Comparing the two, we find

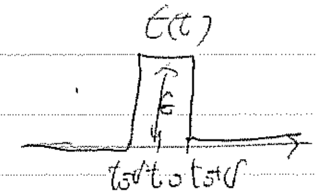
$$\boxed{\hat{H} = \widehat{E(q,\dot{q})} \left(\overset{\text{on}}{=} \widehat{H(q,p)} \right)}$$

i.e. $\hat{H}$ is the operator corresponding to the Hamiltonian!

Example q-translation (non-symmetry in general)

$$\delta q(t) = \epsilon(t)$$

Apply this to $\displaystyle \int_{(t_i,q_i)\to(t_f,q_f)} \mathcal{D}q \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L(q,\dot{q})\,dt} \; q(t_0)$



$$\dot{\epsilon}(t) = \epsilon\,\delta(t-(t_0-\delta)) - \epsilon\,\delta(t-(t_0+\delta))$$

$$0 = \int \delta\left(\mathcal{D}q \; e^{\frac{i}{\hbar}\int L\,dt} \; q(t_0) \right)$$

$$= \int \mathcal{D}q \; e^{\frac{i}{\hbar}\int L\,dt} \left(\frac{i}{\hbar} \underbrace{\delta S[q]}_{} \, q(t_0) + \epsilon \right)$$

$$\delta S[q] = \int_{t_i}^{t_f} \left(\epsilon(t) \frac{\partial L}{\partial q(t)} + \dot{\epsilon}(t) \overbrace{\frac{\partial L}{\partial \dot{q}(t)}}^{p} \right) dt$$

$$= \underbrace{\int_{t_0-\delta}^{t_0+\delta} \epsilon \frac{\partial L}{\partial q(t)}\, dt}_{\to 0 \text{ as } \delta \to 0} + \epsilon\, p(t_0-\delta) - \epsilon\, p(t_0+\delta)$$

$$0 = \int \mathcal{D}q \, e^{\frac{i}{\hbar}\int L \, dt} \left(\frac{i}{\hbar} p(t_0 - \delta)\, q(t_0) - \frac{i}{\hbar} p(t_0 + \delta)\, q(t_0) + 1 \right)$$

$$\hookrightarrow \frac{i}{\hbar} \hat{q}\hat{p} - \frac{i}{\hbar} \hat{p}\hat{q} + 1 = 0$$

i.e. $\boxed{[\hat{q}, \hat{p}] = i\hbar}$ Canonical commutation relation!

Example Harmonic Oscillator

$$S(x) = \int dt \underbrace{\left(\frac{1}{2}\dot{x}^2 - \frac{m^2}{2}x^2\right)}_{L}$$

$p = \dot{x}$

$$E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L = \dot{x}^2 - L = \frac{1}{2}\dot{x}^2 + \frac{m^2}{2}x^2$$

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{m^2}{2}\hat{x}^2 = \frac{1}{2}(\hat{p}+im\hat{x})(\hat{p}-im\hat{x}) - \frac{im}{2}\underbrace{[\hat{x},\hat{p}]}_{i\hbar}$$

$$= m\hbar\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

where $\hat{a} = \frac{1}{\sqrt{2m\hbar}}(\hat{p} - im\hat{x})$, $\hat{a}^+ = \frac{1}{\sqrt{2m\hbar}}(\hat{p}+im\hat{x})$

$$\left.\begin{aligned}[\hat{a},\hat{a}^+] &= 1\\ [\hat{a},\hat{a}] = [\hat{a}^+,\hat{a}^+] &= 0\end{aligned}\right\} \Rightarrow \begin{aligned}[\hat{H},\hat{a}] &= -\hat{a}\,\hbar m\\ [\hat{H},\hat{a}^+] &= \hbar m\,\hat{a}^+\end{aligned}$$

$\hat{a}^+$ increases energy by $\hbar m$ (creation operator)

$\hat{a}$ decreases energy by $\hbar m$ (annihilation operator)

Space of states $\mathcal{H}$ is $|0\rangle$, $\hat{a}^+|0\rangle$, $\hat{a}^{+2}|0\rangle$, ---

$\hat{H} = \frac{\hbar\omega}{2}$ $\frac{\hbar\omega}{2}+\hbar\omega$ $\frac{\hbar\omega}{2}+2\hbar\omega$

$$Z(\beta) = \mathrm{Tr}_{\mathcal{H}}(e^{-\beta\hat{H}}) = e^{-\frac{\hbar\omega}{2}\beta} + e^{-\frac{3\hbar\omega}{2}\beta} + \cdots$$

$Z_E(\beta)$ from

matches with "P.I. ~~result~~".

