§ Symmetry and Ward identity

$$Z = \int d^n x \, e^{-S_E(x)} = \int dx_1 \cdots dx_n \, e^{-S_E(x)}$$

$$\langle f \rangle = \int d^n x \, e^{-S_E(x)} f(x) \Big/ \int d^n x \, e^{-S_E(x)}$$

"Theory" determined by $d^n x \, e^{-S_E(x)}$

Change of integration variable

Single variable case : $y = g(x)$ monotonic function

$$\int_{-\infty}^{\infty} dy \, F(y) = \int_{-\infty}^{\infty} dy \, F(y) = \int_{-\infty}^{\infty} g'(x) \, dx \, F(g(x))$$

multivariable case $g(x) = (g_1(x_1, \cdots, x_n), \cdots, g_n(x_1, \cdots, x_n))$

$$\int d^n x \, F(x) = \int \det\left(\frac{\partial g_i}{\partial x_j}(x)\right) d^n x \, F(g(x))$$

A symmetry of the theory $(d^n x \, e^{-S_E(x)})$ is

a transformation $x \mapsto g(x)$ that leaves

$d^n x \, e^{-S_E(x)}$ invariant.

i.e. $\underbrace{\det \frac{\partial g_i}{\partial x_j}(x)}_{d^n x} e^{-S_E(g(x))} = \underbrace{e^{-S_E(x)}}_{d^n x}$

Classical symmetry : $S_E(g(x)) = S_E(x)$

Symmetry in Quantum theory : The combination $d^n x\, e^{-S_E(x)}$ is invariant

(usually S_E, $d^n x$ both invariant).

e.g. $dx_1 dx_2\, e^{-\frac{1}{2}(x_1 - x_2)^2}$ is invariant under translations
$(x_1, x_2) \to (x_1 + \Delta x, x_2 + \Delta x)$

e.g. $dx_1 dx_2\, e^{-\frac{1}{2}(x_1^2 + x_2^2)}$ is invariant under rotations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If $x \mapsto g(x)$ is a symmetry

$$\int d^n x\, e^{-S_E(x)} f(x) = \int \underbrace{d^n g(x)\, e^{-S_E(g(x))}}_{= d^n x\, e^{-S_E(x)}} f(g(x))$$

i.e. $\boxed{\langle f(x) \rangle = \langle f(g(x)) \rangle}$ — Ward identity.

Infinitesimal form

$x \mapsto x + \delta x$ infinitesimal symmetry

(e.g. $\delta x_i = \epsilon$ for translation , $\delta x_1 = -\epsilon x_2$, $\delta x_2 = \epsilon x_1$ for rotation.)

Then Ward identity $\Rightarrow$ $\boxed{0 = \langle \delta f \rangle}$

<u>Non-symmetry</u> Suppose $d^n x\, e^{-S_E(x)}$ is not invariant under

$x \mapsto g(x)$. e.g. $d^n x$ invariant but $S_E(x)$ not,

$\delta S_E \neq 0$

Then $\int d^n x\, e^{-S_E(x)} f(x) = \int \underbrace{d^n g(x)\, e^{-S_E(g(x))}}_{\parallel} f(g(x))$ still holds

$$d^n x\, e^{-S_E(g(x))}$$

infinitesimal form

$$0 = \int d^n x\, e^{-S_E(x)} \left(-\delta S_E^{(x)} f(x) + \delta f(x) \right)$$

i.e. $\boxed{\langle \delta f \rangle = \langle \delta S_E f \rangle}$

— Ward identity.

Symmetry in classical mechanics (in Lagrangian formalism)

Suppose there is a symmetry transformation $q \to q + \delta q$
$\delta q = \epsilon\, u(q, \dot{q})$

$$\delta L(q, \dot{q}) = \epsilon \frac{d}{dt}\left(\cdots \right) \quad \text{(total derivative)}$$

If we allow variational parameter ϵ to depend on time, $\epsilon(t)$ but require $\epsilon(t_i) = \epsilon(t_f) = 0$, the we always have

$$\delta S = \delta \int_{t_i}^{t_f} L(q, \dot{q})\, dt = \int_{t_i}^{t_f} \dot{\epsilon}(t) \underbrace{Q(q, \dot{q})}_{\text{some expression of } q, \dot{q}}\, dt$$

This $Q = Q(q, \dot{q})$ is called the Noether charge.

<u>Theorem</u> It is conserved. i.e. independent of time for classical solution $q = q_{cl}$.

