Symmetry and Ward identity

Consider a "QFT" with fields $\phi = (\phi_1, \cdots, \phi_n)$,

measure $d^n\phi = d\phi_1 \cdots d\phi_n$

& action $S_E(\phi) = S_E(\phi_1, \cdots, \phi_n)$

Focus of interest :

$$Z = \int d^n\phi \; e^{-S_E(\phi)} \qquad \text{Partition function}$$

$$\langle f \rangle = \frac{1}{Z} \int d^n\phi \; e^{-S_E(\phi)} f(\phi) \qquad \text{Correlation function}$$

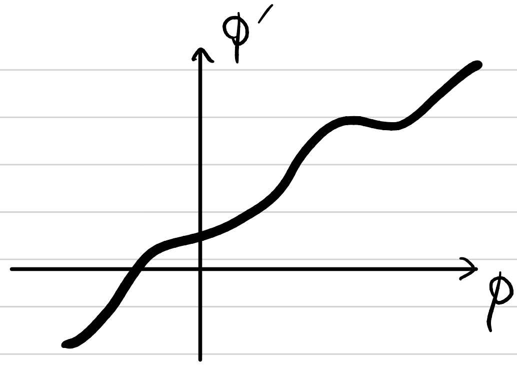
A **symmetry** of the theory is a transformation

$$\phi = (\phi_1, \cdots, \phi_n) \longmapsto g(\phi) = (g_1(\phi), \cdots, g_n(\phi))$$

that leaves $d^n\phi \; e^{-S_E(\phi)}$ invariant.

i.e. $$\det\left(\frac{\partial g_i(\phi)}{\partial \phi_j}\right) e^{-S_E(g(\phi))} = e^{-S_E(\phi)} .$$

Change of integration variables :



Single variable case : $\phi \rightsquigarrow \phi' = g(\phi)$

$$\int_{-\infty}^{\infty} d\phi\, F(\phi) = \int_{-\infty}^{\infty} d\phi'\, F(\phi') = \int_{-\infty}^{\infty} d\phi\, g'(\phi)\, F(g(\phi)).$$

Likewise

$$\int d^n\phi\, e^{-S_E(\phi)} f(\phi) = \int \underbrace{d^n g(\phi)\, e^{-S_E(g(\phi))}}_{\substack{\Vert \;\leftarrow \text{ if } g \text{ is a symmetry.} \\ d^n\phi\, e^{-S_E(\phi)}}} f(g(\phi))$$

$\therefore$ If g is a symmetry, correlation functions satisfy

$$\boxed{\langle f \rangle = \langle f \circ g \rangle.}$$

Ward identity

Infinitesimal form :

$\{ g_\alpha \}_{\alpha \in \mathbb{R}}$: 1-parameter group of transformations.

$$\phi \mapsto \phi + \delta\phi \quad ; \quad \delta\phi = \frac{d}{d\alpha} g_\alpha(\phi) \Big|_{\alpha=0} .$$

... infinitesimal transformation.

If $\{ g_\alpha \}_{\alpha \in \mathbb{R}}$ is a 1-parameter group of symmetries,

Ward identity : $\langle f \rangle = \langle f \circ g_\alpha \rangle \quad \forall \alpha$

$\Rightarrow \frac{d}{d\alpha}$ at $\alpha = 0$:

$$\boxed{0 = \langle \delta f \rangle}$$

(infinitesimal form of) Ward identity

where $\delta f(\phi) := \frac{d}{d\alpha} f(g_\alpha(\phi)) \Big|_{\alpha=0}$

There are Ward identities even for non-symmetries:

$$\int d^n\varphi \, e^{-S_E(\varphi)} f(\varphi) = \int d^n g_\alpha(\varphi) \, e^{-S_E(g_\alpha(\varphi))} f(g_\alpha(\varphi))$$

① Suppose $d^n\varphi$ is invariant but $S_E(\varphi)$ is not.

