Lagrangian $\xleftrightarrow{\text{Legendre}}$ Hamiltonian

$$S = \int_{t_i}^{t_f} L(q, \dot{q})\, dt$$

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

$\downarrow$ extremize fixing $q(t_i)$, $q(t_f)$

$H(p, q)$ Hamiltonian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

$$\frac{d}{dt} A = \{A, H\}$$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

Path-integral $\longleftrightarrow$ Operator

Transition amplitude

$(t_i, q_i) \to (t_f, q_f)$

$$= \int \mathcal{D}q\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L(q, \dot{q})\, dt}$$

$q(t_f) = q_f, \quad q(t_i) = q_i$

$$[\hat{q}, \hat{p}] = i\hbar \widehat{\{q, p\}} = i\hbar$$

States $\leftrightarrow$ vectors in $\mathcal{H}$

Time evolution

$$U_{t_f, t_i} = e^{-i\frac{t_f - t_i}{\hbar}\hat{H}}$$

A reminder : Legendre transform

$\boxed{L \to H}$ $L(q, \dot{q})$ given

Solve $\frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) = p_i$ for $\dot{q}$: $\dot{q}^i = \dot{q}^i(p, q)$

$$H(p, q) := \sum_i p_i \dot{q}^i(p, q) - L(q, \dot{q}(p, q))$$

$\boxed{H \to L}$ $H(p, q)$ given

Solve $\frac{\partial H}{\partial p_i}(p, q) = \dot{q}^i$ for p : $p_i = p_i(q, \dot{q})$

$$L(q, \dot{q}) := \sum_i p_i(q, \dot{q}) \dot{q}^i - H(q, p(q, \dot{q}))$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} \quad \underset{\boxed{L \leftarrow H}}{\overset{\boxed{L \to H}}{\rightleftarrows}} \quad \dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

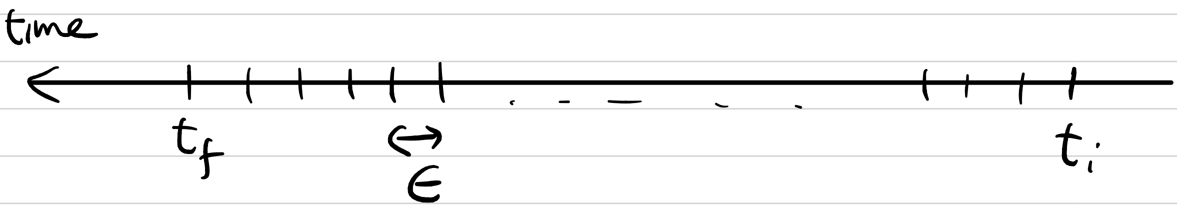
RHS can also obtained by extremizing

$$S = \int_{t_i}^{t_f} dt \, (p\dot{q} - H(p, q))$$ fixing $q(t_i)$ & $q(t_f)$.

Operator → Path-integral

transition amplitude in operator formalism :

$$Z(t_f, q_f\,;\, t_i, q_i) = \langle q_f |\, e^{-\frac{i}{\hbar}(t_f - t_i)\hat{H}} \,| q_i \rangle$$



$$t_f - t_i = N\epsilon$$

$$e^{-\frac{i}{\hbar}(t_f - t_i)\hat{H}} = e^{-\frac{i}{\hbar}N\epsilon\hat{H}} = \underbrace{e^{-\frac{i\epsilon}{\hbar}\hat{H}} \cdots e^{-\frac{i\epsilon}{\hbar}\hat{H}}}_{N}$$

$$Z(t_f, q_f\,;\, t_i, q_i)$$

$$= \langle q_f |\, e^{-\frac{i\epsilon}{\hbar}\hat{H}} \;\; e^{-\frac{i\epsilon}{\hbar}\hat{H}} \cdots\cdots e^{-\frac{i\epsilon}{\hbar}\hat{H}} \;\; e^{-\frac{i\epsilon}{\hbar}\hat{H}} \,| q_i \rangle$$

$$\int dq_{N-1} | q_{N-1} \rangle \langle q_{N-1} | \quad \cdots \quad \int dq_1 | q_1 \rangle \langle q_1 |$$

$$= \int \prod_{j=1}^{N-1} dq_j \; \langle q_f |\, e^{-\frac{i\epsilon}{\hbar}\hat{H}} \,| q_{N-1} \rangle \langle q_{N-1} |\, e^{-\frac{i\epsilon}{\hbar}\hat{H}} \,| q_{N-2} \rangle \cdots$$

$$\cdots \langle q_2 |\, e^{-\frac{i\epsilon}{\hbar}\hat{H}} \,| q_1 \rangle \langle q_1 |\, e^{-\frac{i\epsilon}{\hbar}\hat{H}} \,| q_i \rangle$$

$$\langle q_{j+1} | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | q_j \rangle = \int dp_j \, \langle q_{j+1} | p_j \rangle \langle p_j | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | q_j \rangle$$

