2-d QFT

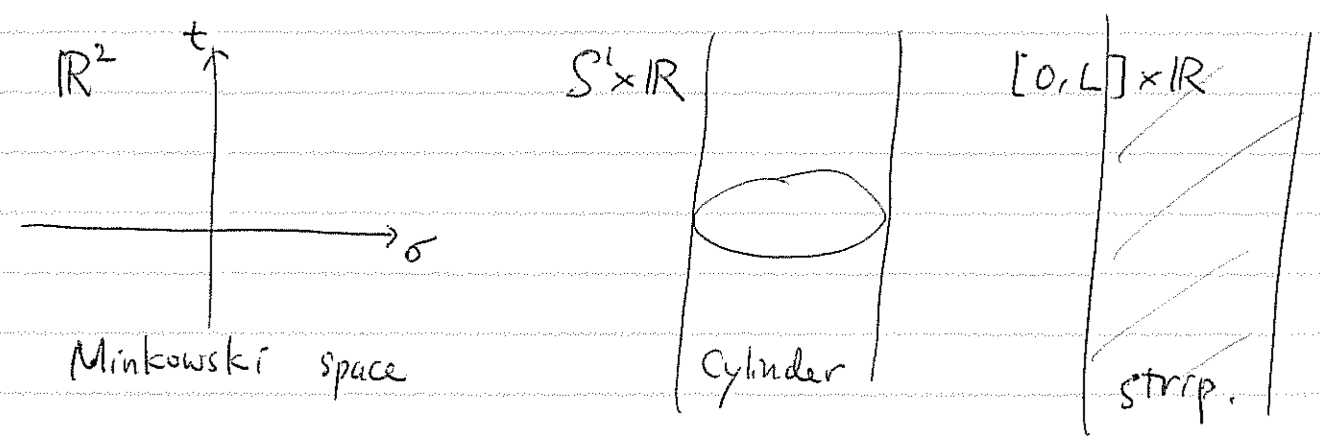
Variable $X(t,\sigma)$ $\quad$ t = time coordinate, σ = space coordinate $\left(\begin{matrix}\text{Sometimes} \\ t=\sigma^0 \\ \sigma=\sigma^1\end{matrix}\right)$

action $$S = \frac{1}{2\pi}\int dt\, d\sigma \left(\frac{1}{2}(\partial_t X)^2 - \frac{1}{2}(\partial_\sigma X)^2 - U(X) \right)$$

[Focus (for October) : free field theory $U(X) = \frac{m^2}{2}X^2$

in particular massless theory $U(X) \equiv 0$.]

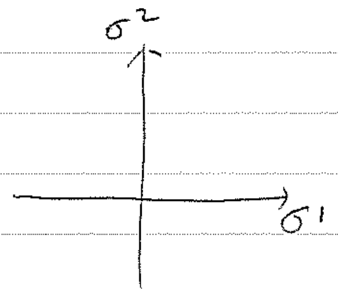
Which space of (t,σ) ?



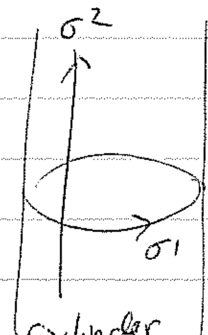
Wick rotation $t \to -i\tau$ $\quad (\sigma,\tau) = (\sigma^1,\sigma^2)$

$$S_E = \frac{1}{2\pi}\int d\sigma^1 d\sigma^2 \left(\frac{1}{2}(\partial_1 X)^2 + \frac{1}{2}(\partial_2 X)^2 + U(X) \right)$$

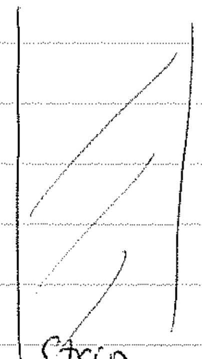
Which space?



