Some Notes on Compact Lie Groups

1 Classical Groups

Examples of compact Lie groups are (the compact form of) the classical groups,

$$\begin{aligned} O(n) &= \left\{ g \in \mathrm{M}_n(\mathbf{R}) \,\middle|\, g^T g = 1_n \right\}, \\ U(n) &= \left\{ g \in \mathrm{M}_n(\mathbf{C}) \,\middle|\, g^\dagger g = 1_n \right\}, \\ Sp(n) &= \left\{ g \in \mathrm{M}_n(\mathbf{H}) \,\middle|\, g^\dagger g = 1_n \right\}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$, where $\mathrm{M}_n(k)$ for a field k is the set of $n \times n$ matrices with entries in k. We also have $SO(n) \subset O(n)$ and $SU(n) \subset U(n)$, the subgroups consisting of matrices of determinant 1.

$\mathbf{H}$ is the set of quaternions, i.e. the associative algebra over $\mathbf{R}$ with unit generated by i, j, k obeying relations $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$ (hence $jk = -kj = i$ and $ki = -ik = j$). We define the conjugation $x \mapsto \overline{x}$ in $\mathbf{H}$ by $x_1 + ix_2 + jx_3 + kx_4 \mapsto x_1 - ix_2 - jx_3 - kx_4$. We have $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$ and $|x|^2 = x\overline{x} = \overline{x}x = \sum_{i=1}^4 (x_i)^2$. In particular, a non-zero element x has the inverse $\overline{x}/|x|^2$. Thus $\mathbf{H}$ is a field. It has $\mathbf{R} = \{x_1\}$ and $\mathbf{C} = \{x_1 + ix_2\}$ as subfields. Note that $\mathbf{H}^n$ can be regarded as a complex vector space, where the scalar multiplication is the multiplication from the right, $(c, x) \in \mathbf{C} \times \mathbf{H}^n \mapsto xc \in \mathbf{H}^n$, and it is isomorphic to $\mathbf{C}^n \oplus \mathbf{C}^n = \mathbf{C}^{2n}$ ($z + jw \in \mathbf{H}^n \leftrightarrow (z, w) \in \mathbf{C}^n \oplus \mathbf{C}^n$). By this, an element of $\mathrm{M}_n(\mathbf{H})$ can be regarded as an element of $\mathrm{M}_{2n}(\mathbf{C})$. For a quaternion matrix A, we define $A^\dagger := \overline{A}^T$. We have a scalar product in $\mathbf{H}^n$, $(x, y) \in \mathbf{H}^n \times \mathbf{H}^n \mapsto x^\dagger y \in \mathbf{H}$. Via $\mathbf{H}^n \cong \mathbf{C}^{2n}$ and $\mathbf{H} \cong \mathbf{C}^2$ given above, it defines a hermitian inner product and a skew-symmetric bilinear form on $\mathbf{C}^{2n}$. The latter is non-degenerate, i.e., a symplectic form. This gives an alternative definition for the group $Sp(n)$:

$$Sp(n) \cong USp(2n) = \left\{ g \in \mathrm{M}_{2n}(\mathbf{C}) \,\middle|\, g^\dagger g = 1_{2n},\, g^T \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} g = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \right\}.$$

$SU(n), SO(n), Sp(n)$ are connected. $SU(n), Sp(n)$ are simply connected but $SO(n)$ has the fundamental group $\mathbf{Z}_2$ (the universal cover is $\mathrm{Spin}(n)$). $SU(n)$ with $n \geq 2$, $SO(n)$ with $n = 3$ and $n \geq 5$, and $Sp(n)$ with $n \geq 1$ are simple.

These groups for low values of n are related. Obviously, $SO(2) \cong U(1)$ and $SU(2) = USp(2) \cong Sp(1)$. Also,

(i) $SO(3) \cong SU(2)/\{\pm 1_2\}$

(ii) $SO(4) \cong (SU(2) \times SU(2))/\{(\pm 1_2, \pm 1_2)\}$

$$\text{(iii)} \qquad SO(5) \cong Sp(2)/\{\pm 1_2\}$$
$$\text{(iv)} \qquad SO(6) \cong SU(4)/\{\pm 1_4\}.$$

(i) is by the adjoint representation of $SU(2)$. (ii) comes from the left and right multiplication of $Sp(1)$ elements on quaternions. (iii) comes from the action of $USp(4)$ on $\wedge^2 \mathbf{C}^4$ with an invariant orthogonal structure inherited from the symplectic form on $\mathbf{C}^4$. Note that $\wedge^2 \mathbf{C}^4$ has a one-dimensional subspace of invariants, and the relevant is its ortho-complement (of dimension $6-1=5$). (iv) comes from the action of $SU(4)$ on $\wedge^2 \mathbf{C}^4$ with an invariant orthogonal structure given by a choice of an element of $(\wedge^4 \mathbf{C}^4)^*$.