∵ Classical solution is s.t. $\delta S = 0$ for any variation δq s.t. $\delta q = 0$ at $t = t_i$ & t_f. For variation $\delta q = \epsilon(t)\, u(q, \dot{q})$

$$\therefore\ 0 = \delta S = \int_{t_i}^{t_f} \dot{\epsilon}(t)\, Q(q, \dot{q})\, dt = -\int_{t_i}^{t_f} \epsilon(t) \frac{dQ}{dt}\, dt$$

$$\Rightarrow \frac{dQ}{dt} = 0 \quad \text{for trajectories solving EOM.}$$

Example $\quad L = \frac{m}{2}\dot{q}^2 \qquad \delta q = \epsilon \quad$ translation

$$\int_{t_i}^{t_f} \delta L = \frac{m}{2} 2\dot{q}\frac{d}{dt}\epsilon = \dot{\epsilon}(m\dot{q})$$ — this is $Q = m\dot{q}$ (momentum)
(~~which is momentum~~)

Example $\quad L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(q_1^2 + q_2^2)$ $\qquad \begin{pmatrix} \delta q_1 = -\epsilon q_2 \\ \delta q_2 = \epsilon q_1 \end{pmatrix}$

rotationary invariant potential.

$$\delta \int_{t_i}^{t_f} L\,dt = \int_{t_i}^{t_f} m\left(\dot{q}_1(-\dot{\epsilon} q_2) + \dot{q}_2(\dot{\epsilon} q_1)\right) dt$$

$$= \int_{t_i}^{t_f} \dot{\epsilon}\left(m q_1 \dot{q}_2 - m q_2 \dot{q}_1\right) dt$$

$$Q = m q_1 \dot{q}_2 - m q_2 \dot{q}_1$$ (angular momentum)

Example $\quad L = L(q, \dot{q})$ general (not explicit t-dependence)

$$\delta q = \epsilon \dot{q}$$

$\dot{\epsilon}\dot{q} + \epsilon\ddot{q}$

$$\int_{t_i}^{t_f} L\,dt = \int_{t_i}^{t_f} \left(\epsilon \dot{q}\frac{\partial L}{\partial q} + \frac{d}{dt}(\epsilon \dot{q})\frac{\partial L}{\partial \dot{q}}\right) dt$$

$$= \int_{t_i}^{t_f} \left(\epsilon \frac{d}{dt} L + \dot{\epsilon}\dot{q}\frac{\partial L}{\partial \dot{q}}\right) dt$$

$$= \int_{t_i}^{t_f} \dot{\epsilon}\left(\underbrace{\dot{q}\frac{\partial L}{\partial \dot{q}} - L}_{= E(q\dot{q})}\right) dt$$

energy. $= H(q, p)$ Hamiltonian

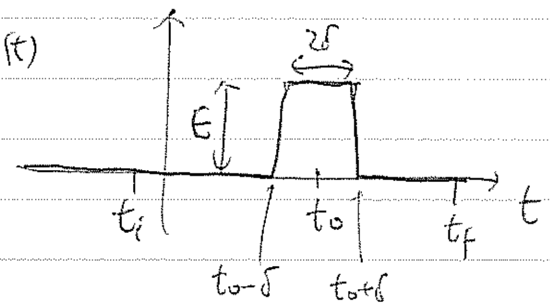
if we use $p = \frac{\partial L}{\partial \dot{q}}$ instead of $\dot{q}$

$\delta q = \epsilon(t)\, U(q, \dot{q})$

Now let us apply this symmetry transformation to the integrand of

$$Z(q_f, t_f ; \mathcal{O}(t_0) ; q_i, t_i) = \int_{q(t_i)=q_i,\ q(t_f)=q_f} \mathcal{D}q \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L(q,\dot{q})\,dt} \; \mathcal{O}(t_0)$$

We choose as $\epsilon(t)$



$$\dot{\epsilon}(t) = \epsilon\,\delta(t-(t_0-\delta)) - \epsilon\,\delta(t-(t_0+\delta))$$

$$0 = \int \delta\left(\mathcal{D}q\; e^{\frac{i}{\hbar}S[q]}\; \mathcal{O}(t_0)\right)$$

$$= \int \mathcal{D}q\; e^{\frac{i}{\hbar}S[q]} \left(\frac{i}{\hbar}\,\delta S[q]\; \mathcal{O}(t_0) + \delta\mathcal{O}(t_0) \right)$$

$$\delta S[q] = \int \dot{\epsilon}(t)\, Q\, dt = \epsilon Q(t_0-\delta) - \epsilon Q(t_0+\delta)$$

i.e. $$Z(q_f, t_f ; \delta\mathcal{O}(t_0) ; q_i, t_i) = Z\left(q_f, t_f ; \left(\frac{i\epsilon}{\hbar} Q(t_0+\delta) - \frac{i\epsilon}{\hbar} Q(t_0-\delta)\right)\mathcal{O}(t_0) ; q_i, t_i\right)$$

operator

$$\widehat{\delta\mathcal{O}(t_0)} = \frac{i\epsilon}{\hbar}\hat{Q}\hat{\mathcal{O}} - \hat{\mathcal{O}}\frac{i\epsilon}{\hbar}\hat{Q} = \frac{i\epsilon}{\hbar}[\hat{Q}, \hat{\mathcal{O}}]$$

