Partition function on a circle $S^1 = [0, T]/_{0 \equiv T}$

$$Z(T) = \int_{X(t+T) = X(t)} \mathcal{D}X \, e^{iS(X)} = \int dx_1 \, Z(x_1, T; x_1, 0)$$

$X(t+T) = X(t)$ ← Periodic boundary condition

After Wick rotation $t \to -i\tau$, $T \to -i\beta$ $(\beta > 0)$

$$Z_E(\beta) = \int_{X(\tau+\beta) = X(\tau)} \mathcal{D}X \, e^{-S_E(X)}$$

$X(\tau+\beta) = X(\tau)$ ← Periodic

Fermionic fields

One can consider $\left\{ \begin{array}{l} \text{periodic} \\ \text{antiperiodic} \end{array} \right.$ boundary conditions.

e.g. in $\underline{E}$

$$Z_P(\beta) = \int_{\psi(\tau+\beta) = \psi(\tau)} \mathcal{D}\psi \, e^{-S_E(\psi)}$$

$$Z_{AP}(\beta) = \int_{\psi(\tau+\beta) = -\psi(\tau)} \mathcal{D}\psi \, e^{-S_E(\psi)}$$

"The right one."

Example Harmonic Oscillator

$$S(X) = \int dt \left(\frac{1}{2}\dot{X}^2 - \frac{m^2}{2} X^2 \right) \qquad \left(\begin{array}{c} \text{normally we write} \\ \omega \text{ for } m \end{array} \right)$$

$$S_E(X) = \int d\tau \left(\frac{1}{2}\left(\frac{dX}{d\tau}\right)^2 + \frac{m^2}{2} X^2 \right)$$

$$Z_E(\beta) = \int_{X(\tau+\beta)=X(\tau)} \mathcal{D}X \; \exp\left(-\frac{1}{2}\int_0^\beta d\tau \, X(\tau) \underbrace{\left[-\frac{d^2}{d\tau^2} + m^2 \right]}_{A} X(\tau) \right)$$

$$= \frac{\# \rightarrow 1}{\sqrt{\det A}} \quad \text{by a choice of } \mathcal{D}X$$

$$\mathcal{F} = \{ X(\tau+\beta) = X(\tau) \} \quad \cdots \quad \text{spanned by } \frac{1}{\sqrt{\beta}}, \sqrt{\frac{2}{\beta}} \cos\left(\frac{2\pi n \tau}{\beta}\right), \sqrt{\frac{2}{\beta}} \sin\left(\frac{2\pi n \tau}{\beta}\right)$$

$$n = 1, 2, 3, \cdots$$

$$X(\tau) = \frac{x_0}{\sqrt{\beta}} + \sum_{n=1}^{\infty} \left\{ x_n^c \sqrt{\frac{2}{\beta}} \cos\left(\frac{2\pi n}{\beta}\tau\right) + x_n^s \sqrt{\frac{2}{\beta}} \sin\left(\frac{2\pi n}{\beta}\tau\right) \right\}$$

$$S_E(X) = \frac{m^2}{2} x_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(\left(\frac{2\pi n}{\beta}\right)^2 + m^2 \right) \left\{ (x_n^c)^2 + (x_n^s)^2 \right\}$$

$$\mathcal{D}X(\tau) := \frac{dx_0}{\sqrt{2\pi}} \prod_{n=1}^{\infty} \frac{dx_n^c}{\sqrt{2\pi}} \frac{dx_n^s}{\sqrt{2\pi}}$$

$$Z_E(\beta) = \frac{1}{\sqrt{\det A}} = \frac{1}{\sqrt{m^2 \prod_{n=1}^{\infty} \left(\left(\frac{2\pi n}{\beta}\right)^2 + m^2 \right)^2}} = \frac{1}{m \prod_{n=1}^{\infty} \left(\left(\frac{2\pi n}{\beta}\right)^2 + m^2 \right)}$$

$$= \frac{1}{m} \underbrace{\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2}}_{[?]} \underbrace{\prod_{n=1}^{\infty} \left(1 + \left(\frac{\beta m}{2\pi n}\right)^2\right)^{-1}}_{\left[\begin{array}{l}\text{pole at } \frac{\beta m}{2} \in \pi i \mathbb{Z}_{\neq 0} \\ = 1 \text{ at } \frac{\beta m}{2} = 0\end{array}\right]}$$