$\hbar = 1$

$$Z_E(\beta) = \int_{X(\tau+\beta)=X(\tau)} \mathcal{D}X \, e^{-S_E(X)}$$

$$Z_E(x_f, \beta_f ; x_i, \tau_i) = \sum_n e^{-(\tau_f-\tau_i)\hat{H}} \Psi_n(x_f)\Psi_n^*(x_i)$$

$$\overset{x}{=} \int dx \, Z_E(x, \beta ; x, 0)$$

$$= \int dx \sum_n \left(e^{-\beta\hat{H}}\Psi_n\right)(x) \cdot \Psi_n(x)^*$$

$$= \sum_n \int dx \, \Psi_n(x)^* \left(e^{-\beta\hat{H}}\Psi_n\right)(x)$$

$$= \sum_n \langle \Psi_n | e^{-\beta\hat{H}} \Psi_n \rangle = \mathrm{Tr}\, e^{-\beta\hat{H}}$$

<u>For fermion</u>

$$Z_E(\beta) = \int d\psi \, Z_E(-\psi, \beta ; \psi, 0)$$

$$= \int d\psi \sum_n \left(e^{-\beta\hat{H}}\Psi_n\right)(-\psi) \cdot \Psi_n(\psi)^*$$

$$= \sum_n \int d\psi \, \Psi_n(\psi)^* e^{-\beta\hat{H}} \Psi_n(\psi) = \mathrm{Tr}\, e^{-\beta\hat{H}}$$

$$\uparrow$$
$$+$$

Example "Fermionic Oscillator"

$$S = \int \underbrace{\left(i\bar{\psi}\dot{\psi} - m\bar{\psi}\psi \right)}_{L} dt$$

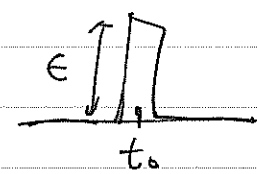
What is E?

$$E = \dot{\psi}\underbrace{\frac{\partial L}{\partial \dot{\psi}}}_{= -i\bar{\psi}} + \bar{\psi}\underbrace{\frac{\partial L}{\partial \dot{\bar{\psi}}}}_{=0} - L = i\bar{\psi}\dot{\psi} - L = m\bar{\psi}\psi$$

Ward id for ψ-translation

$$\delta\psi = \epsilon \qquad (\epsilon \text{ fermionic i.e. anticomm})$$

$$\delta S = \int \left(i\bar{\psi}\dot{\epsilon} - m\bar{\psi}\epsilon \right) dt$$



$$0 = \int \delta\left(\mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{\frac{i}{\hbar}S}\, \psi(t_0) \right)$$

$$= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{\frac{i}{\hbar}S} \left(\underbrace{\frac{i}{\hbar}\delta S}_{\parallel}\, \psi(t_0) + \epsilon \right) dt$$

$$\frac{i}{\hbar} i\bar{\psi}_{(t_0-\delta)}\epsilon - \frac{i}{\hbar} i\bar{\psi}(t_0+\delta)\epsilon + \cancel{\frac{i}{\hbar}\int_{t_0-\delta}^{t_0+\delta} (-m\bar{\psi}\epsilon)\, dt} \xrightarrow{\text{as } \delta \to 0} 0$$

$$\rightsquigarrow \quad -\frac{1}{\hbar}\hat{\psi}\,\hat{\bar{\psi}}\epsilon + \frac{1}{\hbar}\hat{\bar{\psi}}\epsilon\,\hat{\psi} + \epsilon = 0$$

$$\Rightarrow \quad \hat{\psi}\hat{\bar{\psi}} + \hat{\bar{\psi}}\hat{\psi} = \hbar \qquad \text{or } \{\hat{\psi}, \hat{\bar{\psi}}\} = \hbar$$