2 Maximal Torus and Rank

A subgroup T of a compact Lie group G is called a **maximal torus** if it is a maximal Abelian subgroup. A maximal torus always exists, is a torus (i.e. a product of $U(1)$'s), and is unique up to conjugation (i.e., if T_1 and T_2 are maximal tori, there is an element g of G such that $T_2 = gT_1g^{-1}$). For example,

$$G = SU(n):\ T = \left\{ \begin{pmatrix} \mathrm{e}^{i\theta_1} & & \\ & \ddots & \\ & & \mathrm{e}^{i\theta_n} \end{pmatrix} \middle|\ \mathrm{e}^{i\theta_1}\cdots \mathrm{e}^{i\theta_n} = 1 \right\},$$

$$G = SO(2m):\ T = \left\{ \begin{pmatrix} R_{\theta_1} & & \\ & \ddots & \\ & & R_{\theta_m} \end{pmatrix} \right\}, \quad R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$G = SO(2m+1):\ T = \left\{ \begin{pmatrix} R_{\theta_1} & & & \\ & \ddots & & \\ & & R_{\theta_m} & \\ & & & 1 \end{pmatrix} \right\},$$

$$G = Sp(n):\ T = \left\{ \begin{pmatrix} \mathrm{e}^{i\theta_1} & & \\ & \ddots & \\ & & \mathrm{e}^{i\theta_n} \end{pmatrix} \right\}.$$

The **rank** of the group G is the dimension of a maximal torus of G. Namely, a maximal torus of a group of rank r is isomorphic to $U(1)^r = U(1) \times \cdots \times U(1)$ (r times). For example, $U(n)$ has rank n. The ranks for other groups are listed in a Table below.

3 "Tr"

For simple classical groups, we define an invariant bilinear form "Tr(XY)" for the Lie algebra elements X, Y as follows:

$$\begin{aligned} SU(n): &\qquad \mathrm{Tr}(XY) = \mathrm{tr}_{\mathbf{C}^n}(XY), &\qquad n = 2, 3, 4 \ldots \\ SO(n): &\qquad \mathrm{Tr}(XY) = \frac{1}{2}\mathrm{tr}_{\mathbf{R}^n}(XY), &\qquad n = 5, 6, 7, \ldots \\ USp(2n): &\qquad \mathrm{Tr}(XY) = \mathrm{tr}_{\mathbf{C}^{2n}}(XY), &\qquad n = 1, 2, 3, \ldots \end{aligned}$$

where $\mathbf{C}^n$, $\mathbf{R}^n$, $\mathbf{C}^{2n}$ are the defining representations (the "fundamental representations") of the groups $SU(n)$, $SO(n)$, $USp(2n)$ respectively. For $SO(3)$, we use the one for $SU(2)$, which reads $\mathrm{Tr}(XY) = \frac{1}{4}\mathrm{tr}_{\mathbf{R}^3}(XY)$. For the symplectic groups, we may also write

$$Sp(n): \qquad \mathrm{Tr}(XY) = 2\mathrm{tr}_{\mathbf{H}^n}\left(\frac{XY + YX}{2}\right).$$

(Check that the definition of "Tr" is consistent with the relations (iii) and (iv) above.) For a general simple Lie group, we can define the invariant bilinear form "Tr(XY)" by the property that long roots have length squared 2 with respect to the dual metric induced on the root space.

Another way to characterize "Tr" is of topological nature. Let $g: S^3 \to G$ be a smooth map. $g^{-1}\mathrm{d}g$ defines a 1-form on S^3 with values in the Lie algebraof G, and $\mathrm{Tr}(g^{-1}\mathrm{d}g)^3 = \frac{1}{2}\mathrm{Tr}(g^{-1}\mathrm{d}g[g^{-1}\mathrm{d}g, g^{-1}\mathrm{d}g])$ defines a 3-form on S^3. Then,

$$\frac{1}{24\pi^2}\int_{S^3}\mathrm{Tr}(g^{-1}\mathrm{d}g)^3 \tag{3.1}$$

is an integer, and any integer is given by this integral for some map $g: S^3 \to G$. This will be explained below. For now, do

The main exercise: Show that the integral is 1 for $g: S^3 \to SU(2)$ given by

$$g(x) = \begin{pmatrix} x_4 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_4 - ix_3 \end{pmatrix}, \qquad \sum_{i=1}^{4}(x_i)^2 = 1.$$

4 The Third Homotopy Group

An important fact is that for a compact simple Lie group G,

$$\pi_3(G) \cong \mathbf{Z}, \tag{4.1}$$

and the isomorphism is provided by (3.1).

