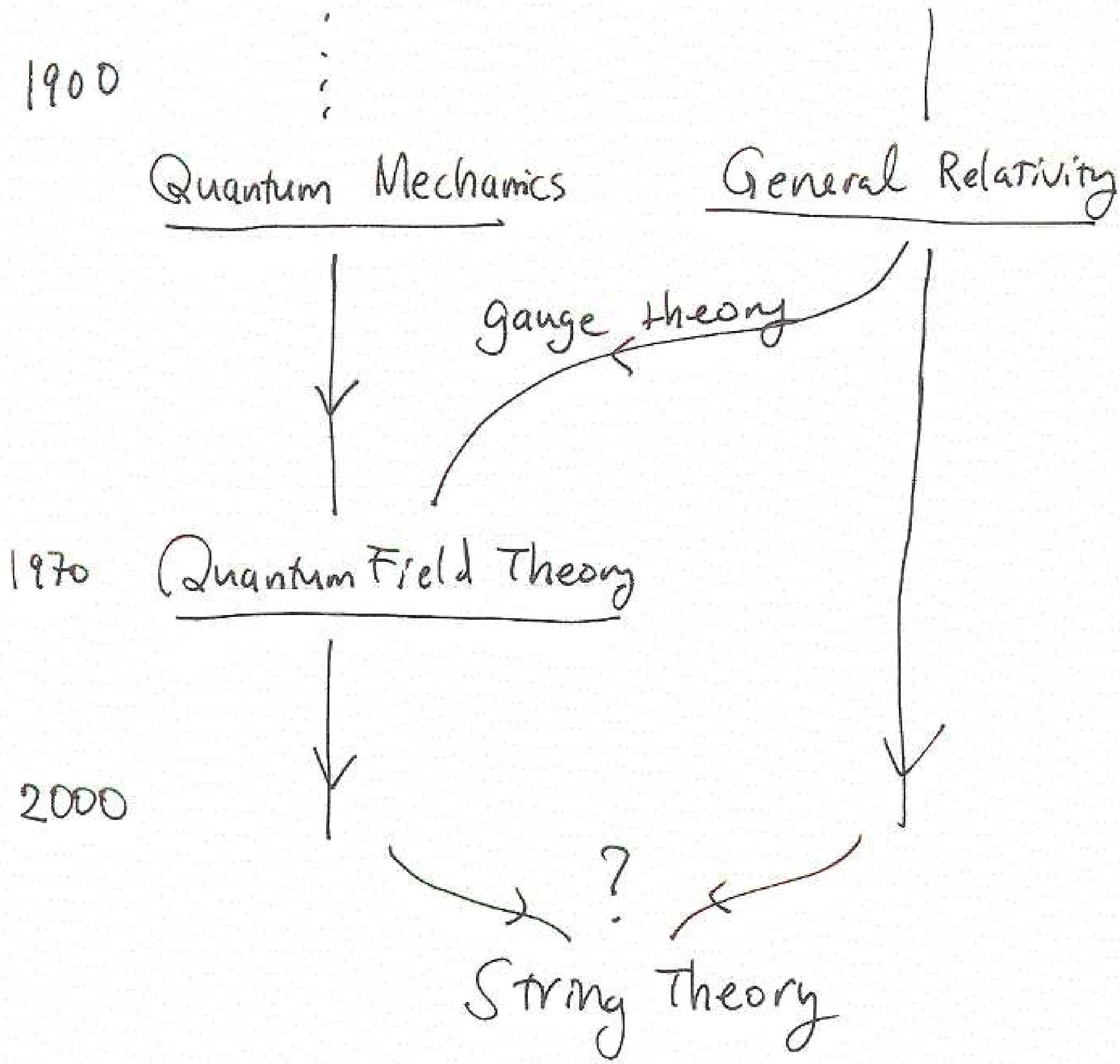
A brief history of physics



At the same time as ~~these~~ developments in physi
or motivated by this, several new
fields of mathematics ~~was~~ was born, emerged
or extensively used. For example,
Riemannian Geometry was new at that
time and was used essentially in General Relativi
Theory of Hibert spaces, Operator algebra
was litterally born by Quantum Mechanics.
I believe that
The real genuin unified theory or framework ~~of~~ of QM
& GR must also unify these
new fields of mathematics,
and generate a new field.
that is best fit to describe the theor

Sideremark

I think Toronto this year is a very exciting period. There are so many talent of really exeelent quality. I hope you can ~~generate~~ contribute in ~~unifying~~ ~~the~~ finding the new mathematics to describe the right physics.

Coming back to the subject

This semester, I will give a course on Mirror Symmetry. As we will see, Mirror Symmetry shows an example where unification of mathematic fields is actually happening.

