Notation (Convention)

$$\delta S_E = -\frac{1}{2\pi}\int_\Sigma d\epsilon\wedge\tilde{J} = -\frac{1}{2\pi}\int_\Sigma \underbrace{(\nabla\epsilon\cdot\mathbb{J})}_{=\,\partial_1\epsilon J^1+\partial_2\epsilon J^2}\, d^2\sigma$$

$$\nabla\epsilon\cdot\mathbb{J} = \partial_\mu\epsilon J^\mu$$

$$d\epsilon\wedge\tilde{J} = (d\sigma^1\partial_1\epsilon + d\sigma^2\partial_2\epsilon)\wedge(d\sigma^1\tilde{J}_1 + d\sigma^2\tilde{J}_2)$$

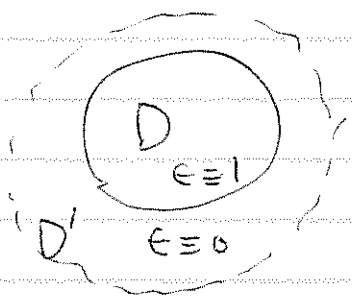
$$= (\partial_1\epsilon\,\tilde{J}_2 - \partial_1\epsilon\,\tilde{J}_1)\, d\sigma^1\wedge d\sigma^2$$

$$= (\nabla\epsilon\times\tilde{\mathbb{J}})\cdot d^2\sigma$$

$$\boxed{\begin{aligned}\tilde{J}^1 &= \tilde{J}_1 = -J^2\\ \tilde{J}^2 &= \tilde{J}_2 = +J^1\end{aligned}}$$

For those who are not familiar with differential forms :

$$\delta S_E = -\frac{1}{2\pi}\int_\Sigma (\nabla\epsilon\times\tilde{\mathbb{J}})\cdot d^2\sigma$$



$$= -\frac{1}{2\pi}\int_{D'}\left\{\nabla\times(\epsilon\tilde{\mathbb{J}}) - \epsilon\,\nabla\times\tilde{\mathbb{J}}\right\}\cdot d^2\sigma$$

Stokes

$$= -\frac{1}{2\pi}\oint_{\partial D'}\underbrace{\epsilon\tilde{\mathbb{J}}\cdot d\ell}_{=\,0} + \frac{1}{2\pi}\int_D(\nabla\times\tilde{\mathbb{J}})\cdot d^2\sigma$$

Stokes

$$= \frac{1}{2\pi}\oint_{\partial D}\tilde{\mathbb{J}}\cdot d\ell = \frac{1}{2\pi}\oint_{\partial D}\tilde{J}$$

$$\tilde{\mathbb{J}}\cdot d\ell = \tilde{J}_1 d\sigma^1 + \tilde{J}_2 d\sigma^2 = \tilde{J}$$

$$= -J^2 d\sigma^1 + J^1 d\sigma^2$$

Example

$$S = \frac{1}{2\pi}\int dt\, d\sigma \left(\frac{1}{2}(\partial_t X)^2 - \frac{1}{2}(\partial_\sigma X)^2 \right) \qquad (U=0)$$

X - translation symmetry $\quad \delta X = \epsilon$

$$\delta S = \frac{1}{2\pi}\int dt\, d\sigma \left(\partial_t \epsilon\, \partial_t X - \partial_\sigma \epsilon\, \partial_\sigma X \right)$$

$$J^t = \partial_t X, \quad J^\sigma = -\partial_\sigma X$$

Conservation eqn $\quad \partial_t J^t + \partial_\sigma J^\sigma = \partial_t^2 X - \partial_\sigma^2 X = 0$

$\iff$ eom.

Noether charge (denoted p)

$$p = \frac{1}{2\pi}\int \partial_t X\, d\sigma$$

Called X-momentum or target space momentum.