$$\rightarrow \quad 0 = \int d^n\varphi \, e^{-S_E(\varphi)} \Big(-\delta S_E(\varphi) f(\varphi) + \delta f(\varphi) \Big)$$

$$\boxed{\langle \delta f \rangle = \langle \delta S_E \cdot f \rangle}$$

② Suppose $S_E(\varphi)$ is invariant but $d^n\varphi$ is not,

and the change is known: $d^n g_\alpha(\varphi) = d^n\varphi \, e^{\alpha a(\varphi)}$

(called anomalous symmetry with anomaly a)

$$\rightarrow \quad 0 = \int d^n\varphi \, e^{-S_E(\varphi)} \big(a(\varphi) \cdot f(\varphi) + \delta f(\varphi) \big)$$

$$\boxed{\langle \delta f \rangle = -\langle a \cdot f \rangle}$$

anomalous
Ward identity

<u>Remark</u> (not needed in the present course)

A continuous symmetry in QFT of dimension ≥ 1

$\rightsquigarrow$ Q Noether charge

Classical: constant of motion.

Quantum: generator of the symmetry,

$$\widehat{\delta \mathcal{O}} = i[\hat{Q}, \hat{\mathcal{O}}].$$

For this and other details, see the additional note.

Quantization of gauge theory

In a gauge theory, a field configuration $(A, \phi, \psi, \dots)$ is identified with its gauge transform $(A^g, \phi^g, \psi^g, \dots)$.

$\mathcal{M}$ = the space of field configurations

$\mathcal{G}$ = the gauge transformation group.

The path-integral is over the quotient space $\mathcal{M}/\mathcal{G}$

$$Z = \int_{\mathcal{M}/\mathcal{G}} \underline{\text{measure}}\; e^{-S_E[A,\phi,\psi,\dots]}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots \rangle$$

$$= \frac{1}{Z} \int_{\mathcal{M}/\mathcal{G}} \underline{\text{measure}}\; e^{-S_E[A,\phi,\psi,\dots]}\, \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots$$

<u>How do we do this ?</u>

… Let us do it in a finite dimensional setting.

M: a manifold, $\dim M = n < \infty$,

G: a Lie group acting on M, $\dim G = d_G < \infty$;

$$g \in G : \phi \in M \longmapsto \phi g \in M$$

(right action : $\phi(gh) = (\phi g)h$)

Assume: the action is <u>free</u>, $\phi g = \phi$ for some $\phi \Rightarrow g = 1$.

Suppose a measure $d\phi$ and a function $S_E(\phi)$ on M are G-invariant, $d(\phi g) = d\phi$, $S_E(\phi g) = S_E(\phi)$.

Want to consider the gauge theory where

$$\begin{cases} \phi \sim \phi g & \text{identified} \\ f(\phi) & \text{physically meaningful when } G\text{-invariant} \end{cases}$$

<u>Question</u> How do we define measure on M/G for

$$Z = \int_{M/G} \text{measure } e^{-S_E(\phi)}$$

$$\langle f \rangle = \frac{1}{Z} \int_{M/G} \text{measure } e^{-S_E(\phi)} f(\phi)$$

?

A naive answer:

