Convention

3+1 dim Minkowski spacetime $\mathbb{R}^{3+1}$

coordinates $x^{\mu=0,1,2,3}$; $x^0 = t$, $(x^{i=1,2,3}) = \mathbf{x}$

metric $ds^2 = dt^2 - d\mathbf{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ "$(+---)$"

$$\{\text{Isometries}\} = \text{Poincaré group} = \underset{\substack{\uparrow \\ \text{translations}}}{\mathbb{R}^4} \rtimes \underset{\substack{\uparrow \\ \text{Lorentz transf.}}}{O(3,1)}$$

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + \xi^\mu \quad ; \quad \Lambda^\rho{}_\mu \eta_{\rho\lambda} \Lambda^\lambda{}_\nu = \eta_{\mu\nu}$$

(infinitesimally, $\delta x^\mu = \omega^\mu{}_\nu x^\nu + \delta\xi^\mu$; $\omega_{\mu\nu} = -\omega_{\nu\mu}$)

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad \text{Clifford algebra}$$

$S \cong \mathbb{C}^4$ the irreducible representation

e.g. $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ $\quad \sigma^0 = \bar{\sigma}^0 = -\mathbb{1}_2$

$\sigma^i = -\bar{\sigma}^i = \sigma_i \quad i=1,2,3$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{Pauli matrices}$$

Inf. Lorentz $\delta\psi = \frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu\psi = \frac{1}{4}\omega_{\mu\nu}\underbrace{\gamma^{\mu\nu}}_{\frac{1}{2}[\gamma^\mu,\gamma^\nu]}\psi.$

S : representation of $Spin(3,1) \xrightarrow{2:1} SO(3,1)$

$$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \quad ; \quad \gamma_5\gamma^\mu = -\gamma^\mu\gamma_5, \quad (\gamma_5)^2 = 1, \quad \text{eigenvalues } \pm 1.$$

$$S_R := \{\gamma_5 = +1 \text{ spinors}\} \quad \text{"right handed"}$$

$$S_L := \{\gamma_5 = -1 \text{ spinors}\} \quad \text{"left handed"}$$

(In the above basis, $\gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$, $S_R \ni \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}$, $S_L \ni \begin{pmatrix} 0 \\ 0 \\ * \\ * \end{pmatrix}$.)

$$\begin{tikzcd} S = S_R \arrow[loop above, "Spin(3{,}1)"] \arrow[r, bend right, "\gamma^\mu"'] & \oplus \; S_L \arrow[loop above] \arrow[l, bend right] \end{tikzcd}$$

· Scalar product.

$$\psi_1, \psi_2 \in S \rightsquigarrow \bar\psi_1\psi_2 \in \mathbb{C}$$

s.t. $\overline{c\psi_1}\psi_2 = \bar{c}\,\bar\psi_1\psi_2$, $\bar\psi_1 c\psi_2 = c\,\bar\psi_1\psi_2$, $c \in \mathbb{C}$

$$\overline{\gamma^\mu\psi_1}\,\psi_2 = \bar\psi_1\gamma^\mu\psi_2$$

(In the above basis, $\bar\psi_1\psi_2 = \psi_1^\dagger\gamma^0\psi_2$.)

$$\rightsquigarrow \overline{\gamma_5\psi_1}\,\psi_2 = -\bar\psi_1\gamma_5\psi_2$$

$\therefore$ $\bar\psi_1\psi_2 = 0$ if ψ_1 & ψ_2 are both in S_R or in S_L.

Wick rotation

We obtain a theory on the Euclidean space $\mathbb{R}^4_E$

$$ds_E^2 = \sum_{\mu=1}^{4} (dx^\mu)^2 \qquad \text{"}(++++)\text{"}$$

by $x^0 \to -ix^4$

$$e^{iS} = e^{i\int d^4x\, \mathcal{L}}$$

$$\to e^{i\int (-i\, d^4x_E)\, \mathcal{L}} =: e^{-\int d^4x_E\, \mathcal{L}_E} =: e^{-S_E}$$

$$\therefore \quad \mathcal{L}_E = -\mathcal{L}\Big|_{x^0 \to -ix^4}$$

$$\gamma^0 \to -i\gamma^4$$

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu} \qquad 1 \le \mu, \nu \le 4 .$$