~~with residue~~

$$= \frac{\beta m}{2\sinh\left(\frac{\beta m}{2}\right)}$$

$$\log[?] = \sum_{n=1}^{\infty} \log\left(\frac{2\pi n}{\beta}\right)^{-2} = \frac{d}{ds}\bigg|_{s=0} \underbrace{\sum_{n=1}^{\infty}\left(\frac{2\pi n}{\beta}\right)^{-2s}}_{=\left(\frac{\beta}{2\pi}\right)^{2s}\zeta(2s)}$$

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \quad \text{Riemann's Zeta fcn.}$$

Convergent if $\mathrm{Re}(z) > 1$, & can be analytically continued on the entire z-plane, with some poles

$z=0$ is regular

$$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2}\log(2\pi)$$

$$\log[?] = \log\left(\frac{\beta}{2\pi}\right)^2 \zeta(0) + 2\zeta'(0) = -\frac{1}{2}\log\left(\frac{\beta}{2\pi}\right)^2 - \log(2\pi) = -\log\beta$$

$$\therefore [?] = e^{-\log\beta} = \beta^{-1}$$

$$\therefore Z_E(\beta) = \frac{1}{m}\beta^{-1} \cdot \frac{\beta m}{2\sinh\left(\frac{\beta m}{2}\right)} = \frac{1}{2\sinh\left(\frac{\beta m}{2}\right)} = \frac{1}{e^{\frac{\beta m}{2}} - e^{-\frac{\beta m}{2}}}$$

$$= \frac{e^{-\frac{\beta m}{2}}}{1 - e^{-\beta m}} = e^{-\frac{\beta m}{2}} + e^{-\frac{3}{2}\beta m} + e^{-\frac{5}{2}\beta m} + \cdots$$

__Example__ Fermionic "Oscillator"

$$S(\psi,\bar\psi) = \int dt \left(i\bar\psi\dot\psi - m\bar\psi\psi \right)$$

$$S_E(\psi,\bar\psi) = \int d\tau \left(+\bar\psi \frac{d}{d\tau}\psi + m\bar\psi\psi \right)$$

$$Z_E(\beta) = Z_{AP}(\beta) = \int_{\substack{\psi(\tau+\beta)=-\psi(\tau) \\ \bar\psi(\tau+\beta)=-\bar\psi(\tau)}} D\psi D\bar\psi \; \exp\left(-\int_0^\beta d\tau\, \bar\psi(\tau)\left(\frac{d}{d\tau}+m\right)\psi(\tau) \right)$$

Spanned by $\underset{n\in\mathbb{Z}}{e^{2\pi i(n+\frac{1}{2})\tau/\beta}} = e^{2\pi i r\tau/\beta} \quad r\in\mathbb{Z}+\frac{1}{2}$

$$\psi(\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r\, e^{2\pi i r\tau/\beta}\Big/\sqrt{\beta}, \quad \bar\psi(\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar\psi_r\, e^{2\pi i r\tau/\beta}\Big/\sqrt{\beta}$$

$$S_E = \int_0^\beta d\tau\, \bar\psi\left(\frac{d}{d\tau}+m\right)\psi = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \left(\frac{2\pi i r}{\beta}+m\right)\bar\psi_{-r}\psi_r$$

__Def__ $D\psi D\bar\psi = \prod_r d\psi_r\, d\bar\psi_{-r} \implies$

$$Z_E(\beta) = \det\left(\frac{\partial}{d\tau}+m\right) = \prod_{r\in\mathbb{Z}+\frac{1}{2}}\left(\frac{2\pi i r}{\beta}+m\right)$$

$$= \underbrace{\prod_{r\in\mathbb{Z}+\frac{1}{2}}\left(\frac{2\pi i r}{\beta}\right)}_{(?)} \prod_{r\in\mathbb{Z}+\frac{1}{2}}\left(1+\frac{\beta m}{2\pi i r}\right)$$

zero at $\frac{\beta m}{2} \in i\pi(\mathbb{Z}+\frac{1}{2})$

1 at $\frac{\beta m}{2}=0$

$$\prod_{r\in\mathbb{Z}+\frac{1}{2}}\left(1+\frac{\beta m}{2\pi i r}\right) = \cosh\left(\frac{\beta m}{2}\right)$$

