Convention

3+1 dim Minkowski spacetime $\mathbb{R}^{3+1}$

coordinates $x^{\mu=0,1,2,3}$; $x^0 = t$, $(x^{i=1,2,3}) = \mathbf{x}$

metric $ds^2 = dt^2 - d\mathbf{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ "$(+---)$"

$$\{\text{Isometries}\} = \text{Poincaré group} = \underset{\substack{\uparrow \\ \text{translations}}}{\mathbb{R}^4} \rtimes \underset{\substack{\uparrow \\ \text{Lorentz transf.}}}{O(3,1)}$$

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + \xi^\mu \quad ; \quad \Lambda^\rho{}_\mu \eta_{\rho\lambda} \Lambda^\lambda{}_\nu = \eta_{\mu\nu}$$

(infinitesimally, $\delta x^\mu = \omega^\mu{}_\nu x^\nu + \delta\xi^\mu$; $\omega_{\mu\nu} = -\omega_{\nu\mu}$)

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad \text{Clifford algebra}$$

$S \cong \mathbb{C}^4$ the irreducible representation

e.g. $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ $\quad \sigma^0 = \bar{\sigma}^0 = -\mathbb{1}_2$

$\sigma^i = -\bar{\sigma}^i = \sigma_i \quad i=1,2,3$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{Pauli matrices}$$

Inf. Lorentz $\delta\psi = \frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu\psi = \frac{1}{4}\omega_{\mu\nu}\underbrace{\gamma^{\mu\nu}}_{\frac{1}{2}[\gamma^\mu, \gamma^\nu]}\psi$.

S : representation of $Spin(3,1) \xrightarrow{2:1} SO(3,1)$

$$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \quad ; \quad \gamma_5\gamma^\mu = -\gamma^\mu\gamma_5, \ (\gamma_5)^2 = 1, \quad \text{eigenvalues } \pm 1.$$

$$S_R := \{ \gamma_5 = +1 \text{ spinors} \} \quad \text{"right handed"}$$

$$S_L := \{ \gamma_5 = -1 \text{ spinors} \} \quad \text{"left handed"}$$

$$\left(\text{In the above basis, } \gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \ S_R \ni \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}, \ S_L \ni \begin{pmatrix} 0 \\ 0 \\ * \\ * \end{pmatrix}. \right)$$

$$S = \begin{tikzcd} S_R \arrow[loop above, "\mathrm{Spin}(3,1)"] \arrow[r, phantom, "\oplus"] \arrow[r, bend right, leftrightarrow, "\gamma^\mu"'] & S_L \arrow[loop above] \end{tikzcd}$$

· Scalar product.

$$\psi_1, \psi_2 \in S \rightsquigarrow \bar{\psi}_1 \psi_2 \in \mathbb{C}$$

$$\text{s.t. } \overline{c\psi_1}\,\psi_2 = \bar{c}\,\bar{\psi}_1\psi_2, \quad \bar{\psi}_1 c\psi_2 = c\,\bar{\psi}_1\psi_2, \quad c \in \mathbb{C}$$

$$\overline{\gamma^\mu\psi_1}\,\psi_2 = \bar{\psi}_1\gamma^\mu\psi_2$$

$$\left(\text{In the above basis, } \quad \bar{\psi}_1\psi_2 = \psi_1^\dagger\gamma^0\psi_2. \right)$$

$$\rightsquigarrow \overline{\gamma_5\psi_1}\,\psi_2 = -\bar{\psi}_1\gamma_5\psi_2$$

$$\therefore \bar{\psi}_1\psi_2 = 0 \quad \text{if } \psi_1 \ \& \ \psi_2 \text{ are both in } S_R \text{ or in } S_L.$$