Euclidean Space



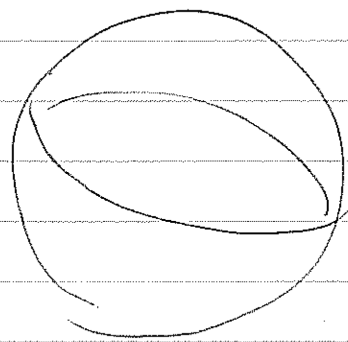
cylinder



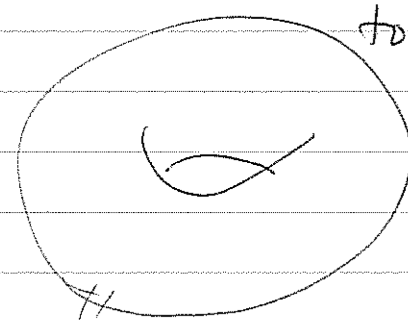
Strip

But one can consider more general space. (Σ, h)

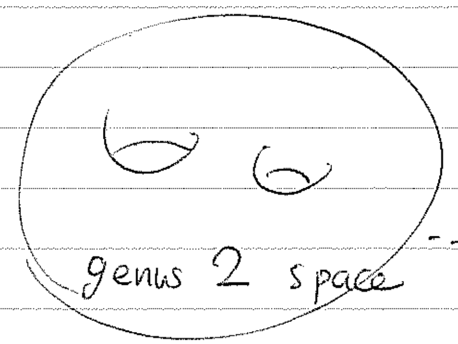
any 2d Riemannian manifold



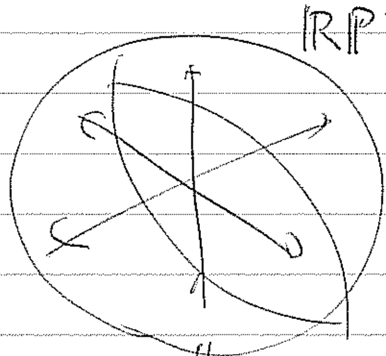
S^2



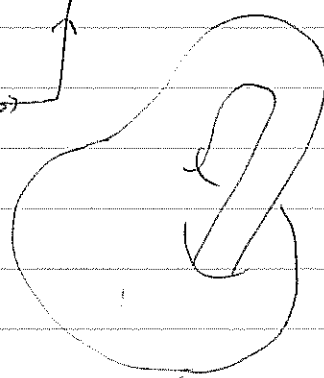
torus



genus 2 space ...

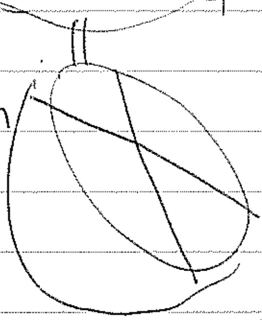


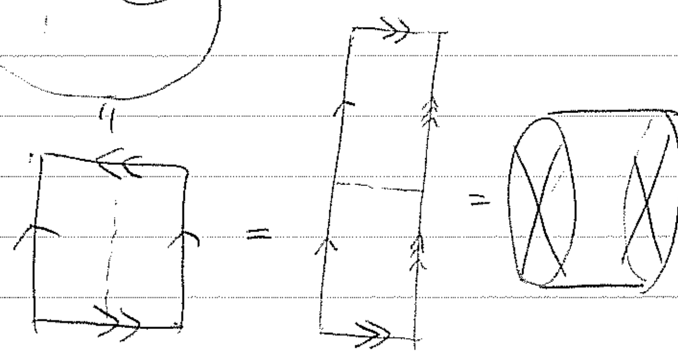
$\mathbb{RP}^2$



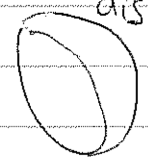
Klein bottle

written

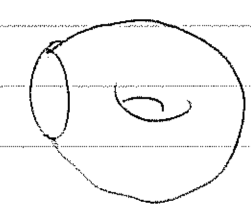




or



disc



...

What is the Action? — we metric $h = h_{\mu\nu} dx^\mu dx^\nu$

$$S_E = \frac{1}{2\pi} \int_\Sigma \sqrt{h}\, d^2\sigma \left(\frac{1}{2} h^{\mu\nu} \partial_\mu X \partial_\nu X + U(X) \right)$$

$$\sqrt{h} = \sqrt{\det(h_{\mu\nu})} = \sqrt{h_{11} h_{22} - h_{12} h_{21}}$$

Symmetry and Ward identity

$$X \to X + \delta X \qquad \delta X = \epsilon\, \mathcal{U}(X, \partial_\mu X, \partial_\mu \partial_\nu X, \cdots)$$

$$\epsilon \to \epsilon(t,\sigma) \quad : \quad \delta S = \frac{1}{2\pi} \int dt\, d\sigma\, \partial_\mu \epsilon\, J^\mu$$

J^μ : Noether current.