The two fields are:

Algebraic (or Complex) Geometry + stringy quantum correction.

and

Symplectic Geometry + stringy quantum correction.

As of today, we know ~~about~~ the relation to a certain extent. Especially at the lev[el of] ~~Su~~ String perturbation theory.

But ~~we~~ do~~n~~ not yet Know the whole picture. & That's why its so interesting.

Also, I want to add that there are other things that suggest unification of Mathematics. MS is just one example. (but prob the most underst at this

Plan

- Supersymmetry & homological algebra
- (2,2) SUSY, NLSM & LG
- Topological Field Theory
- Linear sigma models & Moduli space of theories

Mirror Symmetry

HMS involving D-branes

Supersymmetry

Def Supersymmetric Quantum Mechanics

is a QM with $\mathbb{Z}_2$-graded Hibert Space

$$\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$$

$$(-1)^F = 1 \qquad\qquad (-1)^F = -1$$

with even Hamiltonian H

$$\begin{tikzcd} \mathcal{H}^B \arrow[loop above, "H"] & \mathcal{H}^F \arrow[loop above, "H"] \end{tikzcd}$$

& odd operators Q, $Q^\dagger$
(supercharges)

$$\begin{tikzcd} \mathcal{H}^B \arrow[r, bend left, "{Q, Q^\dagger}"] & \mathcal{H}^F \arrow[l, bend left, "{Q, Q^\dagger}"] \end{tikzcd}$$

obeying

$$Q^2 = Q^{\dagger 2} = 0$$

$$\{Q, Q^\dagger\} = 2H$$

supersymmetry algebra

$$\left(\begin{array}{c} \Rightarrow [H, Q] = [H, Q^\dagger] = 0 \\ (-1)^F Q = -Q(-1)^F \quad \text{etc} \end{array} \right)$$

Example 1 $h(x)$ some function with that $|h'(x)|$ grows at ∞
(say a polynomial of degree ≥ 2)

$$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^2) = L^2(\mathbb{R}) \otimes \underbrace{\mathbb{C}^2}_{\mathbb{C}^B \oplus \mathbb{C}^F} \qquad (\psi_1, \psi_2) = \int_{\mathbb{R}} dx\, \psi_1(x)^\dagger \psi$$

$$\begin{cases} H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ Q = \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + h'(x) & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & -\frac{d}{dx} + h'(x) \\ 0 & 0 \end{pmatrix} \\ (-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}$$

Example 2 (M, g) compact oriented Riemannian mfd

$$\mathcal{H} = \Omega^\cdot(M), \quad \mathcal{H}^{B, F} = \bigoplus_{p:\text{even, odd}} \Omega^p(M)$$

$$(\psi_1, \psi_2) = \int_M \psi_1^* \wedge * \psi_2$$
$\uparrow$ Hodge $*$

$$\begin{cases} H = \frac{1}{2}\Delta = \frac{1}{2}(dd^\dagger + d^\dagger d) \\ Q = d, \quad Q^\dagger = d^\dagger \end{cases}$$

Note If $F = p$ on $\Omega^p(M)$, $[F, Q] = Q$

--- refinement of $(-1)^F Q = -Q($

① Positive Semidefinite Spectrum

$H = \frac{1}{2}\{Q, Q^\dagger\}$

$E = \langle\Psi|\frac{1}{2}\{Q, Q^\dagger\}|\Psi\rangle = \frac{1}{2}\|Q^\dagger\Psi\|^2 + \frac{1}{2}\|Q\Psi\|^2 \geq 0$

$= 0 \iff Q|\Psi\rangle = Q^\dagger|\Psi\rangle = 0.$

such a state is called

a <u>supersymmetric ground state</u>

If $\not\exists$ SUSY ground state $\in \mathcal{H}$,

supersymmetry is <u>spontaneously broken</u>.

② $E > 0$: boson $\rightleftarrows$ fermion isom.

$\mathcal{H} = \bigoplus_{n=0,1,2,} \mathcal{H}_n \qquad H = E_n \text{ on } \mathcal{H}_n$

$E_0 < E_1 < E_2 < \cdots$

$\mathcal{H}_n = \mathcal{H}_n^B \oplus \mathcal{H}_n^F \qquad [H, (-1)^F] = 0.$

<u>Claim</u> $\mathcal{H}_n^B \cong \mathcal{H}_n^F$ if $E_n > 0$.

define $Q_1 := Q + Q^\dagger \qquad Q_1^2 = \{Q, Q^\dagger\} = 2H = 2E_n$ on $\mathcal{H}_n$