Wick rotation

We obtain a theory on the Euclidean space $\mathbb{R}_E^4$

$$ds_E^2 = \sum_{\mu=1}^{4} (dx^\mu)^2 \qquad \text{"}(++++)\text{"}$$

by $x^0 \to -i x^4$

$$e^{iS} = e^{i\int d^4x\, \mathcal{L}}$$

$$\to e^{i\int(-i\, d^4x_E)\mathcal{L}} =: e^{-\int d^4x_E\, \mathcal{L}_E} =: e^{-S_E}$$

$$\therefore \quad \mathcal{L}_E = -\mathcal{L}\Big|_{x^0 \to -i x^4}$$

$$\gamma^0 \to -i\gamma^4$$

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu} \qquad 1 \le \mu, \nu \le 4 .$$

Yang-Mills theory

· Specified by gauge group G : a compact Lie group

e.g. $U(N) = \{ N\times N$ unitary matrix $\}$ unitary group

$SU(N) = \{ N\times N$ unitary, $\det = 1\}$ special unitary group

$\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G "infinitesimal version of G"

e.g. $G = U(N)$: $\mathfrak{g} = \{ N\times N$ antihermitian matrix $\}$

$G = U(1)$: $\mathfrak{g} = i\mathbb{R} \cong \mathbb{R}$

$G = SU(N)$: $\mathfrak{g} = \{ N\times N$ antihermitian, traceless $\}$

field variable $A_\mu(x)$: a vector potential with values in $\mathfrak{g}$

field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

$$S[A] = \int -\frac{1}{4e^2} F^{\mu\nu}\cdot F_{\mu\nu}\, d^4x \qquad \text{Yang-Mills action}$$

e : gauge coupling constant.

"$\cdot$" is a positive definite inner product on $\mathfrak{g}$ which is invariant under the adjoint action of G, $X \mapsto gXg^{-1}$

(the infinitesimal version of conjugation $g_t \mapsto g g_t g^{-1}$):

$$gXg^{-1} \cdot gYg^{-1} = X \cdot Y.$$

e.g. For $G = SU(N)$, a standard choice is $X \cdot Y = -2 \operatorname{Tr} XY$.

e.g. $G = U(1)$: Maxwell theory

$$F_{0i} = \partial_0 A_i - \partial_i A_0 \quad \text{electric field}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i \quad \text{magnetic field}$$

$S[A]$ is invariant under a huge symmetry group:

$g(x)$: G-valued function on spacetime

$$\rightsquigarrow \quad A_\mu \mapsto A_\mu^g = g^{-1} A_\mu g + g^{-1} \partial_\mu g.$$

Under this, the field strength transforms covariantly,

$$F_{\mu\nu} \mapsto F_{\mu\nu}^g = \partial_\mu A_\nu^g - \partial_\nu A_\mu^g + [A_\mu^g, A_\nu^g] = g^{-1} F_{\mu\nu} g,$$

and thus indeed

$$S[A^g] = \int -\frac{1}{4e^2} g^{-1} F^{\mu\nu} g \cdot g^{-1} F_{\mu\nu} g \, d^4x$$

$$= \int -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} \, d^4x = S[A].$$

This is a generalization of invariance of Maxwell action under the gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu \lambda$.

Indeed, for $G = U(1)$, $\mathfrak{g} = i\mathbb{R} \cong \mathbb{R}$ and for $g(x) = e^{i\lambda(x)}$

$$iA_\mu^\lambda = e^{-i\lambda}\, iA_\mu\, e^{i\lambda} + e^{-\lambda}\partial_\mu e^{i\lambda} \Rightarrow A_\mu^\lambda = A_\mu + \partial_\mu \lambda.$$

As in that case, we shall call

$$A_\mu \longmapsto A_\mu^g = g^{-1} A_\mu g + g^{-1}\partial_\mu g$$

the <u>gauge transformation</u> of $A_\mu(x)$ by $g(x)$, and

$$\mathcal{G} := \{ g(x) \mid G\text{-valued function} \}$$

the <u>gauge transformation group</u>. We'd like to regard A and A^g as physically equivalent for any $g \in \mathcal{G}$.