$\boxed{\epsilon = 1}$

$$\boxed{\widehat{\delta \mathcal{O}} = \frac{i}{\hbar}[\hat{Q}, \hat{\mathcal{O}}]}$$ — Ward identity in QM.

In QM, Noether charge generates the symmetry transformtn.

<u>Example</u> Time translation symmetry.

$$\delta q = \dot{q} \quad \Rightarrow \quad \delta \mathcal{O} = \frac{d}{dt}\mathcal{O} \qquad \text{for } \mathcal{O} = \mathcal{O}(q(t), \dot{q}(t), \ddot{q}(t) \cdots)$$

On the other hand, Noether charge is energy

$$E(q, \dot{q}) \quad \text{or} \quad \text{Hamiltonian} \quad H(q, p)$$

$$\therefore \quad \widehat{\frac{d\mathcal{O}}{dt}} = \frac{i}{\hbar}[\underset{\widehat{E(q,\dot{q})}}{\underset{\shortparallel}{\widehat{H(q,p)}}}, \hat{\mathcal{O}}]$$

To find $\widehat{\frac{d}{dt}\mathcal{O}}$, we consider $e^{-\frac{i}{\hbar}(t_f - t)\hat{H}} \widehat{\frac{d}{dt}\mathcal{O}} \cdot e^{-\frac{i}{\hbar}(t - t_i)\hat{H}}$

$$= \hat{Z}_{t_f, t_i}\left(\frac{d}{dt}\mathcal{O}(t)\right) = \frac{d}{dt}\hat{Z}_{t_f, t_i}(\mathcal{O}(t)) = \frac{d}{dt}\left(e^{-\frac{i}{\hbar}(t_f - t)\hat{H}} \hat{\mathcal{O}} e^{-\frac{i}{\hbar}(t - t_i)\hat{H}}\right)$$

$$= e^{-\frac{i}{\hbar}(t_f - t)\hat{H}} \frac{i}{\hbar}[\hat{H}, \hat{\mathcal{O}}] e^{-\frac{i}{\hbar}(t - t_i)\hat{H}}$$

$$\therefore \quad \widehat{\frac{d}{dt}\mathcal{O}} = \frac{i}{\hbar}[\hat{H}, \hat{\mathcal{O}}]$$

Comparing the two, we find

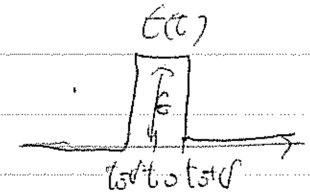
$$\boxed{\hat{H} = \widehat{E(q,\dot{q})} \left(\overset{\text{on}}{=} \widehat{H(q,p)} \right)}$$

i.e. $\hat{H}$ is the operator corresponding to the Hamiltonian!

<u>Example</u> q-translation (<u>non-symmetry</u> in general)

$$\delta q(t) = \epsilon(t)$$

Apply this to $\displaystyle\int_{(t_i,q_i)\to(t_f,q_f)} \mathcal{D}q \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L(q,\dot{q})\,dt} \; q(t_0)$



$$\dot{\epsilon}(t) = \epsilon\,\delta(t-(t_0-\delta)) - \epsilon\,\delta(t-(t_0+\delta))$$

$$0 = \int \delta\left(\mathcal{D}q \; e^{\frac{i}{\hbar}\int L\,dt} \; q(t_0)\right)$$

$$= \int \mathcal{D}q \; e^{\frac{i}{\hbar}\int L\,dt} \left(\frac{i}{\hbar}\,\delta S[q]\; q(t_0) + \epsilon \right)$$

$$\delta S[q] = \int_{t_i}^{t_f} \left(\epsilon(t)\frac{\partial L}{\partial q(t)} + \dot{\epsilon}(t)\overbrace{\frac{\partial L}{\partial \dot{q}(t)}}^{p} \right) dt$$

$$= \underbrace{\int_{t_0-\delta}^{t_0+\delta} \epsilon \frac{\partial L}{\partial q(t)}\,dt}_{\to\, 0 \text{ as } \delta \to 0} + \epsilon\, p(t_0-\delta) - \epsilon\, p(t_0+\delta)$$