Classical Motion (Soln to EOM) $\Rightarrow$ $\delta S = 0 \quad \forall \epsilon(t,\sigma)$ that vanishes at bdy

$$\delta S = \frac{1}{2\pi} \int dt\, d\sigma \left(\partial_\mu (\epsilon J^\mu) - \epsilon\, \partial_\mu J^\mu \right) = -\frac{1}{2\pi} \int dt\, d\sigma\, \epsilon\, \partial_\mu J^\mu$$

$$\Rightarrow \quad \partial_\mu J^\mu = 0 \qquad \underline{\text{Conservation equation}}$$

What is conserved?

$$Q = \frac{1}{2\pi} \underbrace{\int d\sigma}_{\text{whole space}} J^t(t,\sigma)$$

$$\frac{d}{dt} Q = \frac{1}{2\pi} \int d\sigma\, \partial_t J^t \overset{\partial_\mu J^\mu = 0}{=} + \frac{1}{2\pi} \int d\sigma\, (-\partial_\sigma J^\sigma) \overset{*}{=} 0$$

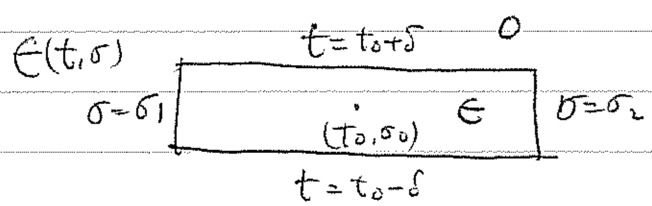
* if there is nothing from the boundary of space $\{\sigma\}$

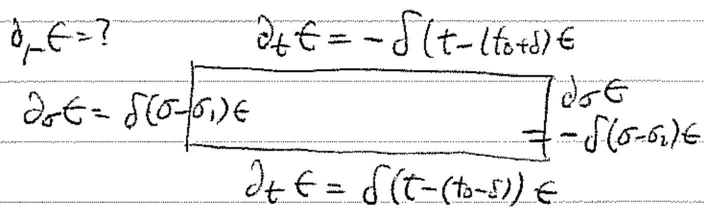
e.g.
$$\begin{cases} \text{Cylinder} : J(t,\sigma+2\pi) = J(t,\sigma) \quad \text{periodic b.c.} \\ \mathbb{R}^2 : J(t,\sigma) \to 0 \quad \text{as} \quad \sigma \to \pm\infty \\ [0,L]\times\mathbb{R} : J(t,\sigma) = 0 \quad \text{at} \quad \sigma = 0, L, \end{cases}$$

Ward identity

$$0 = \int \delta\left(\mathcal{D}X\, e^{iS}\, \mathcal{O}(t_0,\sigma_0) \right)$$

$$= \int \mathcal{D}X\, e^{iS} \left(\frac{i}{2\pi} \int d^2\sigma\, \partial_\mu \epsilon\, J^\mu\, \mathcal{O}(t_0,\sigma_0) + \delta\mathcal{O}(t_0,\sigma_0) \right)$$





$$\therefore \Big\langle \frac{i}{2\pi} \int_{\sigma_1}^{\sigma_2} d\sigma\, \epsilon\left(-J^t(t_0+\delta,\sigma) + J^t(t_0-\delta,\sigma) \right) \mathcal{O}(t_0,\sigma_0)$$

$$+ \frac{i}{2\pi} \int_{t_0-\delta}^{t_0+\delta} dt\, \epsilon\left(-J^\sigma(t,\sigma_2) + J^\sigma(t,\sigma_1) \right) \mathcal{O}(t_0,\sigma_0)$$

$$+ \delta\mathcal{O}(t_0,\sigma_0) \Big\rangle = 0$$

$\delta \to 0$

$\left.\begin{matrix}\sigma_1 \to -\infty \\ \sigma_2 \to \infty\end{matrix}\right)$ in $\mathbb{R}^2$ $\quad \left.\begin{matrix}\sigma_1 \to 0 \\ \sigma_2 \to 2\pi\end{matrix}\right)$ for cylinder $\quad \left.\begin{matrix}\sigma_1 = 0 \\ \sigma_2 = L\end{matrix}\right)$ strip

$$\left\langle \left(-i\epsilon Q(t_0+\delta) + i\epsilon Q(t_0-\delta) \right) \mathcal{O}(t_0,\sigma_0) \right\rangle = 0$$