Similarly $\hat{\psi}\hat{\psi} = \hat{\bar{\psi}}\hat{\bar{\psi}} = 0$

$\cdots$ Clifford algebra $\quad \left(\begin{array}{l} \text{c.f. } [\hat{a}, \hat{a}^{\dagger}] = \hbar \\ [\hat{a}, \hat{a}] = [\hat{a}^{\dagger}, \hat{a}^{\dagger}] = 0 \end{array}\right)$

<u>Representation theory</u>: This algebra has a unique irreducible representation$/\mathbb{C}$ $\quad V \cong \mathbb{C}|0\rangle + \mathbb{C}\hat{\bar{\psi}}|0\rangle$

$|0\rangle \in V$ obey $\hat{\psi}|0\rangle = 0$

Then $\hat{\bar{\psi}}|0\rangle \in V$ obey $\quad \hat{\bar{\psi}}(\hat{\bar{\psi}}|0\rangle) = 0$

$$\hat{\psi}(\hat{\bar{\psi}}|0\rangle) = |0\rangle \; (\times\hbar)$$

"$\hat{\psi}$ annihilation, $\hat{\bar{\psi}}$ creation".

$$\hat{H} = \hat{E} = m\widehat{\bar{\psi}\psi} = ? \overset{\text{want}}{=} -m\widehat{\psi\bar{\psi}}$$

Standard Choice: $\frac{1}{2}m\hat{\bar{\psi}}\hat{\psi} - \frac{1}{2}m\hat{\psi}\hat{\bar{\psi}} = m\left(\hat{\bar{\psi}}\hat{\psi} - \frac{\hbar}{2}\right)$

$$= \begin{pmatrix} -m\frac{\hbar}{2} & 0 \\ 0 & m\frac{\hbar}{2} \end{pmatrix} \text{ on } (|0\rangle, \hat{\bar{\psi}}|0\rangle)$$

<u>Partition function</u>

$$Z_E(\beta) = \mathrm{Tr}_V\left(e^{-\beta\hat{H}}\right) = e^{-m\frac{\hbar}{2}\beta} + e^{m\frac{\hbar}{2}\beta}$$

matches with the "result" of path integral.

$(\hbar = 1)$

Sigma model

(M, g) Riemannian manifold

$\rightsquigarrow$ a 1-d QFT called σ-model

(generalization to $d>1$ is straightforward)

it describes a particle moving in M.

Variable : $X(t)$ a parametrized trajectory in M.

(or $X : \mathbb{R} \longrightarrow M$ map)

$$S = \int_{t_i}^{t_f} \frac{1}{2} g(\dot{X}, \dot{X})\, dt = \int_{t_i}^{t_f} \frac{1}{2} g_{X(t)}(\underset{\substack{\Uparrow \\ T_{X(t)}M}}{\dot{X}(t)}, \dot{X}(t))\, dt$$

velocity vector

(M, g) is the target space of the sigma model.

Simplest

$(M, g) = (\mathbb{R}, dx^2)$: $S = \int \frac{1}{2}\dot{x}^2\, dt$ ... ordinary particle in $\mathbb{R}$ (no potential)

Next simplest $(M, g) = (S^1, dx^2)$ $\quad x \equiv x + 2\pi R$

circle

Sigma model with target $S^1_{2\pi R}$ = circle of radius R

(circumference $2\pi R$)

$$= [0, 2\pi R]/_{0 \equiv 2\pi R}$$

periodic Variable $X(t)$ $\quad X(t) \equiv X(t) + 2\pi R$

action $\quad S(X) = \int_{t_i}^{t_f} \frac{1}{2} \dot{X}(t)^2 \, dt$

Observables $\mathcal{O}(X, \dot{X}, \ddot{X}, \cdots)$ ... must be invariant

under $X \to X + 2\pi R$ $\quad \mathcal{O}(X + 2\pi R, \dot{X}, \cdots) = \mathcal{O}(X, \dot{X}, \cdots)$

eg $\begin{cases} \mathcal{O} = X & \text{not good} \qquad \mathcal{O} = \dot{X} \text{ good} \\ \mathcal{O} = \cos(X/R) & \text{good} \end{cases}$