$$Z = \frac{1}{\mathrm{Vol}\, G} \int_M d\phi \, e^{-S_E(\phi)}$$

See below for the choice of dg

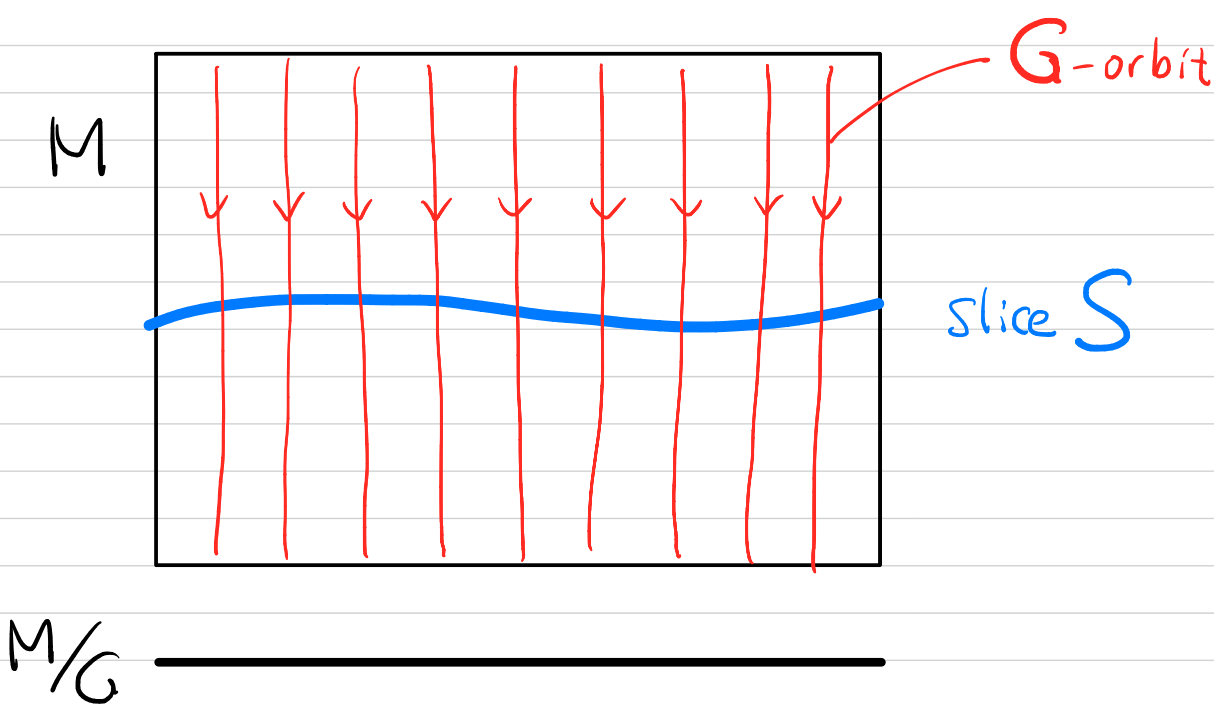
$$\langle f \rangle = \frac{1}{\mathrm{Vol}\, G} \int_M d\phi \, e^{-S_E(\phi)} f(\phi) \Big/ Z,$$

where $\mathrm{Vol}\, G = \int_G dg$ for some measure dg.

A possible problem: $\mathrm{Vol}\, G$ may be infinite

$\int_M d\phi \cdots$ may be infinite.

Suppose we can find a slice $S \subset M$, i.e. a submanifold s.t. any G-orbit has exactly one point in it.



Suppose S is the zero locus of a set of functions of M

$$\phi \in S \Leftrightarrow \chi_1(\phi) = \cdots = \chi_{d_G}(\phi) = 0.$$

$\chi(\phi) = (\chi_a(\phi))_{a=1}^{d_G}$ can be regarded as a function on M with values in $\mathfrak{g} = \mathrm{Lie}(G)$.

Note: For a monotonic function $F(y)$ of a single y,

$$\int_{-\infty}^{\infty} \delta(F(y)) \underbrace{dF(y)}_{F'(y)\,dy} = 1.$$

Multivariable case: $y = (y_1, \cdots, y_m) \mapsto F(y) = (F_1(y), \cdots, F_m(y))$

$$\int_{\mathbb{R}^m} \prod_{a=1}^{m} \delta(F_a(y)) \cdot \det\left(\frac{\partial F_a(y)}{\partial y_b}\right) d^m y = 1.$$

Apply this to the function $F_a(g) = \chi_a(\phi g)$ of G for a fixed $\phi \in M$:

$$\int_G \underbrace{\prod_{a=1}^{d_G} \delta(\chi_a(\phi g))}_{\delta(\chi(\phi g))} \cdot \underbrace{\det\left(\frac{\partial \chi_a(\phi g)}{\partial g_b}\right)}_{\det(\partial^b \chi_a(\phi g))} \cdot dg = 1.$$

insert $1 = \cdots$

$$\int_M d\phi\, e^{-S_E(\phi)} = \int_{M\times G} d\phi\, dg\, e^{-S_E(\phi)} \delta(\chi(\phi g))\, \det(\delta\chi(\phi g))$$