Yang-Mills theory

· Specified by gauge group G : a compact Lie group

e.g. $U(N) = \{ N\times N \text{ unitary matrix} \}$ unitary group

$SU(N) = \{ N\times N \text{ unitary}, \det = 1 \}$ special unitary group

$\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G "infinitesimal version of G"

e.g. $G = U(N)$: $\mathfrak{g} = \{ N\times N \text{ antihermitian matrix} \}$

$G = U(1)$: $\mathfrak{g} = i\mathbb{R} \cong \mathbb{R}$

$G = SU(N)$: $\mathfrak{g} = \{ N\times N \text{ antihermitian, traceless} \}$

field variable $A_\mu(x)$: a vector potential with values in $\mathfrak{g}$

field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

$$S[A] = \int -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} \, d^4x \qquad \text{Yang-Mills action}$$

e : gauge coupling constant.

"$\cdot$" is a positive definite inner product on $\mathfrak{g}$ which is invariant under the adjoint action of G, $X \mapsto gXg^{-1}$

(the infinitesimal version of conjugation $g_t \mapsto g g_t g^{-1}$):

$$gXg^{-1} \cdot gYg^{-1} = X \cdot Y.$$

e.g. For $G = SU(N)$, a standard choice is $X \cdot Y = -2\,\mathrm{Tr}\,XY$.

e.g. $G = U(1)$: Maxwell theory

$$F_{0i} = \partial_0 A_i - \partial_i A_0 \quad \text{electric field}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i \quad \text{magnetic field}$$

$S[A]$ is invariant under a huge symmetry group:

$g(x)$: G-valued function on spacetime

$$\rightsquigarrow \quad A_\mu \mapsto A_\mu^g = g^{-1} A_\mu g + g^{-1} \partial_\mu g.$$

Under this, the field strength transforms covariantly,

$$F_{\mu\nu} \mapsto F^g_{\mu\nu} = \partial_\mu A^g_\nu - \partial_\nu A^g_\mu + [A^g_\mu, A^g_\nu] = g^{-1} F_{\mu\nu} g,$$

and thus indeed

$$S[A^g] = \int -\frac{1}{4e^2}\, g^{-1} F^{\mu\nu} g \cdot g^{-1} F_{\mu\nu} g \; d^4x$$

$$= \int -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} \, d^4x = S[A].$$

This is a generalization of invariance of Maxwell action under the gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu \lambda$.

Indeed, for $G = U(1)$, $\mathfrak{g} = i\mathbb{R} \cong \mathbb{R}$ and for $g(x) = e^{i\lambda(x)}$,

$$iA_\mu^\lambda = e^{-i\lambda}\, iA_\mu\, e^{i\lambda} + e^{-\lambda}\partial_\mu e^{i\lambda} \Rightarrow A_\mu^\lambda = A_\mu + \partial_\mu \lambda .$$

As in that case, we shall call

$$A_\mu \longmapsto A_\mu^g = g^{-1} A_\mu g + g^{-1} \partial_\mu g$$

the <u>gauge transformation</u> of $A_\mu(x)$ by $g(x)$, and

$$\mathcal{G} := \{\, g(x) \mid G\text{-valued function} \,\}$$

the <u>gauge transformation group</u>. We'd like to regard A and A^g as physically equivalent for any $g \in \mathcal{G}$.
I.e. we would like to physically identify them. If we put

$$\mathcal{A} := \{\, A_\mu(x) \mid \mathfrak{g}\text{-valued vector potential} \,\}$$

the space of physically inequivalent field configurations is the quotient space $\mathcal{A}/\mathcal{G}$.

A theory with such an identification of field variables is called a **gauge theory**.

Infinitesimal gauge transformations

A $\mathfrak{g}$-valued function $\epsilon(x)$ generates a one parameter group of gauge transformations: $g_t(x) = e^{t\epsilon(x)}$.

$$A_\mu \mapsto A_\mu^{g_t} = g_t^{-1} A_\mu g + g_t^{-1} \partial_\mu g_t.$$

The infinitesimal transformation is

$$\delta_\epsilon A_\mu = \frac{d}{dt} A_\mu^{g_t} \Big|_{t=0} = -\epsilon A_\mu + A_\mu \epsilon + \partial_\mu \epsilon$$

$$= \partial_\mu \epsilon + [A_\mu, \epsilon] =: D_\mu \epsilon \quad \text{covariant derivative.}$$

The space of such $\epsilon(x)$ may be regarded as the Lie algebra of the gauge transformation group.

$$\{\epsilon(x) \mid \mathfrak{g}\text{-valued function}\} = \mathrm{Lie}(\mathcal{G}).$$

<u>Remark</u> The G-invariant inner product "$\cdot$" on $\mathfrak{g}$ may not be unique.

e.g. $G = SU(N_1) \times SU(N_2)$,

$$\mathcal{L} = -\frac{1}{4e_1^2} F_1^{\mu\nu} \cdot F_{1\mu\nu} - \frac{1}{4e_2^2} F_2^{\mu\nu} \cdot F_{2\mu\nu}$$

e_1 and e_2 can be different.