I.e. we would like to physically identify them. If we put

$$\mathcal{A} := \{ A_\mu(x) \mid \mathfrak{g}\text{-valued vector potential} \}$$

the space of physically inequivalent field configurations is the quotient space $\mathcal{A}/\mathcal{G}$.

A theory with such an identification of field variables is called a gauge theory.

Infinitesimal gauge transformations

A $\mathfrak{g}$-valued function $\epsilon(x)$ generates a one parameter group of gauge transformations: $g_t(x) = e^{t\epsilon(x)}$:

$$A_\mu \mapsto A_\mu^{g_t} = g_t^{-1} A_\mu g + g_t^{-1} \partial_\mu g_t.$$

The infinitesimal transformation is

$$\delta_\epsilon A_\mu = \frac{d}{dt} A_\mu^{g_t} \Big|_{t=0} = -\epsilon A_\mu + A_\mu \epsilon + \partial_\mu \epsilon$$

$$= \partial_\mu \epsilon + [A_\mu, \epsilon] =: D_\mu \epsilon \quad \text{covariant derivative.}$$

The space of such $\epsilon(x)$ may be regarded as the Lie algebra of the gauge transformation group.

$$\{\epsilon(x) \mid \mathfrak{g}\text{-valued function}\} = \mathrm{Lie}(\mathcal{G}).$$

Remark The G-invariant inner product "$\cdot$" on $\mathfrak{g}$ may not be unique.

e.g. $G = SU(N_1) \times SU(N_2)$,

$$\mathcal{L} = -\frac{1}{4e_1^2} F_1^{\mu\nu} \cdot F_{1\mu\nu} - \frac{1}{4e_2^2} F_2^{\mu\nu} \cdot F_{2\mu\nu}$$

e_1 and e_2 can be different.

More generally,

$$G = \underbrace{U(1) \times \cdots \times U(1)}_{k} \times \underbrace{G_1 \times \cdots \times G_\ell}_{\text{"simple" factors}} \Big/ \text{discrete subgroup}$$

$$\mathcal{L} = \sum_{a,b=1}^{k} -\frac{1}{4e_{a,b}^2} F_a^{\mu\nu} F_{b\mu\nu} + \sum_{I=1}^{\ell} -\frac{1}{4e_I^2} F_I^{\mu\nu} \cdot F_{I\mu\nu}$$

$\frac{k(k+1)}{2} + \ell$ gauge coupling constants.

Having this generality in mind, we just write

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu}$$

for simplicity.

Coupling to matter fields

A representation V of a group G is

a vector space$/\mathbb{C}$ or $\mathbb{R}$ on which G acts linearly.

$\exists$ a map $G \times V \to V$; $(g, v) \mapsto gv$

s.t. $\cdot$ $g(hv) = (gh)v$

$\cdot$ $g(cv) = cg(v)$ $\quad c \in \mathbb{C}$ or $\mathbb{R}$

$\cdot$ $g(v+w) = gv + gw$

$\Big\}$ linearity

e.g. $V = \mathbb{C}^N$ for $G = U(N)$ or $SU(N)$ via matrix multiplication.

$V = \mathfrak{g}$ for a general G via adjoint action

$V =$ sum of copies of such, $\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$.

A representation V of a Lie group G

$\rightsquigarrow$ a representation of its Lie algebra $\mathfrak{g}$

$\exists$ a map $\mathfrak{g} \times V \to V$; $(X, v) \mapsto Xv$

s.t. $X(Yv) - Y(Xv) = [X, Y]v$, linearity.

- Scalars

$\phi(x)$: a scalar field with values a representation V of the gauge group G.

Gauge transformation by $g \in \mathcal{G}$:

$$A_\mu \mapsto A_\mu^g , \quad \phi \longmapsto \phi^g = g^{-1}\phi .$$

Infinitesimally, $\delta A_\mu = D_\mu \epsilon$, $\quad \delta\phi = -\epsilon\phi$.