- $$\langle q_{j+1} | p_j \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{p_j q_{j+1}}{\hbar}}$$

- $$\langle p_j | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | q_j \rangle = \langle p_j | \left(1 - \frac{i\epsilon}{\hbar}\hat{H} + O(\epsilon^2) \right) | q_j \rangle$$

  Suppose $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$, then

  $$\langle p_j | \hat{H} | q_j \rangle = \frac{p_j^2}{2m} + V(q_j) = H(p_j, q_j)$$

  $$= \langle p_j | q_j \rangle \left(1 - \frac{i\epsilon}{\hbar} H(p_j, q_j) + O(\epsilon^2) \right)$$

  $$= \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{p_j q_j}{\hbar} - \frac{i\epsilon}{\hbar} H(p_j, q_j)} + O(\epsilon^2)$$

$$= \int \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar} p_j (q_{j+1} - q_j) - \frac{i\epsilon}{\hbar} H(p_j, q_j)} + O(\epsilon^2)$$

$$Z(t_f, q_f : t_i, q_i)$$

$$= \int \prod_{j=1}^{N-1} dq_j \prod_{j=0}^{N-1} \frac{dp_j}{2\pi\hbar} \, e^{\frac{i}{\hbar}\sum_{j=0}^{N-1}\left\{ p_j(q_{j+1}-q_j) - \epsilon H(p_j, q_j) \right\}} + O(N\epsilon^2)$$

$$q_N = q_f, \quad q_0 = q_i$$

$N \to \infty$ holding

$N\epsilon = t_f - t_i$ fixed

$$\sum_{j=0}^{N-1} \epsilon \left\{ p_j \frac{q_{j+1}-q_j}{\epsilon} - H(p_j, q_j) \right\}$$

$$= \int \mathcal{D}q\,\mathcal{D}p \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,(p\dot{q} - H(p,q))}$$

$$q(t_f) = q_f, \quad q(t_0) = q_i$$

[Integrate out p : Solve $\dot{q} - \frac{\partial H}{\partial p}(p,q) = 0$ for p

and insert the answer.

— Legendre transform]

$$= \int \mathcal{D}q \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(q, \dot{q})}$$

$$q(t_f) = q_f, \quad q(t_0) = q_i$$

More concretely, for $H = \frac{p^2}{2m} + V(q)$,

$$\int \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar}\epsilon\left\{ p_j \frac{q_{j+1}-q_j}{\epsilon} - H(p_j, q_j) \right\}}$$

$$= -\frac{p_j^2}{2m} + p_j \frac{q_{j+1}-q_j}{\epsilon} - V(q_j)$$

$$= -\frac{1}{2m}\left(p_j - m\frac{q_{j+1}-q_j}{\epsilon}\right)^2 + \frac{m}{2}\left(\frac{q_{j+1}-q_j}{\epsilon}\right)^2 - V(q_j)$$

$$= C_\epsilon \, e^{\frac{i\epsilon}{\hbar}\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\epsilon}\right)^2 - V(q_j)\right)}$$

$$C_\epsilon := \int \frac{dp_j}{2\pi\hbar} e^{-\frac{i\epsilon}{2m\hbar}p_j^2} = \sqrt{\frac{m}{2\pi\hbar i\epsilon}}$$

$$Z(t_f, q_f; t_i, q_i)$$

$$= \int C_\epsilon^N \prod_{j=1}^{N-1} dq_j \, e^{\frac{i}{\hbar}\sum_{j=0}^{N-1}\epsilon\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\epsilon}\right)^2 - V(q_j)\right)}$$

