Free field theories

A theory is said to be <u>free</u> when the action is quadratic in variables.

e.g. n real variables $\phi = (\phi_1, \cdots, \phi_n)$

$$S_E(\phi) = \frac{1}{2}\sum_{i,j=1}^{n} \phi_i A_{ij} \phi_j \qquad A_{ij} = A_{ji} \text{ symmetric,}$$

positive eigenvalues

$$d^n\phi = d\phi_1 \cdots d\phi_n$$

$$Z = \int d^n\phi \, e^{-S_E(\phi)} = \sqrt{\frac{(2\pi)^n}{\det A}}$$

$$\langle \phi_{i_1} \cdots \phi_{i_s} \rangle = \frac{1}{Z}\int d^n\phi \, e^{-S_E(\phi)} \phi_{i_1} \cdots \phi_{i_s} = ?$$

A trick :

$$f(A, J) := \int d^n\phi \, e^{-S_E(\phi) + \sum_{i=1}^{n} J_i \phi_i}$$

$$\frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_s}} f(A, J) = \int d^n\phi \, e^{-S_E(\phi) + \sum_i J_i \phi_i} \phi_{i_1} \cdots \phi_{i_s}$$

$$\xrightarrow{J \to 0} \quad Z \langle \phi_{i_1} \cdots \phi_{i_s} \rangle$$

But $f(A,J)$ can be computed as

$$f(A,J) = \int d^n\phi \; e^{-\frac{1}{2}(\phi - A^{-1}J)\cdot A(\phi - A^{-1}J) + \frac{1}{2} J\cdot A^{-1}J}$$

$$= \sqrt{\frac{(2\pi)^n}{\det A}} \; e^{\frac{1}{2} J\cdot A^{-1} J} = Z \cdot e^{\frac{1}{2} J\cdot A^{-1}J}$$

$$\therefore \langle \phi_{i_1} \cdots \phi_{i_s} \rangle = \frac{1}{Z} \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_s}} f(A,J) \Big|_{J=0}$$

$$= \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_s}} e^{\frac{1}{2} J\cdot A^{-1}J} \Big|_{J=0}.$$

→ Terms where $\frac{\partial}{\partial J}$ hits only one of the two J's in $\frac{1}{2} J A^{-1} J$ vanish as $J=0$. Terms that survive are those where both J's in $\frac{1}{2} J A^{-1} J$ are hit by $\frac{\partial}{\partial J}$'s. Thus, the result is the sum of terms where the derivatives $\frac{\partial}{\partial J_{i_1}}, \cdots, \frac{\partial}{\partial J_{i_s}}$ form pairs, which is possible only when s is even, each pair $\left\{ \frac{\partial}{\partial J_{i_a}}, \frac{\partial}{\partial J_{i_b}} \right\}$ producing

$$\frac{\partial}{\partial J_{i_a}}\frac{\partial}{\partial J_{i_b}}\left(\frac{1}{2}J\cdot A^{-1}J\right) = A^{-1}_{i_a i_b}.$$ It is the sum of pairwise contractions, called <u>Wick contractions</u>:

$$\langle \phi_i \rangle = 0,$$

$$\langle \phi_i \phi_j \rangle = \overbracket{\phi_i \phi_j} = A^{-1}_{ij},$$

$$\langle \phi_i \phi_j \phi_k \rangle = 0,$$

$$\langle \phi_i \phi_j \phi_k \phi_l \rangle = \overbracket{\phi_i \phi_j}\,\overbracket{\phi_k \phi_l} + \phi_i \phi_j \phi_k \phi_l + \phi_i \phi_j \phi_k \phi_l$$

$$= A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk},$$

$$\vdots$$

- We see that everything is determined by the two point function

$$\langle \phi_i \phi_j \rangle = \overbracket{\phi_i \phi_j} = A^{-1}_{ij}.$$

- The logic holds also when $n = \infty$, i.e. in QFT in dimension $d \geq 1$.