More generally,

$$G = \underbrace{U(1) \times \cdots \times U(1)}_{k} \times \underbrace{G_1 \times \cdots \times G_\ell}_{\text{"simple" factors}} \Big/ \text{discrete subgroup}$$

$$\mathcal{L} = \sum_{a,b=1}^{k} -\frac{1}{4e_{a,b}^2} F_a^{\mu\nu} F_{b\mu\nu} + \sum_{I=1}^{\ell} -\frac{1}{4e_I^2} F_I^{\mu\nu} \cdot F_{I\mu\nu}$$

$\frac{k(k+1)}{2} + \ell$ gauge coupling constants.

Having this generality in mind, we just write

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu}$$

for simplicity.

Coupling to matter fields

A representation V of a group G is

a vector space$/\mathbb{C}$ or $\mathbb{R}$ on which G acts linearly.

$\exists$ a map $G \times V \to V$; $(g, v) \mapsto gv$

s.t. $\cdot$ $g(hv) = (gh)v$

$\cdot$ $g(cv) = cg(v)$ $\quad c \in \mathbb{C}$ or $\mathbb{R}$

$\cdot$ $g(v+w) = gv + gw$

$\Big\}$ linearity

e.g. $V = \mathbb{C}^N$ for $G = U(N)$ or $SU(N)$ via matrix multiplication.

$V = \mathfrak{g}$ for a general G via adjoint action

V = sum of copies of such, $\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$.

A representation V of a Lie group G

$\rightsquigarrow$ a representation of its Lie algebra $\mathfrak{g}$

$\exists$ a map $\mathfrak{g} \times V \to V$; $(X, v) \mapsto Xv$

s.t. $X(Yv) - Y(Xv) = [X, Y]v$, linearity.

- Scalars

$\varphi(x)$: a scalar field with values a representation V of the gauge group G.

Gauge transformation by $g \in \mathcal{G}$:

$$A_\mu \mapsto A_\mu^g, \quad \varphi \longmapsto \varphi^g = g^{-1}\varphi.$$

Infinitesimally, $\delta A_\mu = D_\mu \epsilon$, $\delta\varphi = -\epsilon\varphi$.

Covariant derivative $D_\mu \varphi := \partial_\mu \varphi + A_\mu \varphi$

Its gauge transformation :

$$D_\mu\varphi \mapsto \partial_\mu \varphi^g + A_\mu^g \varphi^g = \underbrace{\partial_\mu (g^{-1}\varphi)}_{\cancel{-g^{-1}\partial_\mu g g^{-1}\varphi} + g^{-1}\partial_\mu\varphi} + (g^{-1}A_\mu g \cancel{+ g^{-1}\partial_\mu g})\, g^{-1}\varphi$$

$$= g^{-1}\partial_\mu\varphi + g^{-1}A_\mu\varphi = g^{-1}D_\mu\varphi$$ "homogeneous" or "covariant".

$(\varphi_1, \varphi_2) \mapsto \varphi_1^\dagger \varphi_2$ G-invariant inner product on V

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu}\cdot F_{\mu\nu} + (D^\mu\varphi)^\dagger D_\mu\varphi - f(\varphi^\dagger\varphi)$$

is gauge invariant.