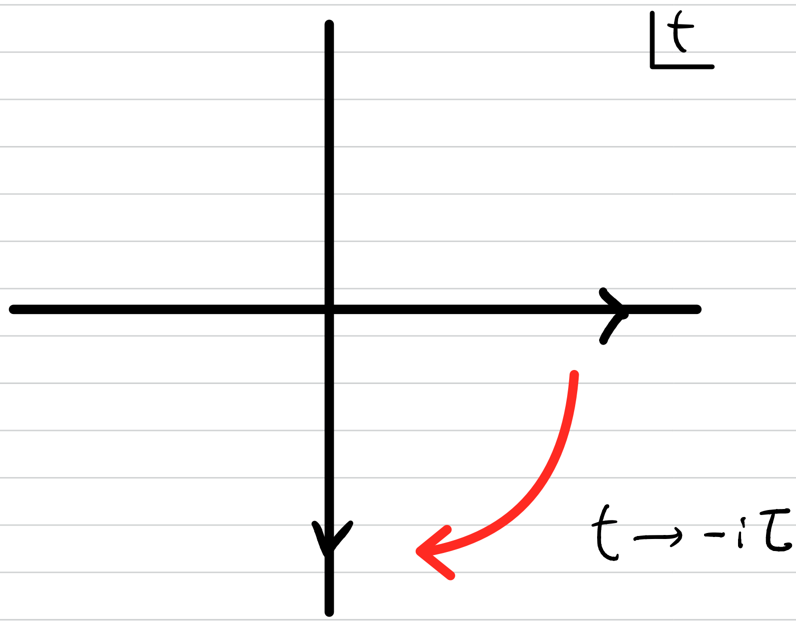
$$q_N = q_f, \quad q_0 = q_i$$

$N \to \infty$

$N\epsilon = t_f - t_i$

$$= \int_{q(t_f)=q_f,\; q(t_i)=q_i} \mathcal{D}q \, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \underbrace{\left(\frac{m}{2}\dot{q}^2 - V(q)\right)}_{L(q,\dot{q})}} .$$

Wick rotation



In the path-integral, $\epsilon \to -i\varepsilon \quad (\varepsilon > 0)$.

Oscillatory integral $\qquad$ Absolutely convergent integral

$$\int \frac{dp_j}{2\pi\hbar}\, e^{\frac{i}{\hbar}\epsilon\left(-\frac{1}{2m}(p_j+\cdots)^2\right)} \to \int \frac{dp_j}{2\pi\hbar}\, e^{-\frac{\varepsilon}{\hbar}\frac{1}{2m}(p_j+\cdots)^2},$$

$$e^{\frac{i\epsilon}{\hbar}\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\epsilon}\right)^2 - V(q_j)\right)}$$

$$\to e^{\frac{i(-i\varepsilon)}{\hbar}\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{-i\varepsilon}\right)^2 - V(q_j)\right)}$$

$$= e^{-\frac{\varepsilon}{\hbar}\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\varepsilon}\right)^2 + V(q_j)\right)}.$$

$$e^{\frac{i}{\hbar}S[q]} = e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left(\frac{m}{2}\dot{q}^2 - V(q)\right)}$$

$$\rightarrow \quad e^{-\frac{1}{\hbar}\int_{\tau_i}^{\tau_f} d\tau \underbrace{\left(\frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q)\right)}_{L_E\left(q, \frac{dq}{d\tau}\right)}} = e^{-\frac{1}{\hbar}S_E[q]}$$

Euclidean Lagrangian/action

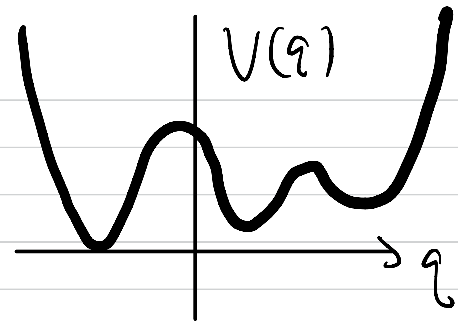
In general, $L_E\left(q, \frac{dq}{d\tau}\right) = -L\left(q, i\frac{dq}{d\tau}\right)$.

$$\langle q_f | e^{-i\frac{t_f - t_i}{\hbar}\hat{H}} | q_i \rangle = \int_{q(t_f)=q_f,\ q(t_i)=q_i} \mathcal{D}q \; e^{-\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(q, \dot{q})}$$

$$\downarrow$$

$$\langle q_f | e^{-\frac{\tau_f - \tau_i}{\hbar}\hat{H}} | q_i \rangle = \int_{q(\tau_f)=q_f,\ q(\tau_i)=q_i} \mathcal{D}q \; e^{-\frac{1}{\hbar}\int_{\tau_i}^{\tau_f} d\tau\, L_E\left(q, \frac{dq}{d\tau}\right)}$$

If $V(q) \nearrow +\infty$ as $|q| \nearrow \infty$



then $S_E[q] \nearrow \infty$ at ∞ of $\{q(\tau)\}$.

The path-integral $\int \mathcal{D}q \, e^{-\frac{1}{\hbar} S_E[q]}$

is well-behaved.

Partition function

$$\mathrm{Tr}_{\mathcal{H}}\left(e^{-\frac{T}{\hbar}\hat{H}}\right) = \int dq \, \langle q | e^{-\frac{T}{\hbar}\hat{H}} | q \rangle$$

$$= \int dq \int_{q(T)=q=q(0)} \mathcal{D}q \, e^{-\frac{1}{\hbar}\int_0^T d\tau \, L_E(q, \frac{dq}{d\tau})}$$

$$= \int_{q(T)=q(0)} \mathcal{D}q \, e^{-\frac{1}{\hbar}\int_0^T d\tau \, L_E(q, \frac{dq}{d\tau})}$$

$q(T) = q(0) \iff q(\tau)$ is periodic under $\tau \to \tau + T$.