Quantization I ... Operator

Ward id for $\delta X = \epsilon$ $\Rightarrow$ $\hat{p} = \hat{\dot{X}}$ obeys

$$[\hat{p}, \widehat{f(X)}] = -i \widehat{\frac{d}{dX} f(X)} \qquad (\hbar = 1)$$

$$\therefore \quad \hat{p} = -i \frac{d}{dX} \quad \text{on} \quad \mathcal{H} = L^2(S^1_R, \mathbb{C}).$$

$$\hat{H} = \frac{1}{2} \hat{p}^2 = -\frac{1}{2} \frac{d^2}{dX^2}$$

Eigen functions $\quad \psi_n(x) = e^{inX/R} \quad n \in \mathbb{Z}$

$$\hat{p}\,\psi_n = \frac{n}{R}\,\psi_n$$

$$\hat{H}\,\psi_n = \frac{1}{2}\left(\frac{n}{R}\right)^2 \psi_n$$

<u>Note</u>: momentum ($\hat{p}$ eigenvalue) is "quantized" (i.e. discrete).

Partition function

$$Z_E(\beta) = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} = \sum_{n \in \mathbb{Z}} e^{-\beta \frac{1}{2}\left(\frac{n}{R}\right)^2}$$

"ϑ-function".

<u>Quantization II</u> --- path integral.

$$Z_E(\beta) = \int \mathcal{D}X\, e^{-S_E(X)}$$

$$X(\tau+\beta) = X(\tau) \quad \underline{\text{mod } 2\pi R}$$

$$S_E(X) = \int_0^\beta d\tau\, \frac{1}{2}\left(\frac{dX}{d\tau}\right)^2$$

What is the space of variables?

$$\mathrm{Map}(S^1_\beta, S^1_{2\pi R}) = \left\{ X : [0,\beta]/_{0\equiv\beta} \to [0,2\pi R]/_{0\equiv 2\pi R} \right\}$$

$$= \left\{ X : \mathbb{R} \to \mathbb{R} \;\middle|\; X(\tau+\beta) = X(\tau) + 2\pi R \underset{\text{winding \#}}{m} \right\} \Big/ X(\tau) \sim X(\tau) + 2\pi R n$$

$$= \coprod_{m\in\mathbb{Z}} \mathrm{Map}^{(m)}(S^1_\beta, S^1_{2\pi R})$$

subspace of maps of winding # $= m$.

$\forall X \in \mathrm{Map}^{(m)}(S^1_\beta, S^1_{2\pi R})$ can be written as

$$X(\tau) = \frac{\tau}{\beta} 2\pi R m + X_0(\tau) \qquad X_0(\tau) \in \mathrm{Map}^{(0)}$$

i.e. periodic $X_0(\tau+\beta) = X_0(\tau)$

$$S_E(X) = \int_0^\beta \frac{1}{2} \dot{X}^2 d\tau = \int_0^\beta \frac{1}{2} \left\{ \frac{2\pi R m}{\beta} + \dot{X}_0(\tau) \right\}^2 d\tau$$

$$= \frac{\beta}{2} \left(\frac{2\pi R m}{\beta} \right)^2 + \int_0^\beta \frac{1}{2} \dot{X}_0^2 \, d\tau$$

$$Z_E(\beta) = \sum_{m\in\mathbb{Z}} \int_{\mathrm{Map}^{(m)}} \mathcal{D}X \, e^{-S_E(X)}$$

$$= \sum_{m\in\mathbb{Z}} e^{-\frac{1}{2}\frac{(2\pi R m)^2}{\beta}} \int_{\mathrm{Map}^{(0)}} \mathcal{D}X_0 \, e^{-\frac{1}{2}\int_0^\beta X_0 \left(-\frac{d}{d\tau}\right)^2 X_0 \, d\tau}$$

What is $\displaystyle\int_{Map^{(0)}} \mathcal{D}X_0 \, e^{-\frac{1}{2}\int_0^\beta X_0 \left(-\frac{d}{d\tau}\right)^2 X_0 \, d\tau}$?

