Perturbative expansion is an asymptotic expansion

We consider the partition finction of a 0-d theory

$$Z(\lambda) = \int_{\mathbf{R}} \mathrm{d}\phi \; \mathrm{e}^{-\frac{1}{2}a\phi^2 - \frac{\lambda}{4!}\phi^4}, \tag{1}$$

and its perturbative expansion

$$Z_{\mathrm{pert}}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!} \right)^n \int_{\mathbf{R}} \mathrm{d}\phi \; \mathrm{e}^{-\frac{1}{2}a\phi^2} \, \phi^{4n}. \tag{2}$$

The latter makes sense at least as a formal power series. We show that $Z_{\mathrm{pert}}(\lambda)$ is an asymptotic expansion of $Z(\lambda)$. That is, for each $N \in \mathbf{Z}_{\geq 1}$, if we define the order N truncation of $Z_{\mathrm{pert}}(\lambda)$ by

$$Z_{\mathrm{pert}}^{\leq N}(\lambda) := \sum_{n=0}^{N} \frac{1}{n!} \left(-\frac{\lambda}{4!} \right)^n \int_{\mathbf{R}} \mathrm{d}\phi \; \mathrm{e}^{-\frac{1}{2}a\phi^2} \, \phi^{4n}, \tag{3}$$

then

$$\lambda^{-N} \left(Z(\lambda) - Z_{\mathrm{pert}}^{\leq N}(\lambda) \right) \longrightarrow 0 \quad \text{as } \lambda \searrow 0. \tag{4}$$

Too see this, we note that

$$Z(\lambda) - Z_{\mathrm{pert}}^{\leq N}(\lambda) = \int_{\mathbf{R}} \mathrm{d}\phi \; \mathrm{e}^{-\frac{1}{2}a\phi^2} \underbrace{\left(\mathrm{e}^{-\frac{\lambda}{4!}\phi^4} - \sum_{n=0}^{N} \frac{1}{n!} \left(-\frac{\lambda}{4!}\phi^4 \right)^n \right)}_{:= f_N(\phi,\lambda)}. \tag{5}$$

Claim. There exists $C_N > 0$ such that

$$\left| \mathrm{e}^{-x} - \sum_{n=0}^{N} \frac{1}{n!}(-x)^n \right| \leq C_N |x|^{N+1} \quad \text{for all } x \in \mathbf{R}. \tag{6}$$

Proof: We separate the analysis to the case $|x| \leq 1$ and the case $|x| \geq 1$. When $|x| \leq 1$, using

$$\mathrm{e}^{-x} - \sum_{n=0}^{N} \frac{1}{n!}(-x)^n = \sum_{n=N+1}^{\infty} \frac{1}{n!}(-x)^n = (-x)^{N+1} \sum_{m=0}^{\infty} \frac{(-x)^m}{(N+1+m)!}, \tag{7}$$

we have

$$\begin{aligned} \left| \mathrm{e}^{-x} - \sum_{n=0}^{N} \frac{1}{n!}(-x)^n \right| &\leq |x|^{N+1} \sum_{m=0}^{\infty} \frac{|x|^m}{(N+1+m)!} \\ &\leq |x|^{N+1} \sum_{m=0}^{\infty} \frac{|x|^m}{m!} = |x|^{N+1} \, \mathrm{e}^{|x|} \leq |x|^{N+1} \, \mathrm{e}^1. \end{aligned} \tag{8}$$

When $|x| \geq 1$, we have $|\,\mathrm{e}^{-x}| \leq 1 \leq |x|^{N+1}$ and $\frac{1}{n!}|x|^n \leq |x|^{N+1}$, and hence

$$\begin{aligned} \left| \mathrm{e}^{-x} - \sum_{n=0}^{N} \frac{1}{n!}(-x)^n \right| &\leq |\,\mathrm{e}^{-x}| + \sum_{n=0}^{N} \frac{1}{n!}|x|^n \\ &\leq (N+2)|x|^{N+1}. \end{aligned} \tag{9}$$

Putting $C_N = N + 2$ which is greater than e^1, we see that (6) holds. **Q.E.D.**

The Claim says

$$|f_N(\phi, \lambda)| \leq C_N \left| \frac{\lambda \phi^4}{4!} \right|^{N+1}, \tag{10}$$

and hence

$$\begin{aligned} \left|\, Z(\lambda) - Z_{\mathrm{pert}}^{\leq N}(\lambda) \,\right| &= \left| \int_{\mathbf{R}} \mathrm{d}\phi \ \mathrm{e}^{-\frac{1}{2}a\phi^2} f_N(\phi, \lambda) \right| \\ &\leq \int_{\mathbf{R}} \mathrm{d}\phi \ \mathrm{e}^{-\frac{1}{2}a\phi^2} \, |f_N(\phi, \lambda)| \\ &\leq \int_{\mathbf{R}} \mathrm{d}\phi \ \mathrm{e}^{-\frac{1}{2}a\phi^2} \, C_N \left| \frac{\lambda \phi^4}{4!} \right|^{N+1} \\ &= C_N \left(\frac{\lambda}{4!} \right)^{N+1} \int_{\mathbf{R}} \mathrm{d}\phi \ \mathrm{e}^{-\frac{1}{2}a\phi^2} \phi^{4(N+1)} = C_N' \lambda^{N+1}, \end{aligned} \tag{11}$$

which means

$$\lambda^{-N} \left|\, Z(\lambda) - Z_{\mathrm{pert}}^{\leq N}(\lambda) \,\right| \leq C_N' \lambda. \tag{12}$$

This proves (4).