Partition function $\mathrm{Tr}_{\mathcal{H}}\, e^{-\frac{T}{\hbar}\hat{H}}$

$=$ Euclidean path-integral over configurations on the circle $S^1_T = \mathbb{R}/T\mathbb{Z}$ of circumference T

$$Z(S^1_T) = \int \mathcal{D}q \; e^{-\frac{1}{\hbar}\int_{S^1_T} d\tau \, L_E\left(q, \frac{dq}{d\tau}\right)}$$

Note: This is well-behaved if the energy spectrum $\{E_n\}_{n=0}^{\infty}$ (i.e. eigenvalues of $\hat{H}$) is bounded below and $E_n \to \infty$ as $n \to \infty$ fast enough.

Symmetry and Ward identity

Consider a "QFT" with fields $\phi = (\phi_1, \cdots, \phi_n)$,

measure $d^n\phi = d\phi_1 \cdots d\phi_n$

& action $S_E(\phi) = S_E(\phi_1, \cdots, \phi_n)$

Focus of interest :

$$Z = \int d^n\phi \; e^{-S_E(\phi)} \qquad \text{Partition function}$$

$$\langle f \rangle = \frac{1}{Z} \int d^n\phi \; e^{-S_E(\phi)} f(\phi) \qquad \text{Correlation function}$$

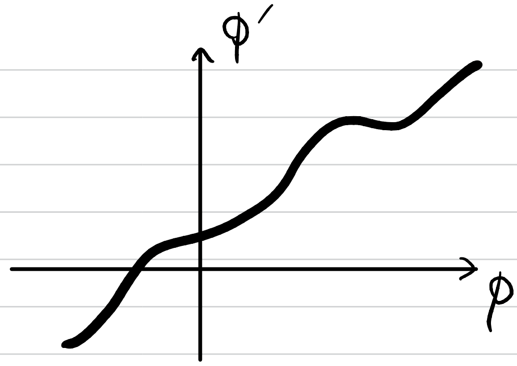
A **symmetry** of the theory is a transformation

$$\phi = (\phi_1, \cdots, \phi_n) \longmapsto g(\phi) = (g_1(\phi), \cdots, g_n(\phi))$$

that leaves $d^n\phi \; e^{-S_E(\phi)}$ invariant.

i.e. $$\det\left(\frac{\partial g_i(\phi)}{\partial \phi_j}\right) e^{-S_E(g(\phi))} = e^{-S_E(\phi)} .$$

Change of integration variables :



Single variable case : $\phi \rightsquigarrow \phi' = g(\phi)$

$$\int_{-\infty}^{\infty} d\phi\, F(\phi) = \int_{-\infty}^{\infty} d\phi'\, F(\phi') = \int_{-\infty}^{\infty} d\phi\, g'(\phi)\, F(g(\phi)).$$

Likewise

$$\int d^n\phi\, e^{-S_E(\phi)} f(\phi) = \int \underbrace{d^n g(\phi)\, e^{-S_E(g(\phi))}}_{\| \\ d^n\phi\, e^{-S_E(\phi)}} f(g(\phi))$$

$\leftarrow$ if g is a symmetry.

$\therefore$ If g is a symmetry, correlation functions satisfy

$$\boxed{\langle f \rangle = \langle f \circ g \rangle .}$$

Ward identity

Infinitesimal form :

$\{g_\alpha\}_{\alpha \in \mathbb{R}}$: 1-parameter group of transformations.

$$\phi \mapsto \phi + \delta\phi \quad ; \quad \delta\phi = \frac{d}{d\alpha} g_\alpha(\phi)\Big|_{\alpha=0}.$$

... infinitesimal transformation.

If $\{g_\alpha\}_{\alpha \in \mathbb{R}}$ is a 1-parameter group of symmetries,

Ward identity : $\langle f \rangle = \langle f \circ g_\alpha \rangle \quad \forall \alpha$

$\Rightarrow \frac{d}{d\alpha}$ at $\alpha = 0$:

$$\boxed{0 = \langle \delta f \rangle}$$

(infinitesimal form of)

Ward identity

where $\delta f(\phi) := \frac{d}{d\alpha} f(g_\alpha(\phi)) \Big|_{\alpha=0}$

There are Ward identities even for non-symmetries:

$$\int d^n\phi\, e^{-S_E(\phi)} f(\phi) = \int d^n g_\alpha(\phi)\, e^{-S_E(g_\alpha(\phi))} f(g_\alpha(\phi))$$

① Suppose $d^n\phi$ is invariant but $S_E(\phi)$ is not.