- Fermions

$\psi(x)$: a Dirac fermion with values in a rep. V of G.

i.e. an anticommuting function on $\mathbb{R}^{3+1}$

with values in $S \otimes V \cong \mathbb{C}^4 \otimes V$

$$\psi(x) = \left(\psi_\alpha^a(x)\right)_{\alpha=1,2,3,4}^{a=1,\cdots,\dim V} \quad \text{in components,}$$

Gauge transformation : $A_\mu \mapsto A_\mu^g$, $\psi \longmapsto g^{-1}\psi$

$$\not{D}_A \psi = \gamma^\mu D_\mu \psi = \gamma^\mu(\partial_\mu \psi + A_\mu \psi) \quad \text{Dirac operator}$$

$$(\not{D}_A \psi)_\alpha^a = \gamma^{\mu}{}_{\alpha}{}^{\beta}\left(\partial_\mu \psi_\beta^a + A_\mu{}^a{}_b \psi_\beta^b\right) \quad \text{in components}$$

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + i\bar{\psi}\not{D}_A\psi - m\bar{\psi}\psi$$

is gauge invariant.

e.g. QED with electrons of charge $Q_1, \cdots, Q_{N_f}$:

$$G = U(1), \quad e^{i\lambda} : \psi_i \longmapsto e^{iQ_i\lambda}\psi_i \quad (i=1,\cdots,N_f)$$

e.g. QCD with color N_c and flavor N_f :

$$G = SU(N_c), \quad V = \mathbb{C}^{N_c} \oplus \cdots \oplus \mathbb{C}^{N_c} \quad (N_f \text{ copies})$$

- More generally, the representations for right-handed & left-handed fermions can be different:

ψ_R valued in $S_R \otimes V_R$, ψ_L valued in $S_L \otimes V_L$.

Then, $\not{D}_A \psi_R$ valued in $S_L \otimes V_R$, $\not{D}_A \psi_L$ valued in $S_R \otimes V_L$.

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} \cdot F_{\mu\nu} + i \bar{\psi}_R \not{D}_A \psi_R + i \bar{\psi}_L \not{D}_A \psi_L$$

makes sense & is gauge invariant.

Such a theory is called "**chiral**".

- Suppose $\exists$ a <u>G-equivariant</u> bilinear map

$$V_B \times V_R \to V_L \, , \quad (v_B, v_R) \mapsto v_B \cdot v_R$$

$$g v_B \cdot g v_R = g (v_B \cdot v_R).$$

Then, for a V_B-valued scalar ϕ & a $S_{R,L} \otimes V_{R,L}$-valued fermion $\psi_{R,L}$, Yukawa coupling

$$\bar{\psi}_L \phi \cdot \psi_R + \overline{\phi \cdot \psi_R} \psi_L \quad \text{makes sense.}$$

Quantization of gauge theory (path integral)

In a gauge theory, a field configuration $(A, \phi, \psi, \cdots)$ is identified with its gauge transform $(A^g, \phi^g, \psi^g, \cdots)$.

$\mathcal{M}$ = the space of field configurations

$\mathcal{G}$ = the gauge transformation group.

The path-integral is over the quotient space $\mathcal{M}/\mathcal{G}$

$$Z = \int_{\mathcal{M}/\mathcal{G}} \underline{\text{measure}}\; e^{-S_E[A, \phi, \psi, \cdots]}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots \rangle = \frac{1}{Z} \int_{\mathcal{M}/\mathcal{G}} \underline{\text{measure}}\; e^{-S_E[A, \phi, \psi, \cdots]}\, \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots$$

<u>How do we do this?</u>

... Let us do it in a finite dimensional setting.

An answer:

Introduce $\mathfrak{g}$-valued scalar fields $\underbrace{c, \bar{c}}_{\text{fermionic}}, \underbrace{B}_{\text{bosonic}}$

and change the Lagrangian to $\tilde{\mathcal{L}}_E = \mathcal{L}_E + \mathcal{L}_{g.f.}$

$$\mathcal{L}_{g.f.} = \frac{e^2 \xi}{2} B^2 - i B \cdot \partial^\mu A_\mu + \bar{c} \cdot \partial^\mu D_\mu c.$$

Then, the partition/correlation functions are given by

$$Z = \int \mathcal{D}A \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}B \, e^{-\tilde{S}_E[A, \phi, \psi, c, \bar{c}, B]}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots \rangle$$

$$= \frac{1}{Z} \int \mathcal{D}A \cdots \mathcal{D}B \, e^{-\tilde{S}_E[A, \cdots, B]} \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots$$

I.e. the <u>measure</u> in question is

$$\mathcal{D}A \mathcal{D}\phi \mathcal{D}\psi \cdot \int \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}B \, e^{-S_{g.f.}[A, c, \bar{c}, B]}$$

$c, \bar{c}$: Faddeev-Popov ghosts

$\xi > 0$ is called the "gauge parameter" ($\xi \searrow 0$ also considered)
physics should not depend on its value.