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cf Euclidean

$$\delta S_E = \frac{1}{2\pi}\int d\sigma^1 d\sigma^2 \left(\partial_1 \epsilon\, \overbrace{\partial_1 X}^{-J^1} + \partial_2 \epsilon\, \overbrace{\partial_2 X}^{-J^2} \right)$$

$$= -\frac{1}{2\pi}\int \left(d\sigma^1 \partial_1 \epsilon + d\sigma^2 \partial_2 \epsilon \right)\left(-d\sigma^2 \partial_1 X + d\sigma^1 \partial_2 X \right)$$

$$\therefore\ J = d\sigma^1 \underbrace{\partial_2 X}_{\tilde{J}_1 = -J^2} \underbrace{- d\sigma^2 \partial_1 X}_{\tilde{J}_2 = J^1}$$

Example

$$\frac{1}{2}\eta^{\mu\nu}\partial_\mu X \partial_\nu X$$

$$S = \frac{1}{2\pi}\int dt\, d\sigma \left(\frac{1}{2}(\partial_t X)^2 - \frac{1}{2}(\partial_\sigma X)^2 - \frac{m^2}{2} X^2 \right)$$

time and space translation $X(t,\sigma) \to X(t+\Delta t, \sigma + \Delta\sigma)$

infinitesimal

$$\rightsquigarrow \delta X = \epsilon^\mu \partial_\mu X$$

$$\delta S = \frac{1}{2\pi}\int \Bigg(\underbrace{\eta^{\mu\nu}\partial_\mu(\epsilon^\lambda \partial_\lambda X)\partial_\nu X}_{\substack{\eta^{\mu\nu}\partial_\mu \epsilon^\lambda \partial_\lambda X \partial_\nu X \\ + \epsilon^\lambda \partial_\mu \partial_\lambda X \partial_\nu X \eta^{\mu\nu}}} - m^2 \underbrace{\epsilon^\lambda \partial_\lambda X \cdot X}_{= \frac{1}{2}\partial_\lambda(\epsilon^\lambda X^2) - \frac{1}{2}\partial_\lambda \epsilon^\lambda X^2} \Bigg) d^2\sigma$$

$$\underbrace{\epsilon^\lambda \partial_\mu \partial_\lambda X \partial_\nu X \eta^{\mu\nu}}_{=\frac{1}{2}\partial_\lambda(\epsilon^\lambda \eta^{\mu\nu}\partial_\mu X \partial_\nu X) - \frac{1}{2}\partial_\lambda \epsilon^\lambda \eta^{\mu\nu}\partial_\mu X \partial_\nu X}$$

$$= \frac{1}{2\pi}\int \left(\partial_\mu \epsilon^\lambda \partial_\lambda X \partial_\nu X \eta^{\mu\nu} - \frac{1}{2}\partial_\mu \epsilon^\mu \eta^{\rho\nu}\partial_\rho X \partial_\nu X + \frac{m^2}{2}\partial_\mu \epsilon^\mu X^2 \right) d^2\sigma$$

$$= \frac{1}{2\pi}\int \partial_\mu \epsilon^\lambda \underbrace{\left(\eta^{\mu\nu}\partial_\lambda X \partial_\nu X - \frac{1}{2}\delta^\mu_\lambda \eta^{\rho\nu}\partial_\rho X \partial_\nu X + \frac{m^2}{2}\delta^\mu_\lambda X^2 \right)}_{T^\mu_\lambda \text{ energy-momentum tensor}} d^2\sigma$$

$$T^t_{\ t} = (\partial_t X)^2 - \frac{1}{2}\left((\partial_t X)^2 - (\partial_\sigma X)^2\right) + \frac{m^2}{2}X^2 = \frac{1}{2}\left((\partial_t X)^2 + (\partial_\sigma X)^2 + m^2 X^2\right)$$

$$T^\sigma_{\ t} = -\partial_t X \partial_\sigma X$$

$$T^t_{\ \sigma} = \partial_\sigma X \partial_t X$$

$$T^\sigma_{\ \sigma} = -(\partial_\sigma X)^2 - \frac{1}{2}\left((\partial_t X)^2 - (\partial_\sigma X)^2\right) + \frac{m^2}{2}X^2 = -\frac{1}{2}(\partial_t X)^2 - \frac{1}{2}(\partial_\sigma X)^2 + \frac{m^2}{2}X^2$$