[Change the variable $\phi g \to \phi$ and use G-invariance of $d\phi$ & $S_E(\phi)$

$$= \int_{M\times G} d\phi\, dg\, e^{-S_E(\phi)}\, \delta(\chi(\phi))\, \det(\delta\chi(\phi))$$

$$= \underbrace{\int_G dg}_{\mathrm{Vol}\, G} \int_M d\phi\, e^{-S_E(\phi)} \delta(\chi(\phi))\, \det(\delta\chi(\phi))$$

$$\therefore Z = \text{“}\frac{1}{\mathrm{Vol}\, G} \int_M d\phi\, e^{-S_E(\phi)}\text{”}$$

$$= \int_M d\phi\, e^{-S_E(\phi)}\, \delta(\chi(\phi))\, \det(\delta\chi(\phi)).$$

Similarly

$$\langle f \rangle = \text{"} \frac{1}{\mathrm{Vol}\, G} \int_M d\phi\, e^{-S_E(\phi)} f(\phi) \Big/ Z \text{"}$$

$$= \int_M d\phi\, e^{-S_E(\phi)} \delta(\chi(\phi)) \det(\delta\chi(\phi)) f(\phi) \Big/ Z .$$

$\chi(\phi) = (\chi_a(\phi))_{a=1}^{d_G}$ ... gauge fixing function

$\chi_1(\phi) = \cdots = \chi_{d_G}(\phi) = 0$ ... gauge fixing condition

$$\det(\delta\chi(\phi)) = \det\left(\frac{\partial \chi_a(\phi g)}{\partial g_b} \Big|_{g=1} \right)$$

$$= \det\left(\frac{\partial \chi_a(\phi e^{\epsilon})}{\partial \epsilon_b} \Big|_{\epsilon=0} \right) \qquad \{e^a\}_{a=1}^{d_G} \subset \mathfrak{g} \text{ basis}$$

$$\epsilon = \sum_a e^a \epsilon_a$$

... <u>Faddeev-Popov determinant</u>

The results for Z & $\langle f \rangle$ do not depend on the choice of slice S or gauge fixing fcn χ.

Rewriting

① Use independence on the choice of $\chi(\phi)$.

② $\det(\delta\chi(\phi))$ via integration of anticommuting variables

① Replace $\chi(\phi) \to \chi(\phi) - \omega$, $\omega \in \mathfrak{g}$.

Also,
$$\int_{\mathfrak{g}} d\omega \, e^{-\frac{1}{2\xi}\omega^2} = (2\pi\xi)^{d_G/2}$$

$$Z = \int_M d\phi \, e^{-S_E(\phi)} \, \delta(\chi(\phi)) \det(\delta\chi(\phi))$$

$$1 = \frac{1}{(2\pi\xi)^{d_G/2}} \int_{\mathfrak{g}} d\omega \, e^{-\frac{1}{2\xi}\omega^2}$$

$\chi(\phi) - \omega$

$\delta\chi(\phi)$ intact

$$= \frac{1}{(2\pi\xi)^{d_G/2}} \int_{M \times \mathfrak{g}} d\phi \, d\omega \, e^{-S_E(\phi) - \frac{1}{2\xi}\omega^2} \, \delta(\chi(\phi) - \omega) \det(\delta\chi(\phi))$$

(#)

$$= \frac{1}{(2\pi\xi)^{d_G/2}} \int_M d\phi \, e^{-S_E(\phi) - \frac{1}{2\xi}(\chi(\phi))^2} \det(\delta\chi(\phi))$$

Digression: integration of anti commuting variables

- η anti commuting variable

$$\int d\eta \cdot 1 := 0\,, \qquad \int d\eta \cdot \eta = 1$$

- $\eta^1, \dots \eta^m$ anticommuting variables

$$\int d\eta^1 \cdots d\eta^m \cdot \eta^1 \cdots \eta^m = \int d\eta^1 \cdots d\eta^m \cdot \eta^m \cdots \eta^1 \,(-1)^{m-1+m-2+\dots+1} = (-1)^{\frac{m(m-1)}{2}}$$

$$\int d\eta^1 \cdots d\eta^m \cdot \text{lower powers} = 0$$

- $\eta, \bar\eta$ pair of anticomm. var.

