} - \frac{12}{(m+1)^3} + O\left(\frac{1}{m^4}\right)$$

Note that $C_{m+\Delta m} = 1 - \dfrac{6}{(m+\Delta m)(m+\Delta m+1)} = 1 - \dfrac{6}{m(m+1)+(2m+1)\Delta m+\Delta m^2}$

$$= 1 - \frac{6}{m(m+1)} + \frac{12\Delta m}{m^2(m+1)^2} + O\left(\frac{1}{m^4}\right).$$

This is consistent with $C_{IR} = C_{m+\Delta m}$ with $\Delta m = -1$.

i.e. The IR fixed point is the $(m-1)$-th minimal model.

(This is what is expected from the LG description.)

Let us compute the dimension of $\mathcal{O}$ at the IR fixed point.

$$d_{\mathcal{O}}^{IR} = 2 + \underbrace{(2 - d_{\mathcal{O}}^{UV})}_{2\epsilon = \frac{4}{m+1}} + O(\epsilon^2)$$

$$= 2 + \frac{4}{m+1} + O\left(\frac{1}{m^2}\right)$$

Check the least irrelevant operator $\phi_{\Delta_{3,1}}$

$$d_{3,1}^{IR} = 2\Delta_{3,1}^{IR} = 2\,\frac{(m-1)^2 + 3(m-1) + 2}{(m-1)m} = 2\,\frac{m+1}{m-1}$$

$$= 2 + \frac{4}{m-1} = 2 + \frac{4}{m+1} + O\left(\frac{1}{m^2}\right) \qquad \underline{\text{matches !}}$$

Thus, we may identify $\mathcal{O}$ as the least irrelevant operator $\phi_{\Delta_{3,1}}$ at the IR fixed point.