Complex scalar

A finite system:

Variables : $\varphi = (\varphi^1, \dots, \varphi^n) \in \mathbb{C}^n$

notation $\bar{\varphi} = (\bar{\varphi}_1, \dots, \bar{\varphi}_n) := (\varphi^{1*}, \dots, \varphi^{n*})$

$$S_E(\varphi) = \sum_{i,j=1}^{n} \bar{\varphi}_i A^i_j \varphi^j =: \bar{\varphi} A \varphi$$

A hermitian, positive eigenvalues

$$d\bar{\varphi}\, d\varphi = d\bar{\varphi}_1 d\varphi^1 \cdots d\bar{\varphi}_n d\varphi^n = d\bar{\varphi}_n \cdots d\bar{\varphi}_1\, d\varphi^1 \cdots d\varphi^n$$

$$Z = \int d\bar{\varphi}\, d\varphi\, e^{-S_E(\varphi)} = \frac{(2\pi i)^n}{\det A}$$

$$\langle \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_t} \rangle = ?$$

$$f(A, \bar{J}, J) := \int d\bar{\varphi}\, d\varphi\, e^{-S_E(\varphi) + \sum_i (\bar{J}_i \varphi^i + \bar{\varphi}_i J^i)}$$

$$\frac{\partial}{\partial \bar{J}_{i_1}} \cdots \frac{\partial}{\partial \bar{J}_{i_s}} \frac{\partial}{\partial J^{j_1}} \cdots \frac{\partial}{\partial J^{j_t}} f(A, J)$$

$$= \int d\bar{\varphi}\, d\varphi\, e^{-S_E(\varphi) + \bar{J}\varphi + \bar{\varphi} J}\, \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_t}$$

$$\xrightarrow{\bar{J}, J \to 0} Z \langle \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_t} \rangle$$

$f(A, \bar{J}, J)$ can be computed:

$$= \int d\bar{\varphi}\, d\varphi\; e^{-(\bar{\varphi} - \bar{J}A^{-1})A(\varphi - A^{-1}J) + \bar{J}A^{-1}J} = Z\, e^{\bar{J}A^{-1}J}$$

$$\therefore \langle \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_t} \rangle = \underbrace{\frac{\partial}{\partial \bar{J}_{i_1}} \cdots \frac{\partial}{\partial \bar{J}_{i_s}}}_{\text{do this first}} \frac{\partial}{\partial J^{j_1}} \cdots \frac{\partial}{\partial J^{j_t}}\, e^{\bar{J}A^{-1}J} \Bigg|_{J, \bar{J} \to 0}$$

$$= \frac{\partial}{\partial J^{j_1}} \cdots \frac{\partial}{\partial J^{j_t}} (A^{-1}J)^{i_1} \cdots (A^{-1}J)^{i_s}\, e^{\bar{J}A^{-1}J} \Bigg|_{J, \bar{J} \to 0}$$

$$= \frac{\partial}{\partial J^{j_1}} \cdots \frac{\partial}{\partial J^{j_t}} (A^{-1}J)^{i_1} \cdots (A^{-1}J)^{i_s} \Bigg|_{J \to 0}$$

- When $s \neq t$, this vanishes.
- When $s = t$, this is the sum of $i_a - j_b$ parings $1 \leq a, b \leq s$.

e.g.

$$\langle \varphi^i \bar{\varphi}_j \rangle = \varphi^i \bar{\varphi}_j = {A^{-1}}^i{}_j$$

$$\langle \varphi^i \varphi^j \bar{\varphi}_k \bar{\varphi}_l \rangle = \varphi^i \varphi^j \bar{\varphi}_k \bar{\varphi}_l + \varphi^i \varphi^j \bar{\varphi}_k \bar{\varphi}_l$$

$$= {A^{-1}}^i{}_k {A^{-1}}^j{}_l + {A^{-1}}^i{}_l {A^{-1}}^j{}_k$$

$\vdots$

$$\langle \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_s} \rangle = \varphi^{i_1} \cdots \varphi^{i_s} \bar{\varphi}_{j_1} \cdots \bar{\varphi}_{j_s} + \cdots$$

$$= \sum_{\sigma \in \mathfrak{S}_s} {A^{-1}}^{i_1}{}_{j_{\sigma(1)}} \cdots {A^{-1}}^{i_s}{}_{j_{\sigma(s)}}.$$