$$\int d\bar\eta\, d\eta\, \eta\bar\eta = 1$$

$$\int d\bar\eta\, d\eta\; e^{-\bar\eta M \eta} = \int d\bar\eta\, d\eta\,(1 - \bar\eta M \eta) = M$$

- $\eta^1, \bar\eta_1, \dots, \eta^m, \bar\eta_m$ m-pairs

$$\int d\bar\eta_1 d\eta^1 \cdots d\bar\eta_m d\eta^m \; e^{-\sum_{a,b=1}^{m} \bar\eta_a M^a{}_b \eta^b} = \det(M^a{}_b)$$

$\uparrow$ exercise.

End of digression (For more details, see the additional notes.)

② $$\det(\delta^b \chi_a(\phi)) = \int \prod_{a=1}^{d_G} d\bar{c}^a dc_a \, e^{-\sum_{a,b} \bar{c}^a \delta^b \chi_a(\phi) c_b}$$

$$\sum_b \delta^b \chi_a(\phi) c_b = \delta_c \chi_a(\phi)$$

infinitesimal transformation of $\chi_a(\phi)$

by $c = \sum_{a=1}^{d_G} e^a c_a$

$$= \int \prod_{a=1}^{d_G} d\bar{c}^a dc_a \, e^{-\sum_a \bar{c}^a \delta_c \chi_a(\phi)}$$

$$\overset{\text{or}}{=} \int_{\mathfrak{g}\times\mathfrak{g}} d\bar{c}\, dc \, e^{-\bar{c}\cdot\delta_c \chi(\phi)}$$ in a simplified form.

$c, \bar{c}$: Faddeev-Popov ghosts

Also

$$\delta(\chi(\phi)-\omega) = \int \prod_{a=1}^{d_G} \frac{dB_a}{2\pi} e^{iB^a(\chi_a(\phi)-\omega_a)}$$

$$\overset{\text{or}}{=} \frac{1}{(2\pi)^{d_G}} \int_{\mathfrak{g}} dB \, e^{iB\cdot(\chi(\phi)-\omega)}$$

$\therefore \delta(\chi(\phi)-\omega)\det(\delta\chi(\phi))$

$$= \frac{1}{(2\pi)^{d_G}} \int_{\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}} dB\, d\bar{c}\, dc \, e^{iB\cdot(\chi(\phi-\omega)) - \bar{c}\cdot\delta_c \chi(\phi)}$$

Insert this in (#):

$$Z = \frac{1}{(2\pi\xi)^{d_a/2}} \int_{M\times\mathfrak{g}} d\phi\, d\omega\; e^{-S_E(\phi) - \frac{1}{2\xi}\omega^2}$$

$$\times \frac{1}{(2\pi)^{d_a}} \int_{\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}} dB\, d\bar{c}\, dc\; e^{iB\cdot(\chi(\phi-\omega)) - \bar{c}\cdot\delta_c\chi(\phi)}$$

Perform ω - integral

$$\int d\omega\; e^{-\frac{1}{2\xi}\omega^2 - iB\cdot\omega} = (2\pi\xi)^{d_a/2}\, e^{-\frac{\xi}{2}B^2}$$

We end up with

$$Z = \frac{1}{(2\pi)^{d_a}} \int_{M\times\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}} d\phi\, dB\, d\bar{c}\, dc\; e^{-\tilde{S}_E(\phi, B, \bar{c}, c)}$$

$$\tilde{S}_E = S_E(\phi) + \frac{\xi}{2}B^2 - iB\cdot\chi(\phi) + \bar{c}\cdot\delta_c\chi(\phi)$$

... gauge fixed action

Similarly for $\langle f \rangle$.