$$\rightarrow \quad 0 = \int d^n\phi\, e^{-S_E(\phi)} \left(-\delta S_E(\phi)\, f(\phi) + \delta f(\phi) \right)$$

$$\boxed{\langle \delta f \rangle = \langle \delta S_E \cdot f \rangle}$$

② Suppose $S_E(\phi)$ is invariant but $d^n\phi$ is not,

and the change is known: $d^n g_\alpha(\phi) = d^n\phi\, e^{\alpha a(\phi)}$

(called anomalous symmetry with anomaly a)

$$\rightarrow \quad 0 = \int d^n\phi\, e^{-S_E(\phi)} \left(a(\phi)\cdot f(\phi) + \delta f(\phi) \right)$$

$$\boxed{\langle \delta f \rangle = -\langle a \cdot f \rangle}$$

anomalous Ward identity

Path-integral $\longrightarrow$ Operator

$$Z(t_f, q_f \,;\, t_i, q_i) = \int_{q(t_f)=q_f,\ q(t_i)=q_i} \mathcal{D}q \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(q,\dot{q})}$$

$\rightsquigarrow$ An operator $\hat{Z}_{t_f,t_i}$ on the space $\mathcal{H}$ of functions on q:

$$\Psi(q) \longmapsto \left(\hat{Z}_{t_f,t_i}\Psi\right)(q) = \int dq'\, Z(t_f,q\,;\,t_i,q')\,\Psi(q').$$

- $\hat{Z}_{t_f,t_i}$ depends only on $t_f - t_i$, so can be written as

$$\hat{Z}_{t_f,t_i} =: \hat{Z}_{t_f-t_i}. \qquad t_3 > t_2 > t_1$$

- $$\int dq_2\, Z(t_3,q_3\,;\,t_2,q_2)\, Z(t_2,q_2\,;\,t_1,q_1) = Z(t_3,q_3\,;\,t_1,q_1).$$

$$\therefore\quad \hat{Z}_{t_3-t_2} \circ \hat{Z}_{t_2-t_1} = \hat{Z}_{t_3-t_1}$$

$\rightsquigarrow$ One can write $\hat{Z}_T = e^{-\frac{i}{\hbar}T\hat{H}}$

for some operator $\hat{H}$ on $\mathcal{H}$

At this moment, we don't know the relation to Hamiltonian. (We'll see it later.)

Let $\mathcal{O}$ be an expression of $q, \dot{q}, \ddot{q}, \cdots$

e.g. $\mathcal{O} = q^3, \dot{q}^2, q\ddot{q}, \dot{q}^2 + q^2, \cdots$ "local observable"

For $t_i < t < t_f$, write

$$Z(t_f, q_f; \mathcal{O}(t); t_i, q_i) = \int_{q(t_f)=q_f,\ q(t_i)=q_i} \mathcal{D}q\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(q,\dot{q})} \mathcal{O}(t)$$

⇝ an operator on $\mathcal{H}$:

$$\Psi(q) \mapsto \left(\hat{Z}_{t_f,t_i}(\mathcal{O}(t))\Psi\right)(q) = \int dq'\, Z(t_f, q; \mathcal{O}(t); t_i, q')\Psi(q')$$

- $$\hat{Z}_{t_f+\Delta t_f,\ t_i-\Delta t_i}(\mathcal{O}(t)) = \hat{Z}_{\Delta t_f} \circ \hat{Z}_{t_f,t_i}(\mathcal{O}(t)) \circ \hat{Z}_{\Delta t_i}$$

- $$\hat{Z}_{t_f+\Delta t,\ t_i+\Delta t}(\mathcal{O}(t+\Delta t)) = \hat{Z}_{t_f,t_i}(\mathcal{O}(t))$$

<u>Define</u> $$\hat{\mathcal{O}} := \lim_{\substack{t_f \searrow t \\ t_i \nearrow t}} \hat{Z}_{t_f,t_i}(\mathcal{O}(t)) \quad \cdots \text{indep of } t.$$

Then,

$$\hat{Z}_{t_f,t_i}(\mathcal{O}(t)) = \hat{Z}_{t_f-t} \circ \hat{\mathcal{O}} \circ \hat{Z}_{t-t_i} = e^{-\frac{i}{\hbar}(t_f-t)\hat{H}} \circ \hat{\mathcal{O}} \circ e^{-\frac{i}{\hbar}(t-t_i)\hat{H}}$$

For $t_i < t_1, t_2 < t_f$ and local observables $\mathcal{O}_1$ & $\mathcal{O}_2$,

$$Z(t_f, q_f; \mathcal{O}_1(t_1)\mathcal{O}_2(t_2); t_i, q_i) = \int_{q(t_f)=q_f,\ q(t_i)=q_i} \mathcal{D}q \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(q,\dot{q})} \mathcal{O}_1(t_1)\mathcal{O}_2(t_2).$$

$\rightsquigarrow$ Operator $\hat{Z}_{t_f, t_i}(\mathcal{O}_1(t_1)\mathcal{O}_2(t_2))$ on $\mathcal{H}$

is defined in the similar way.