Note $T^\mu_\mu = m^2 X^2$ $\left[\begin{array}{l}\text{Vanishes if } m=0 \leftarrow \text{IMPORTANT!} \\ \text{non-vanishing if } m \neq 0.\end{array}\right.$

Euclidean case

$$S_E = \frac{1}{2\pi}\int \underbrace{\left(\frac{1}{2}(\partial_\tau X)^2 + \frac{1}{2}(\partial_\sigma X)^2 + \frac{m^2}{2}X^2\right)}_{\frac{1}{2}h^{\mu\nu}\partial_\mu X\partial_\nu X + \frac{m^2}{2}X^2} d\sigma d\tau \qquad h^{\mu\nu} = \delta^{\mu\nu} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\rightarrow \delta S_E = \frac{1}{2\pi}\int \partial_\mu \epsilon^\lambda \underbrace{\left(h^{\mu\nu}\partial_\lambda X\partial_\nu X - \frac{1}{2}\delta^\mu_\lambda h^{\rho\nu}\partial_\rho X\partial_\nu X - \frac{m^2}{2}\delta^\mu_\lambda X^2\right)}_{-T^\mu_\lambda} d\sigma d\tau$$

$$T^1_1 = -\partial_1 X\partial_1 X + \frac{1}{2}\left((\partial_1 X)^2 + (\partial_2 X)^2\right) + \frac{m^2}{2}X^2$$

$$= -\frac{1}{2}(\partial_1 X)^2 + \frac{1}{2}(\partial_2 X)^2 + \frac{m^2}{2}X^2$$

$$T^2_1 = -\partial_1 X\partial_2 X$$

$$T^1_2 = -\partial_2 X\partial_1 X$$

$$T^2_2 = -(\partial_2 X)^2 + \frac{1}{2}\left((\partial_1 X)^2 + (\partial_2 X)^2\right) + \frac{m^2}{2}X^2$$

$$= \frac{1}{2}(\partial_1 X)^2 - \frac{1}{2}(\partial_2 X)^2 + \frac{m^2}{2}X^2$$

$T^\mu_\mu = +m^2 X^2$ again $\left\{\begin{array}{l}\text{Vanishes if } m=0 \\ \text{non-vanishing if } m \neq 0\end{array}\right.$

Noether charges

$$H = \frac{1}{2\pi}\int_{S^1} T^t{}_t \, d\sigma = \frac{1}{2\pi}\int \frac{1}{2}\left((\partial_t X)^2 + (\partial_\sigma X)^2\right) d\sigma$$

WS energy

$$P = \frac{1}{2\pi}\int_{S^1} T^t{}_s \, d\sigma = \frac{1}{2\pi}\int \partial_t X \, \partial_\sigma X \; d\sigma$$

WS Momentum

Mode expansions

at fixed t $\quad X(t,-) : S^1 \to \mathbb{R}$

decompose into Fourier modes

$$X(t,\sigma) = x_0(t) + \sum_{n\neq 0} x_n(t)\, e^{in\sigma}$$

X real $\Leftrightarrow$ $x_n(t)^* = x_{-n}(t)$

Plug this into S :

$$S = \int dt \left\{ \frac{1}{2}(\dot{x}_0)^2 + \sum_{n=1}^{\infty}\left(|\dot{x}_n|^2 - n^2 |x_n|^2\right) \right\}$$

free particle in a line $\mathbb{R}$

(Complex) Harmonic Oscilator with frequency n

$$\boxed{\begin{array}{l}\text{It is a decoupled system of} \\ 1 \text{ free particle } \& \ \infty \text{ many Harmonic Oscillators}\end{array}}$$

Quantization

quantize each constituent system & add them up.