The gauge fixed system has a symmetry δ_B called

BRST symmetry

$$\delta_B \phi = \delta_c \phi$$
$$\delta_B B = 0$$
$$\delta_B \bar{c} = iB$$
$$\delta_B c = -\frac{1}{2}[c, c]$$

$\cdot\, c = e^a c_a$

$$[c, c] = [e^a c_a, e^b c_b] = [e^a, e^b] c_a c_b$$

If $[e^a, e^b] = e^d f_d^{ab}$,

$$\delta c_a = -\frac{1}{2} f_a^{bc} c_b c_d$$

It is a fermionic symmetry $\begin{cases} \delta_B \text{ bosonic is fermionic} \\ \delta_B \text{ fermionic is bosonic.} \end{cases}$

$$\delta_B(\mathcal{O}_1 \mathcal{O}_2) = \delta_B \mathcal{O}_1 \cdot \mathcal{O}_2 + (-1)^{|\mathcal{O}_1|} \mathcal{O}_1 \cdot \delta_B \mathcal{O}_2$$

$$\delta_B \tilde{S}_E = \cancel{\delta_c S_E(\phi)}^{\,0} - iB \cdot \cancel{\delta_c \chi(\phi)} + iB \cdot \cancel{\delta_c \chi(\phi)} - \bar{c}\, \underbrace{\delta c_a}_{-\frac{1}{2} f_a^{bd} c_b c_d} \delta^a \chi(\phi) + \bar{c}\, c_a \underbrace{\delta_c \delta^a \chi(\phi)}_{c_b \delta^b \delta^a \chi(\phi)}$$

$$\left[f_a^{bd} \delta^a \chi = \delta^b \delta^d \chi - \delta^d \delta^b \chi \quad (\because \text{right action}) \right.$$

$$= \frac{1}{2} \bar{c}\, c_b c_d (\delta^b \delta^d - \delta^d \delta^b) \chi(\phi) + \bar{c}\, c_a c_b \delta^b \delta^a \chi(\phi) = 0.$$

Remarks

- $\delta_B \circ \delta_B = 0$ (exercise)

  $\mathcal{O}$ is said to be **BRST closed** when $\delta_B \mathcal{O} = 0$

  BRST exact when $\mathcal{O} = \delta_B(-)$.

  By $\delta_B \circ \delta_B = 0$, BRST exact $\Rightarrow$ BRST closed.

- $\tilde{S}_E = S_E - \delta_B\left(\bar{c} \cdot \left(\chi(\phi) - \frac{i\xi}{2} B\right)\right)$

  $\therefore$ The gauge fixing term is BRST exact.

- $\langle \delta_B h \rangle = 0$ by ward identity.

  If $\delta_B f = 0$, then

  $$\langle f \cdot \delta_B h \rangle = (-1)^{|f|} \langle \delta_B (f \cdot h) \rangle = 0.$$

  In particular, if $f_1, \cdots, f_n$ are BRST closed,

  $\langle f_1 \cdots f_n \rangle$ does not change under change of f_i's

  by BRST exact ones, $f_i \to f_i + \delta_B h_i$

There motivate us to consider BRST cohomology :

$$H_{BRST} = \{\text{BRST closed}\} / \{\text{BRST exact}\}$$

A proposal :

Physical observables (states) are BRST cohomology classes (states).

There is another symmetry : ghost number N_{gh}

	ϕ	B	$\bar{c}$	c
N_{gh}	0	0	-1	1

δ_B increases N_{gh} by 1, $[N_{gh}, \delta_B] = 1$

$$\mathcal{F}^i = \{\text{observable of } N_{gh} = i\}$$

$$\Rightarrow \quad \delta_B : \mathcal{F}^i \to \mathcal{F}^{i+1}$$

$$H^i_{BRST}(\mathcal{F}) = \mathrm{Ker}(\delta_B : \mathcal{F}^i \to \mathcal{F}^{i+1}) / \mathrm{Im}(\delta_B : \mathcal{F}^{i-1} \to \mathcal{F}^i).$$

We may also integrate-out B :

$$Z = \frac{1}{(2\pi\xi)^{d_G/2}} \int_{M\times\mathfrak{g}\times\mathfrak{g}} d\phi\, d\bar{c}\, dc\; e^{-\tilde{S}_E(\phi,\bar{c},c)}$$

$$\tilde{S}_E = S_E(\phi) + \frac{1}{2\xi}\chi(\phi)^2 + \bar{c}\cdot\delta_c\chi(\phi)$$

This is also obtained directly from (1) & (2).