$$X_0(\tau) = x_0 + \sum_{n=1}^{\infty} \sqrt{\frac{2}{\beta}} \left(\cos\left(\frac{2\pi n \tau}{\beta}\right) x_n^c + \sin\left(\frac{2\pi n \tau}{\beta}\right) x_n^s \right)$$

$$(\delta x_0, \delta x_0) = \int_0^\beta (\delta x_0)^2 d\tau = \beta (\delta x_0)^2$$

$$\rightsquigarrow \quad \mathcal{D}X_0 = \frac{\sqrt{\beta}\, dx_0}{\sqrt{2\pi}} \prod_{n=1}^{\infty} \frac{dx_n^c}{\sqrt{2\pi}} \frac{dx_n^s}{\sqrt{2\pi}}$$

$$= \frac{\sqrt{\beta}}{\sqrt{2\pi}} \cdot \underset{\uparrow\ \int dx_0 = 2\pi R.}{2\pi R} \cdot \underbrace{\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2}}_{= \beta^{-1}} = \sqrt{\frac{2\pi}{\beta}} \cdot R$$

$$\therefore \quad Z_E(\beta) = \sqrt{\frac{2\pi}{\beta}} \cdot R \cdot \sum_{m \in \mathbb{Z}} e^{-\frac{(2\pi R m)^2}{2\beta}}$$

$$\underset{\substack{\uparrow \\ \text{Poisson resummation}}}{=} \sum_{n \in \mathbb{Z}} e^{-\frac{\beta}{2}\left(\frac{n}{R}\right)^2}$$

PI & OF matches !

Poisson resummation.

$$\sum_{n\in\mathbb{Z}} \delta(x+2\pi n) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{imx}$$

☺ $f(x)$ supported at, say, $-\pi \leq x \leq \pi$

then $$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}} \delta(x+2\pi n)\, dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x) \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{imx} = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{imx} f(x)\, dx$$

$$= \sum_{m\in\mathbb{Z}} f_m = f(0)$$

if $f(x) = \sum_{m'\in\mathbb{Z}} e^{-im'x} f_{m'}$ //

multiply $e^{-\frac{\alpha}{2}x^2}$ and integrate over $\mathbb{R}$

$$\int_{-\infty}^{\infty} dx \sum_{n\in\mathbb{Z}} \delta(x+2\pi n)\, e^{-\frac{\alpha}{2}x^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{m\in\mathbb{Z}} e^{imx} e^{-\frac{\alpha}{2}x^2} dx$$

$$\sum_{n\in\mathbb{Z}} e^{-\frac{\alpha}{2}(-2\pi n)^2} \qquad\qquad \frac{1}{2\pi}\sqrt{\frac{2\pi}{\alpha}} \sum_{m\in\mathbb{Z}} e^{-\frac{m^2}{2\alpha}}$$

$$\boxed{\therefore \sum_{n\in\mathbb{Z}} e^{-\frac{\alpha}{2}(2\pi n)^2} = \frac{1}{\sqrt{2\pi\alpha}}\, e^{-\frac{m^2}{2\alpha}}}$$

Theta function

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\alpha)^2} e^{2\pi i(\nu+\beta)(n+\alpha)}$$

$$\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(0,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} = \sum_{n\in\mathbb{Z}} e^{\frac{2\pi i\tau n^2}{2}}$$

$\vartheta_3(0,\tau)$

$$= \sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}(2\pi(\tau_2 - i\tau_1))n^2}$$

$$\therefore \quad 2\pi\tau_2 = \frac{\beta}{R^2}$$

$$\therefore Z_E(\beta) = \vartheta\begin{bmatrix}0\\ 0\end{bmatrix}\left(0, i\frac{\beta}{2\pi R^2}\right) = \vartheta_3\left(0, i\frac{\beta}{2\pi R^2}\right)$$

Poisson resummation : This $= \sqrt{\frac{2\pi}{\beta}}\cdot R \cdot \sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}\frac{(2\pi R)^2}{\beta}m^2}$ $\qquad 2\pi\tau_2' = \frac{(2\pi R)^2}{\beta}$

$$= \sqrt{\frac{2\pi}{\beta}}\cdot R \cdot \vartheta_3\left(0, i\frac{2\pi R^2}{\beta}\right)$$

<u>Note</u> $\vartheta_3\left(\frac{\nu}{\tau}, -\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{\pi i \nu^2/\tau} \vartheta_3(\nu,\tau)$

"Modular transformation property".