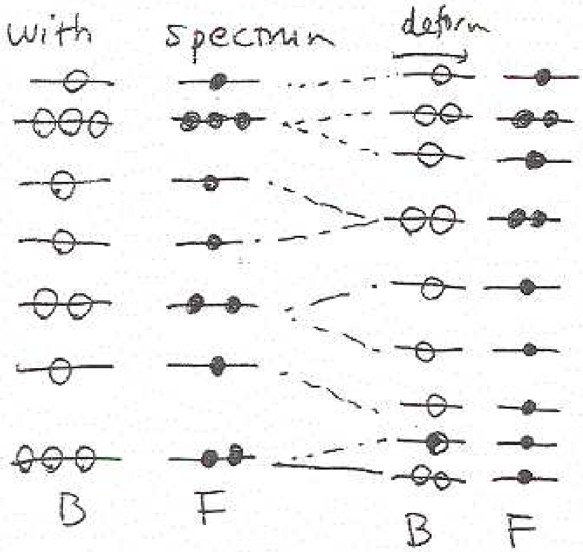
$E_n > 0 \Rightarrow Q_1 : \mathcal{H}_n^B \rightleftarrows \mathcal{H}_n^F$ isomorphism

$E_0 = 0 \Rightarrow Q_1^2 = 0 \quad \therefore$ it's not an isomophism.

③ Property under deformation

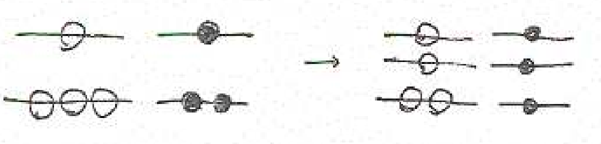
Start with a SUSY QM

with spectrum

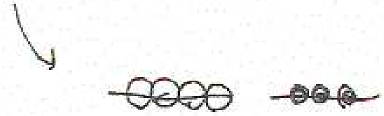


bosonic & fermionic states with $E>0$ must move tog

Some level may split, but the # of bosonic & fermionic states must stay the same at each $E>0$ level.



Some SUSY ground states may become excited state but bosonic & fermionic states must move in pairs



Some excited states may become SUSY ground states but bosonic & fermionic at the same time

④ Witten index

bosonic SUSY ground states − # fermonic SUSY ground states

$= \dim \mathcal{H}_0^B - \dim \mathcal{H}_0^F$

is invariant under deformation preserving supersymmetry.

(an example of deformation invariance)

$$\mathrm{Tr}(-1)^F := \mathrm{Tr}_{\mathcal{H}} (-1)^F e^{-\beta H}$$

$$= \sum_{n=0}^{\infty} \Big\{ \underbrace{\dim \mathcal{H}_n^B e^{-\beta E_n} - \dim \mathcal{H}_n^F e^{-\beta E_n}}_{\text{if } E_n > 0} \Big\}$$

$$= \dim \mathcal{H}_0^B - \dim \mathcal{H}_0^F$$

This is called the <u>Witten index</u> of the system

⑤ Cohomology & SUSY ground states

$$\mathcal{H}^B \underset{Q}{\overset{Q}{\rightleftarrows}} \mathcal{H}^F \qquad Q^2 = 0$$

$(\mathcal{H}^B, \mathcal{H}^F, Q)$ forms a $\mathbb{Z}_2$-graded complex.

One can define a cohomology of the complex:

$$H^B(Q) := \frac{\mathrm{Ker}(Q : \mathcal{H}^B \to \mathcal{H}^F)}{\mathrm{Im}(Q : \mathcal{H}^F \to \mathcal{H}^B)}$$

$$H^F(Q) := \frac{\mathrm{Ker}(Q : \mathcal{H}^F \to \mathcal{H}^B)}{\mathrm{Im}(Q : \mathcal{H}^B \to \mathcal{H}^F)}$$

Note: $\mathrm{Im}(Q : \mathcal{H}^F \to \mathcal{H}^B) \subset \mathrm{Ker}(Q : \mathcal{H}^B \to \mathcal{H}^F)$ by $Q^2 = 0$

<u>Theorem</u> ~~Claim~~

$$H^B(Q) \cong \mathcal{H}^B_0$$

$$H^F(Q) \cong \mathcal{H}^F_0$$

<u>proof</u> $[H, Q] = 0 \Rightarrow \mathcal{H}^B_n \underset{Q}{\overset{Q}{\rightleftarrows}} \mathcal{H}^F_n \quad Q^2 = 0$

define $H^B(Q|_{\mathcal{H}_n})$, $H^F(Q|_{\mathcal{H}_n})$ similarly

$$H^{B,F}(Q) = \bigoplus_{n=0}^{\infty} H^{B,F}(Q|_{\mathcal{H}_n})$$

We show $H^{B,F}(Q|_{\mathcal{H}_n}) = 0$ if $E_n > 0$.