$\rightsquigarrow$ Operator form $\quad \left[i\epsilon \widehat{Q} \,, \widehat{\mathcal{O}}(t_0,\sigma_0) \right] = \delta \mathcal{O}(t_0,\sigma_0)$

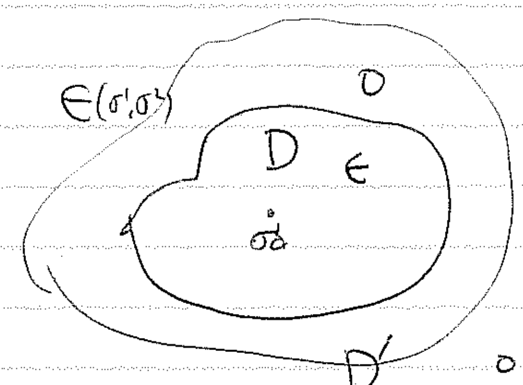
<u>Euclidean version</u> using differential form

$$\delta S_E = -\frac{1}{2\pi} \int_\Sigma d\epsilon \wedge J$$

J Noether current (1-form)

$$0 = \int \delta\left(\mathcal{D}X\, e^{-S_E}\, \mathcal{O}(\sigma_0^1 \sigma_0^2) \right)$$

$$= \int \mathcal{D}X\, e^{-S_E} \left(\frac{1}{2\pi} \int_\Sigma d\epsilon \wedge J \ \mathcal{O}(\sigma_0^1 \sigma_0^2) + \delta \mathcal{O}(\sigma_0^1,\sigma_0^2) \right)$$

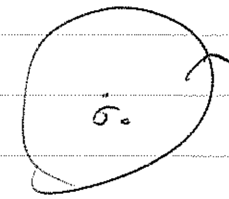


$$\int_\Sigma d\epsilon \wedge J = \int_{D'} d\epsilon \wedge J$$

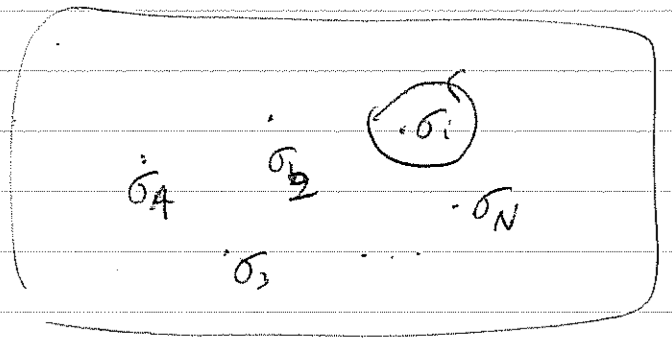
$$= \int_{D'} \left\{ d(\epsilon J) - \epsilon\, dJ \right\} \qquad \text{Stokes thm.}$$

$$= \overset{0}{\overset{\|}{\int_{\partial D'} \epsilon J}} - \int_{D'} \epsilon\, dJ = -\epsilon \int_D dJ = -\epsilon \oint_{\partial D} J$$

$$\therefore \quad \langle \delta \mathcal{O}(\sigma_0) \rangle = \left\langle \frac{1}{2\pi} \oint_{\sigma_0} J \, \mathcal{O}(\sigma_0) \right\rangle$$



↑ any contour surrounding σ_0



more general situation.

$$\langle \mathcal{O}_1(\sigma_1) \, \mathcal{O}_2(\sigma_2) \cdots \delta \mathcal{O}_i(\sigma_i) \cdots \mathcal{O}_N(\sigma_N) \rangle$$

$$= \left\langle \frac{1}{2\pi} \oint_{\sigma_i} J \; \mathcal{O}_1(\sigma_1) \, \mathcal{O}_2(\sigma_2) \cdots \mathcal{O}_i(\sigma_i) \cdots \mathcal{O}_N(\sigma_N) \right\rangle$$

↑ contour surrounding only σ_i

If contour C surrounds $\sigma_1, \cdots, \sigma_k$, not $\sigma_{k+1}, \cdots, \sigma_N$:

$$\left\langle \frac{1}{2\pi} \oint_C J \; \mathcal{O}_1(\sigma_1) \cdots \mathcal{O}_N(\sigma_N) \right\rangle$$

$$= \sum_{i=1}^{k} \langle \mathcal{O}_1(\sigma_1) \cdots \delta_\epsilon \mathcal{O}_i(\sigma_i) \cdots \mathcal{O}_N(\sigma_N) \rangle$$