Free fermions

A finite system: n pairs of anticommuting variables

$$\psi_1, \bar{\psi}^1, \psi_2, \bar{\psi}^2, \dots, \psi_n, \bar{\psi}^n$$

$$S_E = \sum_{i,j} \bar{\psi}^i A_i{}^j \psi_j$$

$$d\bar{\psi}\, d\psi = d\bar{\psi}^n \cdots d\bar{\psi}^1 d\psi_1 \cdots d\psi_n = d\bar{\psi}^1 d\psi_1 \cdots d\bar{\psi}^n d\psi_n$$

Partition function is

$$Z = \int d\bar{\psi}\, d\psi\; e^{-S_E} = \det A$$

To compute correlation functions, let us introduce

$$f(A, \bar{\eta}, \eta) := \int d\bar{\psi}\, d\psi\; e^{-S_E + \sum_i (\bar{\eta}^i \psi_i + \bar{\psi}^i \eta_i)}.$$

Note:

$$\frac{\partial}{\partial \bar{\eta}^{i_1}} \frac{\partial}{\partial \bar{\eta}^{i_2}} \cdots \frac{\partial}{\partial \bar{\eta}^{i_s}}\, e^{\bar{\eta}\psi + \bar{\psi}\eta}\, \frac{\overleftarrow{\partial}}{\partial \eta_{j_1}} \frac{\overleftarrow{\partial}}{\partial \eta_{j_2}} \cdots \frac{\overleftarrow{\partial}}{\partial \eta_{j_t}}$$

$$= e^{\bar{\eta}\psi}\, \psi_{i_1} \cdots \psi_{i_s} \bar{\psi}^{j_1} \cdots \bar{\psi}^{j_t}\, e^{\bar{\psi}\eta}$$

Thus,

$$\frac{\partial}{\partial \bar{\eta}^{i_1}} \frac{\partial}{\partial \bar{\eta}^{i_2}} \cdots \frac{\partial}{\partial \bar{\eta}^{i_s}}\, f(A, \bar{\eta}, \eta)\, \frac{\overleftarrow{\partial}}{\partial \eta_{j_1}} \frac{\overleftarrow{\partial}}{\partial \eta_{j_2}} \cdots \frac{\overleftarrow{\partial}}{\partial \eta_{j_t}} \Bigg|_{\bar{\eta} = \eta = 0}$$

$$= Z \langle \psi_{i_1} \cdots \psi_{i_s} \bar{\psi}^{j_1} \cdots \bar{\psi}^{j_t} \rangle$$

$$f(A,\bar\eta,\eta) = \int d\bar\Psi\, d\Psi\, e^{-(\bar\Psi - \bar\eta A^{-1})A(\Psi - A^{-1}\eta) + \bar\eta A^{-1}\eta}$$

$$= Z\, e^{\bar\eta A^{-1}\eta}$$

$$\therefore \langle \Psi_{i_1} \cdots \Psi_{i_s} \bar\Psi^{j_1} \cdots \bar\Psi^{j_t} \rangle$$

$$= \frac{\partial}{\partial\bar\eta^{i_1}} \frac{\partial}{\partial\bar\eta^{i_2}} \cdots \frac{\partial}{\partial\bar\eta^{i_s}}\, e^{\bar\eta A^{-1}\eta}\, \frac{\overleftarrow{\partial}}{\partial\eta_{j_1}} \frac{\overleftarrow{\partial}}{\partial\eta_{j_2}} \cdots \frac{\overleftarrow{\partial}}{\partial\eta_{j_t}} \Bigg|_{\bar\eta=\eta=0}$$

$$= (A^{-1}\eta)_{i_1} \cdots (A^{-1}\eta)_{i_s} \frac{\overleftarrow{\partial}}{\partial\eta_{j_1}} \frac{\overleftarrow{\partial}}{\partial\eta_{j_2}} \cdots \frac{\overleftarrow{\partial}}{\partial\eta_{j_t}} \Bigg|_{\eta=0}$$