B is an auxiliary field without kinetic term
$\rightarrow$ it can be integrated out. Then

$$\mathcal{L}_{g.f.} = \frac{1}{2e^2\xi}(\partial^\mu A_\mu)^2 + \bar{c}\cdot\partial^\mu D_\mu c$$

$c, \bar{c}$ – integration $\longrightarrow$ determinant of $\partial^\mu D_\mu$.

$$\therefore \text{measure} = \mathcal{D}A\,\mathcal{D}\phi\,\mathcal{D}\psi\; e^{-\int \frac{1}{2e^2\xi}(\partial^\mu A_\mu)^2 d^4x}$$

$$\times \operatorname{Det}(\partial^\mu D_\mu)$$

$$\xrightarrow{\xi \searrow 0} \mathcal{D}A\,\mathcal{D}\phi\,\mathcal{D}\psi\;\delta(\partial^\mu A_\mu)\cdot \underbrace{\operatorname{Det}(\partial^\mu D_\mu)}_{}$$

Faddeev-Popov determinant

The extended system is called the gauge fixed system. It no longer has a gauge symmetry. Instead, it has BRST symmetry

$$\delta_B A_\mu = D_\mu c,$$

$$\delta_B \phi = -c\phi,$$

$$\delta_B \psi = -c\psi,$$

$$\delta_B c = -\frac{1}{2}[c, c],$$

$$\delta_B \bar{c} = iB,$$

$$\delta_B B = 0.$$

It is a fermionic symmetry: $\begin{cases} \delta_B \text{ bosonic is fermionic} \\ \delta_B \text{ fermionic is bosonic.} \end{cases}$

$$\delta_B(\mathcal{O}_1 \cdot \mathcal{O}_2) = \delta_B \mathcal{O}_1 \cdot \mathcal{O}_2 + (-1)^{|\mathcal{O}_1|} \mathcal{O}_1 \cdot \delta_B \mathcal{O}_2.$$

It is nilpotent, $\delta_B \circ \delta_B = 0$.

$\mathcal{O}$ is said to be $\begin{cases} \underline{\text{BRST closed}} & \text{when } \delta_B \mathcal{O} = 0 \\ \underline{\text{BRST exact}} & \text{when } \mathcal{O} = \delta_B(-). \end{cases}$

By $\delta_B \circ \delta_B = 0$, BRST exact $\Rightarrow$ BRST closed.

This allows us to consider BRST cohomology :

$$H_{BRST} = \{\text{BRST closed}\} / \{\text{BRST exact}\}$$

There is another symmetry : ghost number N_{gh}

	φ	B	$\bar{c}$	c
N_{gh}	0	0	-1	1

δ_B increases N_{gh} by 1, $[N_{gh}, \delta_B] = 1$

$\mathcal{F}^i = \{\text{observable of } N_{gh} = i\}$

$\Rightarrow \delta_B : \mathcal{F}^i \to \mathcal{F}^{i+1}$

$$H^i_{BRST}(\mathcal{F}) = \mathrm{Ker}(\delta_B : \mathcal{F}^i \to \mathcal{F}^{i+1}) / \mathrm{Im}(\delta_B : \mathcal{F}^{i-1} \to \mathcal{F}^i).$$

The gauge fixed system may be used as a new starting point of quantization:

We may do reverse Wick rotation to real time, convert it to Hamiltonian formulation via Legendre, and then perform the operator quantization.

However A_0 has wrong sign kinetic term (note $\xi > 0$)

$-\frac{1}{2e^2\xi}(\dot{A}_0)^2$ which yields negative norm states.

Also the ghosts with kinetic term $i\dot{\bar{C}}\dot{C}$ also yield zero & negative norm states.

As the existence of such negative/zero norm states indicates, the gauge fixed system has a huge number of unphysical degrees of freedom.

This is the quantum counterpart of the huge gauge symmetry in the classical system: the gauge transformations $(A, \Phi, \Psi, \ldots) \mapsto (A^g, \Phi^g, \Psi^g, \ldots)$ are regarded as unphysiscal change of field configurations.

A proposal:

Physical degrees of freedom are

BRST cohomology classes.

For example, the space of physical states is the BRST cohomology of states

$$\mathcal{H}_{phys} := H_{BRST}(\mathcal{H}).$$

It is expected that this consists of positive norm states only.