$$Z(t_f, q_f; \mathcal{O}_1(t_1)\mathcal{O}_2(t_2); t_i, q_i)$$

$$= \begin{cases} \int dq' \, Z(t_f, q_f; \mathcal{O}_2(t_2); t', q')\, Z(t', q'; \mathcal{O}_1(t_1); t_i, q_i) \\ \quad \text{if} \quad t_2 > t_1 \quad (\text{for any } t' \in (t_1, t_2)) \\ \int dq' \, Z(t_f, q_f; \mathcal{O}_1(t_1); t', q')\, Z(t', q'; \mathcal{O}_2(t_2); t_i, q_i) \\ \quad \text{if} \quad t_1 > t_2 \quad (\text{for any } t' \in (t_2, t_1)) \end{cases}$$

$$\hat{Z}_{t_f, t_i}(\mathcal{O}_1(t_1)\mathcal{O}_2(t_2))$$

$$= \begin{cases} e^{-\frac{i}{\hbar}(t_f - t_2)\hat{H}} \hat{\mathcal{O}}_2 \, e^{-\frac{i}{\hbar}(t_2 - t_1)\hat{H}} \hat{\mathcal{O}}_1 \, e^{-\frac{i}{\hbar}(t_1 - t_i)\hat{H}} & \text{if } t_2 > t_1 , \\ e^{-\frac{i}{\hbar}(t_f - t_1)\hat{H}} \hat{\mathcal{O}}_1 \, e^{-\frac{i}{\hbar}(t_1 - t_2)\hat{H}} \hat{\mathcal{O}}_2 \, e^{-\frac{i}{\hbar}(t_2 - t_i)\hat{H}} & \text{if } t_1 > t_2 . \end{cases}$$

Correlation function of product of local observables corresponds to the time ordered product of the corresponding operators.

Symmetry in classical mechanics (in Lagrangian)

Suppose $\exists$ a symmetry $q \mapsto q + \delta q$ $\left(\delta q = \epsilon\, u(q, \dot{q}) \right)$

$$\delta L(q, \dot{q}) = \epsilon \frac{d}{dt}\left(\cdots \right) \quad \text{total derivative}$$

Allow variational parameter ϵ to depend on time, $\epsilon(t)$,

s.t. $\epsilon(t_f) = \epsilon(t_i) = 0$:

$$\delta S = \delta \int_{t_i}^{t_f} dt\, L(q, \dot{q}) = \int_{t_i}^{t_f} dt\, \dot{\epsilon}(t)\, Q(q, \dot{q})$$

This $Q = Q(q, \dot{q})$ is called the <u>Noether charge</u>.

<u>Noether's theorem</u> Q is conserved. I.e. it is time-independent for a solution to equation of motion.

<u>proof</u> A solution is s.t. $\delta S = 0$ for $\forall \delta q$ s.t. $\delta q|_{t_f, t_i} = 0$.

For $\forall \epsilon(t)$ s.t. $\epsilon(t_f) = \epsilon(t_i) = 0$, under $q \to q + \epsilon(t)\, u(q, \dot{q})$,

$$0 = \delta S = \int_{t_i}^{t_f} dt\, \dot{\epsilon}(t)\, Q = -\int_{t_i}^{t_f} dt\, \epsilon(t) \frac{dQ}{dt}.$$

$$\therefore \quad \frac{dQ}{dt} = 0 \qquad \underline{\text{Q.E.D.}}$$

Example $L = \frac{m}{2}\dot{q}^2$: a free particle without potential

$\delta q = \epsilon$: translation in q

$$\delta S = \int_{t_i}^{t_f} dt \, \frac{m}{2} 2\dot{q}\dot{\epsilon} = \int_{t_i}^{t_f} dt \, \dot{\epsilon} \, m\dot{q}$$

$\therefore Q = m\dot{q}$: momentum.

Example $L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(q_1^2 + q_2^2)$

$$g_\alpha : \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$ rotational symmetry

Infinitesimal version:

$$\delta q_1 = -\epsilon q_2, \quad \delta q_2 = \epsilon q_1$$

$$\delta S = \int_{t_i}^{t_f} dt \, \frac{m}{2}\left(2\dot{q}_1(-\dot{\epsilon} q_2) + 2\dot{q}_2(\dot{\epsilon} q_1)\right)$$

$$= \int_{t_i}^{t_f} dt \, \dot{\epsilon} \, m(q_1\dot{q}_2 - q_2\dot{q}_1)$$

$\therefore Q = mq_1\dot{q}_2 - mq_2\dot{q}_1$: angular momentum.

<u>Example</u> $L(q, \dot q)$ general (no explicit t-dependence).