This is non-zero only if $s=t$.

e.g. $\langle \Psi_i \bar\Psi^j \rangle = \underbrace{(A^{-1}\eta)_i}_{A^{-1}{}_i{}^k \eta_k} \frac{\overleftarrow{\partial}}{\partial\eta_j} = A^{-1}{}_i{}^j$

$$\langle \Psi_i \Psi_j \bar\Psi^k \bar\Psi^l \rangle = (A^{-1}\eta)_i (A^{-1}\eta)_j \frac{\overleftarrow{\partial}}{\partial\eta_k} \frac{\overleftarrow{\partial}}{\partial\eta_l}$$

$$= A^{-1}{}_i{}^l A^{-1}{}_j{}^k - A^{-1}{}_i{}^k A^{-1}{}_j{}^l$$

$\frac{\overleftarrow{\partial}}{\partial\eta_k}$ passes through η in $(A^{-1}\eta)_j$

The result can also be presented as the sum of Wick contractions, with the understanding that a (-1) is produced each time two fermionic objects are swapped:

$$\langle \psi_i \bar{\psi}^j \rangle = \psi_i \bar{\psi}^j = A^{-1}{}_i{}^j$$

$$\langle \psi_i \psi_j \bar{\psi}^k \bar{\psi}^l \rangle = \underbrace{\psi_i \psi_j \bar{\psi}^k \bar{\psi}^l}_{+} + \underbrace{\psi_i \psi_j \bar{\psi}^k \bar{\psi}^l}_{-}$$

$$= \psi_i \bar{\psi}^l \psi_j \bar{\psi}^k - \psi_i \bar{\psi}^k \psi_j \bar{\psi}^l$$

$$= A^{-1}{}_i{}^l A^{-1}{}_j{}^k - A^{-1}{}_i{}^k A^{-1}{}_j{}^l$$

$\vdots$

- We see that everything is determined by the two point functions $\langle \psi_i \bar{\psi}^j \rangle = \psi_i \bar{\psi}^j = A^{-1}{}_i{}^j$.

- The logic holds also when $n = \infty$, e.g. in QFT in dimension $d \geq 1$.

Unpaired fermions

We also encounter systems of unpaired fermions
(e.g. Majorana fermions)

$\psi_1, \cdots, \psi_{2n}$

with action $S_E = \frac{1}{2} \sum_{i,j} \psi_i A_{ij} \psi_j$ and measure
$d\psi = d\psi_1 \cdots d\psi_{2n}$. By anticommutativity of ψ_i's, we may assume antisymmetry $A_{ij} = -A_{ji}$.

The partition function is

$$Z = \int d\psi \; e^{-S_E} = \frac{1}{n!\,2^n} \epsilon^{i_1 j_1 \cdots i_n j_n} A_{i_1 j_1} \cdots A_{i_n j_n}$$

where $\epsilon^{k_1 \cdots k_{2n}}$ is totally antisymmetric and $\epsilon^{1 \cdots 2n} = 1$.

This is called the Pfaffian of the antisymmetric matrix $A = (A_{ij})$ and is denoted by $\mathrm{Pf}\, A$.

$$\therefore \; Z = \mathrm{Pf}\, A$$

It has the property $(\mathrm{Pf}\, A)^2 = \det A$.

Thus it is $\sqrt{\det A}$ with a specific sign.

For computation of correlation functions, we introduce

$$f(A,\eta) = \int d\psi \; e^{-S_E + \sum_i \eta_i \psi_i}$$

for anticommuting $\eta_1, \cdots, \eta_{2n}$.