This system also has BRST symmetry

$$\delta_B\phi = \delta_c\phi,$$

$$\delta_B\bar{c} = -\frac{1}{\xi}\chi(\phi) \quad \leftarrow \text{from EOM: } B = \frac{i}{\xi}\chi(\phi)$$

$$\delta_B c = -\frac{1}{2}[c,c].$$

But $\delta_B\circ\delta_B = 0$ holds only on-shell

(EOM $\delta_c\chi(\phi)=0$ is needed).

Back to the case of gauge theory:

$$M \rightsquigarrow \mathcal{M} = \{ (A_\mu(x), \varphi(x), \psi(x), \cdots) \text{ field config.} \}$$

$$G \rightsquigarrow \mathcal{G} = \{ g(x) \mid G\text{-valued function} \}$$

$$\mathfrak{g} \rightsquigarrow \mathrm{Lie}(\mathcal{G}) = \{ \epsilon(x) \mid \mathfrak{g}\text{-valued function} \}$$

As gauge fixing function, we can take

$$\chi[A](x) = \partial^\mu A_\mu(x) \qquad \text{Lorentz gauge}$$

$$\delta_\epsilon \chi[A](x) = \partial^\mu D_\mu \epsilon(x)$$

gauge fixed Lagrangian

$$\tilde{\mathcal{L}}_E = \mathcal{L}_E + \frac{\xi}{2} B^2 - i B \cdot \partial^\mu A_\mu + \bar{c} \cdot \partial^\mu D_\mu c$$

Inverse Wick rotation to real time

(with $B \to -iB$, $\bar{c} \to -i\bar{c}$ & $\xi \to e^2 \xi$ for convenience)

$$\tilde{\mathcal{L}} = \mathcal{L} + \frac{e^2 \xi}{2} B^2 - B \cdot \partial^\mu A_\mu - i \bar{c}\, \partial^\mu D_\mu c$$

BRST symmetry :

$$\delta_B A_\mu = D_\mu c \,,\quad \delta_B \phi = -c\phi \,,\quad \delta_B \psi = -c\psi$$

$$\delta_B B = 0$$

$$\delta_B \bar{c} = iB$$

$$\delta_B c = -\frac{1}{2}[c,c]$$

The version where B is integrated out :

$$\tilde{\mathcal{L}} = \mathcal{L} - \frac{1}{2e^2\xi}(\partial^\mu A_\mu)^2 - i\bar{c}\cdot\partial^\mu D_\mu c$$

$$\delta_B \bar{c} = \frac{i}{e^2\xi}\partial^\mu A_\mu \,,$$

δ_B(others) = same as above.

$\xi \in (0,\infty)$ is called the "gauge parameter".

physics should not depend on its value.

($\xi \searrow 0$ is also considered.)

We may use this as the new starting point for quantization.

For example, we may convert this via Legendre transform to Hamiltonian formulation and then perform the operator quantization.

[※ This is now possible thanks to $-\frac{1}{2e^2\xi}(\partial^\mu A_\mu)^2$:
Without that, A_0 would have no kinetic term and hence no conjugate momentum.]

However A_0 has wrong sign kinetic term (note $\xi > 0$)

$-\frac{1}{2e^2\xi}(\dot{A}_0)^2$ which yields negative norm states.

Also the ghosts with kinetic term $i\dot{\bar{C}}\dot{C}$ also yield zero & negative norm states. [Lec 3, Exercise (c)]

As the existence of such negative/zero norm states indicates, the gauge fixed system has a huge number of unphysical degrees of freedom.

This is the quantum counterpart of the huge gauge symmetry in the classical system: the gauge transformations $(A, \Phi, \psi, \ldots) \mapsto (A^g, \Phi^g, \psi^g, \ldots)$ are regarded as unphysical change of field configurations.

The proposal is to take the **BRST cohomology** to select physical degrees of freedom.

For example, the space of physical states is the BRST cohomology of states

$$\mathcal{H}_{phys} := H_{BRST}(\mathcal{H}).$$

It is expected that this consists of positive norm states only.

<u>Remark</u> (not needed in the present course)

There is a way to quantize gauge theories without introducing ghosts.

See the additional notes for details.