- free particle : $x_0 \rightsquigarrow p_0 = \dot{x}_0$

$$[x_0, p_0] = i$$

$|k\rangle_0$ p_0-eigenstate $\quad p_0|k\rangle_0 = k|k\rangle_0$

$$H_0 = \frac{1}{2}p_0^2 \qquad H_0|k\rangle_0 = \frac{k^2}{2}|k\rangle_0 .$$

ground state $|0\rangle_0$; $H_0 = 0$

- n-th complex Oscillator $(n = 1, 2, \cdots)$

$$L_n = |\dot{x}_n|^2 - n^2|x_n|^2$$

$$x_n = \frac{1}{\sqrt{2}}(x_{1n} + i x_{2n}) \qquad L_n = \frac{1}{2}\dot{x}_{1n}^2 - \frac{n^2}{2}x_{1n}^2 + \frac{1}{2}\dot{x}_{2n}^2 - \frac{n^2}{2}x_{2n}^2$$

$x_{in} \Leftrightarrow$ creation $a_{in}^\dagger$ & annihilation a_{in}

$$H_n = n\left(a_{1n}^\dagger a_{1n} + \frac{1}{2}\right) + n\left(a_{2n}^\dagger a_{2n} + \frac{1}{2}\right).$$

rewriting it (~~following convention~~) for making left-right decomposition manifest

$$\alpha_n := \sqrt{\frac{n}{2}}(a_{1n} + i a_{2n}) \qquad \alpha_{-n} := \alpha_n^\dagger$$

$$\tilde{\alpha}_n = \sqrt{\frac{n}{2}}(a_{1n} - i a_{2n}) \qquad \tilde{\alpha}_{-n} := \tilde{\alpha}_n^\dagger$$

$$H_n = \alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n + n$$

$$[\alpha_n, \alpha_{-n}] = [\tilde{\alpha}_n, \tilde{\alpha}_{-n}] = n, \quad [\alpha_{\pm n}, \tilde{\alpha}_{\pm n}] = 0.$$

$$\left.\begin{array}{l}\alpha_{n>0}\\ \tilde{\alpha}_{n>0}\end{array}\right\}\text{annihilation}\qquad \left.\begin{array}{l}\alpha_{(-n)<0}\\ \tilde{\alpha}_{(-n)<0}\end{array}\right\}\text{creation.}$$

ground state $|0\rangle_n$: $H_n = \frac{n}{2} + \frac{n}{2} = n$

Space of states : ~~tensor product of ∞ systems~~

~~ground (lowest energy) state :~~

~~product of ground states~~

define $|k\rangle := |k\rangle_0 \otimes \bigotimes_{n=1}^{\infty} |0\rangle_n$

Other states are obtained by considering oscillator excitations (multiply α_{-n_1}, $\tilde{\alpha}_{-n_2}$ $n_1 > 0$, ...)

Hamiltonian

$$H = H_0 + \sum_{n=1}^{\infty} H_n$$

$$= \frac{1}{2} p_0^2 + \sum_{n=1}^{\infty} (\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n + n)$$

What is the ground state energy ?

$|0\rangle_0 \otimes \bigotimes_{n=1}^{\infty} |0\rangle_n = |0\rangle$ is the ground state

$$E_0 = \sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12} \qquad \zeta\text{-fun reg.}$$

$$\left(\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\right)$$

WS momentum

$$P = \frac{1}{2\pi}\int_{S^1} \partial_t X \partial_\sigma X \, d\sigma = - \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n$$

(Exercise)

target space momentum

$$p = \frac{1}{2\pi}\int_{S^1} \partial_t X \, d\sigma = p_0$$

The excited states $\prod_{n=1}^{\infty} (\alpha_{-n})^{m_n} (\tilde{\alpha}_{-n})^{\tilde{m}_n} |k\rangle$

($m_n \neq 0$, $\tilde{m}_n \neq 0$ only for finite # of n's.)

has
$$H = \frac{k^2}{2} + \sum_{n=1}^{\infty} n(m_n + \tilde{m}_n) - \frac{1}{12}$$

$$P = \sum_{n=1}^{\infty} n(-m_n + \tilde{m}_n)$$

$$p = k.$$

$$X(t,\sigma) = e^{iHt} X(0,\sigma) e^{-iHt}$$

$$= x_0 + t p_0 + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n e^{-in(t-\sigma)} + \tilde{\alpha}_n e^{-in(t+\sigma)}\right)$$