Since
$$\frac{\partial}{\partial \eta_{i_1}} \cdots \frac{\partial}{\partial \eta_{i_s}} e^{\sum_i \eta_i \psi_i} = e^{\sum_i \eta_i \psi_i} \psi_{i_1} \cdots \psi_{i_s},$$

$$\frac{\partial}{\partial \eta_{i_1}} \cdots \frac{\partial}{\partial \eta_{i_s}} f(A,\eta)\Big|_{\eta=0} = Z \langle \psi_{i_1} \cdots \psi_{i_s} \rangle$$

On the other hand

$$f(A,\eta) = \int d\psi \; e^{\underbrace{-\frac{1}{2}\sum_{i,j} \psi_i A_{ij} \psi_j + \sum_i \eta_i \psi_i}}$$

$$= -\frac{1}{2}\sum_{i,j} \left(\psi_i - \sum_k A^{-1}_{ik}\eta_k\right) A_{ij} \left(\psi_j - \sum_l A^{-1}_{jl}\eta_l\right)$$

$$+ \frac{1}{2} \underbrace{\sum_{i,j,k,l} A^{-1}_{ik}\eta_k A_{ij} A^{-1}_{jl} \eta_l}_{\sum_{k,l} \eta_k A^{-1}_{lk} \eta_l = -\sum_{k,l} \eta_k A^{-1}_{kl} \eta_l}$$

$$= \int d\psi \; e^{-\frac{1}{2}\sum_{i,j} (\psi_i + \cdots) A_{ij} (\psi_j + \cdots) - \frac{1}{2}\sum_{k,l} \eta_k A^{-1}_{kl} \eta_l}$$

$$= \mathrm{Pf} A \cdot e^{-\frac{1}{2}\eta^T A^{-1} \eta}$$

$$\therefore \langle \psi_{i_1} \cdots \psi_{i_s} \rangle = \underbrace{\frac{\partial}{\partial \eta_{i_1}} \cdots \frac{\partial}{\partial \eta_{i_s}} e^{-\frac{1}{2}\eta^T A^{-1} \eta}}\Big|_{\eta=0}$$

Here we proceed just as in the real boson case, though anticommutativity taken into account :

→ Terms where $\frac{\partial}{\partial \eta}$ hits only one of the two η's in $-\frac{1}{2}\eta^T A^{-1}\eta$ vanish as $\eta = 0$. Terms that survive are those where both η's in $-\frac{1}{2}\eta^T A^{-1}\eta$ are hit by $\frac{\partial}{\partial \eta}$'s.

Thus, the result is the sum of terms where the derivatives $\frac{\partial}{\partial \eta_{i_1}}, \cdots, \frac{\partial}{\partial \eta_{i_s}}$ form pairs, which is possible only when s is even, each pair $\left\{\frac{\partial}{\partial \eta_{i_a}}, \frac{\partial}{\partial \eta_{i_b}}\right\}$ producing

$$\frac{\partial}{\partial \eta_{i_a}} \frac{\partial}{\partial \eta_{i_b}} \left(-\frac{1}{2}\eta^T A^{-1} \eta\right) = -\frac{1}{2} A^{-1}_{i_b i_a} + \frac{1}{2} A^{-1}_{i_a i_b} = A^{-1}_{i_a i_b}$$

It is the sum of such Wick contractions.

But we should keep track of a sign (-1) that appears each time $\frac{\partial}{\partial \eta_i}$ & $\frac{\partial}{\partial \eta_j}$ are swapped.

For example,

$$\langle \psi_i \rangle = 0,$$

$$\langle \psi_i \psi_j \rangle = \wick{\c\psi_i \c\psi_j} = A^{-1}_{ij},$$

$$\langle \psi_i \psi_j \psi_k \rangle = 0,$$

$$\langle \psi_i \psi_j \psi_k \psi_l \rangle = \wick{\c1\psi_i \c1\psi_j \c2\psi_k \c2\psi_l} + \underbrace{\wick{\c1\psi_i \c2\psi_j \c1\psi_k \c2\psi_l}}_{-1} + \underbrace{\wick{\c1\psi_i \c2\psi_j \c2\psi_k \c1\psi_l}}_{+1}$$

$$= A^{-1}_{ij} A^{-1}_{kl} - A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk},$$

$$\vdots$$

The free field theory (either bosonic or fermionic) in dimension ≥ 1 can be considered also in operator formalism.

Comparison of path-integral & operator formalism is done in Lecture 6 and 7 in QFT II along with the additional notes for them.

If you are interested, please have a look.