$\delta q = \epsilon \dot q$: time translation.

$$\delta S = \int_{t_i}^{t_f} dt \left(\epsilon \dot q \frac{\partial L}{\partial q} + \underbrace{\frac{d}{dt}(\epsilon \dot q)}_{\epsilon \ddot q + \dot\epsilon \dot q} \frac{\partial L}{\partial \dot q} \right)$$

$$\epsilon \frac{d}{dt} L + \dot\epsilon \dot q \frac{\partial L}{\partial \dot q}$$

$$= \int_{t_i}^{t_f} dt\, \dot\epsilon \left(\dot q \frac{\partial L}{\partial \dot q} - L \right) dt$$

$$\therefore\ Q = \dot q \frac{\partial L}{\partial \dot q}(q, \dot q) - L(q, \dot q) =: E(q, \dot q) \quad \text{energy}$$

c.f. If we solve $\frac{\partial L}{\partial \dot q}(q, \dot q) \overset{!}{=} p$ for $\dot q$

and plug the solution $\dot q = \dot q(p, q)$, then

$$E(q, \dot q(p,q)) = \dot q(p,q)\, p - L(q, \dot q(p,q))$$

$$= H(p,q) \quad \text{Hamiltonian}$$

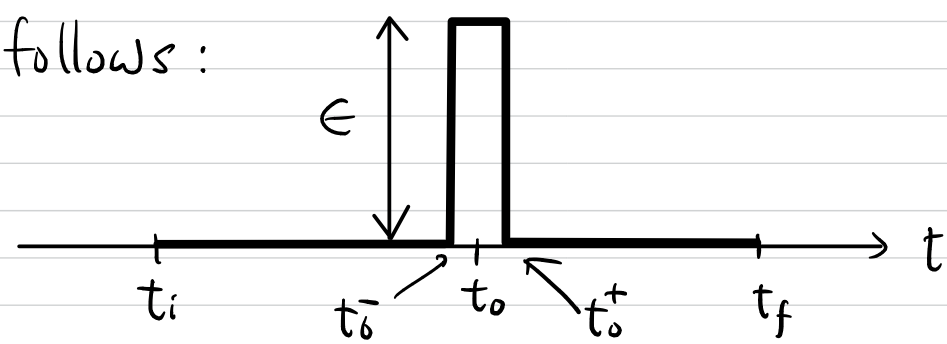
Symmetry in quantum mechanics

Suppose $\exists$ a symmetry $\delta q = \epsilon\, \mathcal{U}(q,\dot q)$ of the classical system & it is also a symmetry of the path-integral measure $\mathcal{D}q$.

Apply $\delta q = \epsilon(t)\, \mathcal{U}(q,\dot q)$ in the integrand of

$$Z(t_f, q_f; \mathcal{O}(t_0); t_i, q_i) = \int_{q(t_f)=q_f,\ q(t_i)=q_i} \mathcal{D}q\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\, L(q,\dot q)}\ \mathcal{O}(t_0)$$

for $\epsilon(t)$ as follows:



Note: $\dot\epsilon(t) = \epsilon\,\delta(t-t_0^-) - \epsilon\,\delta(t-t_0^+)$

Ward id

$$0 \overset{\downarrow}{=} \int \delta\left(\mathcal{D}q\ e^{\frac{i}{\hbar}S[q]}\ \mathcal{O}(t_0)\right)$$

$$= \int \mathcal{D}q\ e^{\frac{i}{\hbar}S[q]} \left(\frac{i}{\hbar}\delta S[q]\, \mathcal{O}(t_0) + \delta\mathcal{O}(t_0)\right)$$

$$0 = \int \mathcal{D}q \, e^{\frac{i}{\hbar}S[q]} \left(\frac{i}{\hbar} \underbrace{\delta S[q]}_{} \mathcal{O}(t_0) + \delta\mathcal{O}(t_0) \right)$$

$$\int_{t_i}^{t_f} dt \, \dot{\epsilon} Q = \epsilon Q(t_0^-) - \epsilon Q(t_0^+)$$

i.e.

$$Z(t_f, q_f : \delta\mathcal{O}(t_0) ; t_i, q_i)$$

$$= Z\left(t_f, q_f ; \left(\frac{i\epsilon}{\hbar} Q(t_0^+) - \frac{i\epsilon}{\hbar} Q(t_0^-) \right) \mathcal{O}(t_0) ; q_i, t_i \right)$$

$$\hat{Z}_{t_f, t_i}(\delta\mathcal{O}(t_0)) = \hat{Z}_{t_f, t_i}\left(\left(\frac{i\epsilon}{\hbar} Q(t_0^+) - \frac{i\epsilon}{\hbar} Q(t_0^-) \right) \mathcal{O}(t_0) \right)$$

Take the limit $t_0^+ \searrow t_0$ and $t_0^- \nearrow t_0$:

$$\widehat{\delta\mathcal{O}} = \frac{i\epsilon}{\hbar} \hat{Q} \circ \hat{\mathcal{O}} - \hat{\mathcal{O}} \circ \frac{i\epsilon}{\hbar} \hat{Q}$$

Put $\epsilon \to 1$:

$$\boxed{\widehat{\delta\mathcal{O}} = \frac{i}{\hbar} [\hat{Q}, \hat{\mathcal{O}}]}$$

Ward identity in quantum mechanics

(in operator formalism)

In classical mechanics, a continuous symmetry yields a conserved charge (Noether charge).