$$\log[?] = \log\left[\prod_{n=0}^{\infty}\left(\frac{2\pi(n+\frac{1}{2})}{\beta}\right)^2\right] = \sum_{n=0}^{\infty}\log\left(\frac{2\pi(n+\frac{1}{2})}{\beta}\right)^2$$

$$= -\frac{d}{ds}\bigg|_{s=0}\underbrace{\sum_{n=0}^{\infty}\left(\frac{2\pi(n+\frac{1}{2})}{\beta}\right)^{-2s}}_{=\left(\frac{2\pi}{\beta}\right)^{-2s}\underbrace{\zeta(2s,\frac{1}{2})}_{=(2^{2s}-1)\zeta(2s)}}$$

$$= \frac{d}{ds}\left[\left(\left(\frac{2\pi}{\beta}\right)^{-2s}-\left(\frac{\pi}{\beta}\right)^{-2s}\right)\zeta(2s)\right]_{s=0}$$

$$= \left(-2\log\left(\frac{2\pi}{\beta}\right)+2\log\left(\frac{\pi}{\beta}\right)\right)\underbrace{\zeta(0)}_{=-\frac{1}{2}} = \log\left(\frac{2\pi}{\beta}\right)-\log\left(\frac{\pi}{\beta}\right) = \log 2$$

$$\therefore [?] = 2$$

$$\therefore Z_E(\beta) = 2\cdot\cosh\left(\frac{\beta m}{2}\right) = e^{\frac{\beta m}{2}} + e^{-\frac{\beta m}{2}}$$

$\boxed{\S\ \text{PI} \to \text{OF}}$

$$Z(q_f, t_f ; q_i, t_i) = \int_{q(t_i)=q_i,\ q(t_f)=q_f} \mathcal{D}q \ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L(q,\dot{q})\,dt}$$

This defines an operator $\hat{Z}_{t_f, t_i}$ on the space $\mathcal{H}$ of functions of q:

$$\Psi(q) \mapsto \left(\hat{Z}_{t_f,t_i}\Psi\right)(q) = \int dq' \, Z(q, t_f ; q', t_i)\,\Psi(q')$$

<u>Note</u> $\int dq_2 \, Z(q_3, t_3 ; q_2, t_2)\, Z(q_2, t_2 ; q_1, t_1) = \cancel{\int d}\, Z(q_3, t_3 ; q_1, t_1)$

$$\Leftrightarrow \quad \hat{Z}_{t_3,t_2} \circ \hat{Z}_{t_2,t_3} = \hat{Z}_{t_3,t_1} \qquad \text{——} \quad (C)$$

<u>Note</u> $L(q,\dot{q})$ does not explictly depend on time $\Rightarrow Z(q_f, t_f ; q_i, t_i)$ – invariant under $(t_f, t_i) \to (t_f + \Delta t, t_i + \Delta t)$

$$\Leftrightarrow \quad \hat{Z}_{t_f+\Delta t,\ t_i+\Delta t} = \hat{Z}_{t_f, t_i}$$

$\rightsquigarrow$ One can write $\hat{Z}_{t_f,t_i} = \hat{Z}_{t_f - t_i, 0} =: \hat{Z}_{t_f - t_i}$

Then (C) means $\hat{Z}_{\Delta t} \circ \hat{Z}_{\Delta t'} = \hat{Z}_{\Delta t + \Delta t'}$.

Then one can write $\hat{Z}_{\Delta t} = e^{-\frac{i}{\hbar}\Delta t \hat{H}}$

for <u>some</u> operator $\hat{H}$ on $\mathcal{H}$

(This is how $\hat{H}$ is <u>defined</u>. At this point we don't yet know the relation to Hamiltonian)

Let $\mathcal{O}$ be an expression of $q, \dot{q}, \ddot{q}, \frac{d^3q}{dt^3}, \cdots$

e.g. $\mathcal{O} = \dot{q}$, $\mathcal{O} = q^2 + \dot{q}^3$, $\cdots$ etc

For $t_i < t < t_f$, write

$$Z(q_f, t_f ; \mathcal{O}(t) ; q_i, t_i) = \int_{q(t_i)=q_i,\ q(t_f)=q_f} \mathcal{D}q \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L(q,\dot{q})\,dt} \mathcal{O}(t).$$