Covariant derivative $D_\mu \phi := \partial_\mu \phi + A_\mu \phi$

Its gauge transformation :

$$D_\mu \phi \mapsto \partial_\mu \phi^g + A_\mu^g \phi^g = \partial_\mu (g^{-1}\phi) + (g^{-1}A_\mu g \cancel{+ g^{-1}\partial_\mu g})\, g^{-1}\phi$$

$$-\cancel{g^{-1}\partial_\mu g}\, g^{-1}\phi + g^{-1}\partial_\mu \phi$$

$$= g^{-1}\partial_\mu \phi + g^{-1}A_\mu \phi = g^{-1} D_\mu \phi$$

"homogeneous" or "covariant".

$(\phi_1, \phi_2) \mapsto \phi_1^\dagger \phi_2$ $\quad$ G-invariant inner product on V

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + (D^\mu \phi)^\dagger D_\mu \phi - f(\phi^\dagger \phi)$$

is gauge invariant.

- Fermions

$\Psi(x)$: a Dirac fermion with values in a rep. V of G.

i.e. an anticommuting function on $\mathbb{R}^{3+1}$

with values in $S \otimes V \cong \mathbb{C}^4 \otimes V$

$$\Psi(x) = \left(\Psi^a_\alpha(x) \right)^{a=1,\cdots,\dim V}_{\alpha=1,2,3,4} \quad \text{in components}$$

Gauge transformation : $A_\mu \mapsto A^g_\mu$, $\Psi \longmapsto g^{-1}\Psi$

$$\not{D}_A \Psi = \gamma^\mu D_\mu \Psi = \gamma^\mu (\partial_\mu \Psi + A_\mu \Psi) \quad \text{Dirac operator}$$

$$(\not{D}_A \Psi)^a_\alpha = \gamma^{\mu}{}_\alpha{}^\beta \left(\partial_\mu \Psi^a_\beta + A_\mu{}^a{}_b \Psi^b_\beta \right) \quad \text{in components}$$

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + i\bar{\Psi} \not{D}_A \Psi - m\bar{\Psi}\Psi$$

is gauge invariant.

e.g. QED with electrons of charge $Q_1, \cdots, Q_{N_f}$:

$$G = U(1), \ e^{i\lambda} : \Psi_i \longmapsto e^{iQ_i\lambda} \Psi_i \quad (i=1,\cdots,N_f)$$

eg. QCD with color N_c and flavor N_f:

$$G = SU(N_c), \ V = \mathbb{C}^{N_c} \oplus \cdots \oplus \mathbb{C}^{N_c} \quad (N_f \text{ copies})$$

- More generally, the representations for right-handed & left-handed fermions can be different:

ψ_R valued in $S_R \otimes V_R$, ψ_L valued in $S_L \otimes V_L$.

Then, $\not{D}_A \psi_R$ valued in $S_L \otimes V_R$, $\not{D}_A \psi_L$ valued in $S_R \otimes V_L$.

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + i \bar{\psi}_R \not{D}_A \psi_R + i \bar{\psi}_L \not{D}_A \psi_L$$

makes sense & is gauge invariant.

Such a theory is called "**chiral**".

- Suppose $\exists$ a <u>G-equivariant</u> bilinear map

$$V_B \times V_R \to V_L \,, \quad (v_B, v_R) \mapsto v_B \cdot v_R$$

$$g v_B \cdot g v_R = g(v_B \cdot v_R).$$

Then, for a V_B-valued scalar ϕ & a $S_{R,L} \otimes V_{R,L}$-valued fermion $\psi_{R,L}$, Yukawa coupling

$$\bar{\psi}_L \phi \cdot \psi_R + \overline{\phi \cdot \psi_R}\, \psi_L \quad \text{makes sense.}$$