After quantization, the Noether charge generates the symmetry transformation.

The Case of time translation symmetry:

$$\widehat{\frac{d}{dt}\mathcal{O}} = \frac{i}{\hbar}\left[\widehat{H(p,q)}, \hat{\mathcal{O}}\right].$$

On the other hand, using $\hat{H}$ defined by $\hat{Z}_{t_f,t_i} = e^{-\frac{i}{\hbar}(t_f-t_i)\hat{H}}$,

we also know

$$e^{-\frac{i}{\hbar}(t_f-t)\hat{H}}\,\widehat{\frac{d}{dt}\mathcal{O}}\,e^{-\frac{i}{\hbar}(t-t_i)\hat{H}} = \hat{Z}_{t_f,t_i}\left(\frac{d}{dt}\mathcal{O}(t)\right)$$

$$= \frac{d}{dt}\hat{Z}_{t_f,t_i}\left(\mathcal{O}(t)\right) = \frac{d}{dt}\left(e^{-\frac{i}{\hbar}(t_f-t)\hat{H}}\,\hat{\mathcal{O}}\,e^{-\frac{i}{\hbar}(t-t_i)\hat{H}}\right)$$

$$= e^{-\frac{i}{\hbar}(t_f-t)\hat{H}}\,\frac{i}{\hbar}\left[\hat{H}, \hat{\mathcal{O}}\right]e^{-\frac{i}{\hbar}(t-t_i)\hat{H}}$$

$$\therefore\quad \widehat{\frac{d}{dt}\mathcal{O}} = \frac{i}{\hbar}\left[\hat{H}, \hat{\mathcal{O}}\right].$$

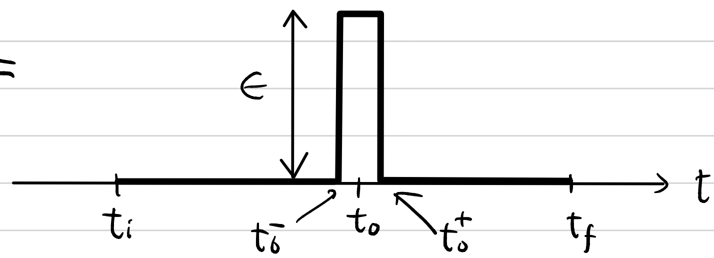
Comparison $\Rightarrow$ $\hat{H} = \widehat{H(p,q)}$ + c-number.

$\hat{H}$ is the operator corresponding to Hamiltonian

(modulo a c-number shift).

The case of q-translation (not a symmetry in general).

Apply $\delta q(t) = \epsilon(t) =$



in the integrand of $Z(t_f, q_f ; q(t_0) ; t_i, q_i)$:

$$0 = \int \delta\left(\mathcal{D}q \; e^{\frac{i}{\hbar} S[q]} \; q(t_0)\right)$$

$$= \int \mathcal{D}q \; e^{\frac{i}{\hbar} S[q]} \left(\frac{i}{\hbar} \underline{\delta S[q]} \; q(t_0) + \epsilon\right)$$

$$\delta S[q] = \int_{t_i}^{t_f} dt \left(\epsilon(t) \frac{\partial L}{\partial q} + \dot{\epsilon}(t) \frac{\partial L}{\partial \dot{q}}\right)$$

Conjugate momentum p

$$= \int_{t_0^-}^{t_0^+} dt \; \epsilon \frac{\partial L}{\partial q} + \epsilon \, p(t_0^-) - \epsilon \, p(t_0^+)$$

$$= \epsilon \int \mathcal{D}q \; e^{\frac{i}{\hbar} S[q]} \left\{ \frac{i}{\hbar} \left(\int_{t_0^-}^{t_0^+} dt \frac{\partial L}{\partial q} + p(t_0^-) - p(t_0^+) \right) q(t_0) + 1 \right\}$$

Take the limit $t_0^+ \searrow t_0$, $t_0^- \nearrow t_0$:

The part $\int_{t_0^-}^{t_0^+} dt \frac{\partial L}{\partial q} q(t_0)$ vanishes in this limit.

$$\therefore \quad 0 = \frac{i}{\hbar}\left(\hat{q} \circ \hat{p} - \hat{p} \circ \hat{q}\right) + 1$$

$$\boxed{\therefore \quad [\hat{q}, \hat{p}] = i\hbar}$$

The canonical commutation relation!

- Remark on terminology

We used "local observable" for $\mathcal{O}(t)$, but it is common to call it "local operator" even inside path-integral.

We have chosen "observable" to emphasize the distinction between path-integral & operator formalisms.

- It is instructive to do path-integrals in explicit examples. Please do it yourself. For your convenience, a note on it is uploaded.

- In a classical field theory in a general dimension, a continuous symmetry yields a conserved current (Noether current). Just like in quantum mechanics one can derive Ward identity involving Noether current. A note on it will be uploaded.