This also defines an operator

$$\Psi(q) \mapsto \left(\widehat{Z_{t_f,t_i}(\mathcal{O}(t))}\,\Psi\right)(q) := \int dq'\, Z(q, t_f ; \mathcal{O}(t) ; q', t_i)\, \Psi(q')$$

<u>Note</u> $\hat{Z}_{t_f+\Delta t_f,\, t_i+\Delta t_i}(\mathcal{O}(t)) = \hat{Z}_{\Delta t_f} \circ \hat{Z}_{t_f, t_i}(\mathcal{O}(t)) \circ \hat{Z}_{-\Delta t_i}$

<u>Note</u> $\hat{Z}_{t_f+\Delta t,\, t_i+\Delta t}(\mathcal{O}(t+\Delta t)) = \hat{Z}_{t_f, t_i}(\mathcal{O}(t))$

Define an operator $\hat{O}$ by

$$\hat{O} := \lim_{\substack{t_f \searrow t \\ t_i \nearrow t}} \hat{Z}_{t_f, t_i}(O(t)) \qquad \left[\begin{array}{l} \text{it is independent of } t \\ \text{by the second of the above} \end{array}\right]$$

Then $\hat{Z}_{t_f, t_i}(O(t)) = \hat{Z}_{t_f - t} \circ \hat{O} \circ \hat{Z}_{t - t_i} = e^{-\frac{i}{\hbar}(t_f - t)\hat{H}} \circ \hat{O} \circ e^{-\frac{i}{\hbar}(t - t_i)\hat{H}}$

For $t_i < t_1 \,\&\, t_2 < t_f$, O_1, O_2

$$Z(q_f, t_f ; O_1(t_1) O_2(t_2) ; q_i, t_i) := \int_{q(t_i) = q_i,\ q(t_f) = q_f} \mathcal{D}q \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L(q, \dot{q}) dt} \, O_1(t_1) O_2(t_2)$$

$\rightsquigarrow$ Operator $\hat{Z}_{t_f, t_i}(O_1(t_1) O_2(t_2))$

$$Z(q_f, t_f ; O_1(t_1) O_2(t_2) ; q_i, t_i) = \begin{cases} \int dq' \, Z(q_f, t_f ; O_2(t_2) ; q', t') Z(q', t' ; O_1(t_1) ; q_i, t_i) \\ \qquad \text{if } t_2 > t_1 \quad (t_2 \overset{\forall}{>} t' > t_1) \\ \int dq' \, Z(q_f, t_f ; O_1(t_1) ; q', t') Z(q', t' ; O_2(t_2) ; q_i, t_i) \\ \qquad \text{if } t_2 < t_1 \quad (t_1 \overset{\forall}{>} t' > t_2) \end{cases}$$

$$\Rightarrow \hat{Z}_{t_f, t_i}(O_1(t_1) O_2(t_2)) = \begin{cases} e^{-\frac{i}{\hbar}(t_f - t_2)\hat{H}} \, \hat{O}_2 \, e^{-\frac{i}{\hbar}(t_2 - t_1)\hat{H}} \, \hat{O}_1 \, e^{-\frac{i}{\hbar}(t_1 - t_i)\hat{H}} & \text{if } t_2 > t_1 \\ & \text{if } t_2 < t_1 \end{cases}$$

Thus time ordering is buid-in in the definition.

More generally

$$\hat{Z}_{t_f, t_i}(O_1(t_1) \cdots O_n(t_n))$$

$$\text{if } t_i < t_1 < t_2 \cdots < t_n < t_f \quad e^{-\frac{i}{\hbar}(t_f - t_n)\hat{H}} \hat{O}_n \, e^{-\frac{i}{\hbar}(t_n - t_{n-1})\hat{H}} \hat{O}_{n-1} \, e^{-\frac{i}{\hbar}(t_{n-1} - t_{n-2})\hat{H}} \cdots \hat{O}_1 e^{-\frac{i(t_1 - t_i)}{\hbar}\hat{H}}.$$

However we have not yet shown what $\hat{O}$ are

when O involves $\dot{q}$,

we have not yet shown $[\hat{q}^i, \hat{p}_j] = i\hbar \delta^i_j$ $\left(p_j = \frac{\partial L}{\partial \dot{q}_j}\right)$

We have not shown $\hat{H} = ?$ relation to Hamiltonian?

— They can be done using Ward identity