A proof goes as follows. For any root α, we can find a subalgebra of the Lie algebra of G which is isomorphic to the Lie algebra of $SU(2)$, and we can construct an $SU(2)$ or $SO(3)$ subgroup of G. This defines a map $\varphi_\alpha : S^3 \to G$. The main point is that if α is a long root, φ_α induces an isomorphism at the level of the third homotopy groups.

For classical groups, this can be shown "directly" as follows. The main tool is the homotopy exact sequence (for a Lie group G and its subgroup H)

$$\cdots \to \pi_4(G/H) \to \pi_3(H) \to \pi_3(G) \to \pi_3(G/H) \to \pi_2(H) \to \cdots$$

Let

$$\begin{aligned} SU(n) &\hookrightarrow SU(n+1), \\ SO(n) &\hookrightarrow SO(n+1), \\ Sp(n) &\hookrightarrow Sp(n+1), \end{aligned}$$

be the embeddings associated with $k^n \hookrightarrow k^{n+1}$ for $k = \mathbf{C}, \mathbf{R}, \mathbf{H}$. Then, we have

$$\begin{aligned} SU(n+1)/SU(n) &\cong S^{2n+1}, \\ SO(n+1)/SO(n) &\cong S^n, \\ Sp(n+1)/Sp(n) &\cong S^{4n+3}. \end{aligned}$$

Note that $\pi_4(S^i)$ and $\pi_3(S^i)$ are trivial if $i = 5, 6, 7, \ldots$. Then, using the homotopy exact sequence, we find

$$\begin{aligned} &\pi_3(SU(2)) \cong \pi_3(SU(3)) \cong \pi_3(SU(4)) \cong \cdots, \\ &\pi_3(SO(5)) \cong \pi_3(SO(6)) \cong \pi_3(SO(7)) \cong \cdots, \\ &\pi_3(Sp(1)) \cong \pi_3(Sp(2)) \cong \pi_3(Sp(3)) \cong \cdots. \end{aligned}$$

In particular, the embeddings $SU(2) \hookrightarrow SU(n)$ and $Sp(1) \hookrightarrow Sp(n)$ induce isomorphisms at the level of π_3. For $SO(n)$, we use $\pi_3(Sp(2)) \cong \pi_3(SO(5))$ that follows from the relation (iii) and the homotopy exact sequence (for $G = Sp(2)$, $H = \{\pm 1_2\}$). By this we find a map $Sp(1) \to SO(n)$ for $n = 5, 6, 7, \ldots$ that induces an isomorphism at the level of π_3.

What we have seen explains, given **the main exercise**, the fact that the integral (3.1) for any map $g : S^3 \to G$ is an integer and any integer is realized by the integral (3.1) for some map $g : S^3 \to G$ (at least for simple classical groups).

5 The Dual Coxeter Number

The trace in any representation R of a Lie group G gives an invariant bilinear form on the Lie algebra. If G is simple, all invariant bilinear forms are proportional to one another. We define the number T_R by the proportionality to "Tr",

$$\mathrm{tr}_R(XY) = 2T_R\,\mathrm{Tr}(XY). \tag{5.1}$$

For example, $T_R = \frac{1}{2}, 1, \frac{1}{2}$ for the defining representations $R = \mathbf{C}^n, \mathbf{R}^n, \mathbf{C}^{2n}$ of $SU(n)$, $SO(n)$, $USp(2n)$ respectively. The **dual Coxeter number** $h^\vee$ of G is defined by $T_{\mathbf{adj}}$ for the adjoint representation

$$\mathrm{tr}_{\mathbf{adj}}(XY) = 2h^\vee\,\mathrm{Tr}(XY). \tag{5.2}$$

In Table below, we list the dimension, the rank and the dual Coxeter number of simple Lie groups.

G	dimension	rank	$h^\vee$
$SU(n)$	n^2-1	$n-1$	n
$SO(2m+1)$	$(2m+1)m$	m	$2m-1$
$Sp(n)$	$n(2n+1)$	n	$n+1$
$SO(2m)$	$m(2m-1)$	m	$2m-2$
E_6	78	6	12
E_7	133	7	18
E_8	248	8	30
F_4	52	4	9
G_2	14	2	4