$v \in \mathcal{H}_n \quad E_n > 0 \Rightarrow 2H = \{Q, Q^\dagger\}$

$$v = \frac{1}{2E_n}\{Q, Q^\dagger\} v$$

$$Qv = 0 \Rightarrow v = \frac{1}{2E_n} Q Q^\dagger v = Q\left(\frac{1}{2E_n} Q^\dagger v\right)$$

$$\therefore \quad \mathrm{Ker}\, Q|_{\mathcal{H}_n} = \mathrm{Im}\, Q|_{\mathcal{H}_n} \quad \therefore H^{B,F}(Q|_{\mathcal{H}_n}) = 0$$

$$\forall v \in \mathcal{H}_0 \quad E_0 = 0 \Rightarrow Qv = 0$$

$$\therefore \left.\begin{array}{l} \mathrm{Ker}\, Q|_{\mathcal{H}_0} = \mathcal{H}_0 \\ \mathrm{Im}\, Q|_{\mathcal{H}_0} = 0 \end{array}\right\} H^{B,F}(Q|_{\mathcal{H}_0}) = \mathcal{H}_0^{B,F}$$

$$\therefore \quad H^{B,F}(Q) = H^{B,F}(Q|_{\mathcal{H}_0}) = \mathcal{H}_0^{B,F} \quad //$$

⑥ Witten index as Euler characteristic.

For a $\mathbb{Z}_2$-graded complex $\mathcal{H}^B \underset{Q}{\overset{Q}{\rightleftarrows}} \mathcal{H}^F$, $Q^2 = 0$

it's Euler characteristic is defined as

$$\chi(Q) = \dim H^B(Q) - \dim H^F(Q).$$

We see by the theorem $\underline{\mathrm{Tr}(-1)^F = \chi(Q)}$.

$\mathbb{Z}$-graded case.

Suppose $\exists$ charge F s.t. $[F, Q] = Q$.

& eigenvalues of F are integers.

define $\underbrace{\mathcal{H}^p}_{\text{for } p \in \mathbb{Z}} \subset \mathcal{H}$ subspace of states with $F = p$

$$\cdots \xrightarrow{Q} \mathcal{H}^p \xrightarrow{Q} \mathcal{H}^{p+1} \xrightarrow{Q} \mathcal{H}^{p+2} \xrightarrow{Q} \cdots \qquad Q^2 = 0$$

: $\mathbb{Z}$-graded complex

its cohomology is defined as

$$H^p(Q) = \frac{\mathrm{Ker}(Q : \mathcal{H}^p \to \mathcal{H}^{p+1})}{\mathrm{Im}(Q : \mathcal{H}^{p-1} \to \mathcal{H}^p)}$$

$$\mathcal{H}^B = \bigoplus_{p:\text{even}} \mathcal{H}^p, \quad \mathcal{H}^F = \bigoplus_{p:\text{odd}} \mathcal{H}^p$$

$$\Rightarrow \quad H^B(Q) = \bigoplus_{p:\text{even}} H^p(Q), \quad H^F(Q) = \bigoplus_{p:\text{odd}} H^p(Q)$$

$$\therefore \quad \mathrm{Tr}(-1)^F = \sum_{p \in \mathbb{Z}} (-1)^p \dim H^p(Q) =: \chi(Q)$$

(7) Witten index as index of Fredholm Operator

$$Q_1\big|_{\mathcal{H}^B} = Q + Q^\dagger \ : \ \mathcal{H}^B \to \mathcal{H}^F$$

Its Kernel & Cokernel $\left(= \mathcal{H}^F / \mathrm{Im}\, Q_1\big|_{\mathcal{H}^B}\right)$

are both finite dimensional.

Such an operator is called a <u>Fredholm operator</u>

Index of Fredholm Operator D is defined as

$$\mathrm{Index}\, D = \dim \mathrm{Ker}\, D - \dim \mathrm{Coker}\, D$$

In our case, $\mathrm{Ker}\, Q_1\big|_{\mathcal{H}^B} = \mathcal{H}^B_0$

$$\mathrm{Coker}\, Q_1\big|_{\mathcal{H}^B} \cong \mathrm{Ker}\left(Q_1\big|_{\mathcal{H}^B}\right)^\dagger = \mathrm{Ker}\, Q_1\big|_{\mathcal{H}^F} = \mathcal{H}^F_0$$

$$\therefore \ \mathrm{Index}\left(Q_1\big|_{\mathcal{H}^B}\right) = \dim \mathcal{H}^B_0 - \dim \mathcal{H}^F_0 = \mathrm{Tr}(-1)^F$$

Example 2

$F = p$ on $\Omega^p(M)$

$$\mathcal{H}_0^p = \{{}^{\mathbb{C}}\text{Harmonic } p\text{-forms}\}$$

$$H^p(Q) = H^p_{DR}(M;\mathbb{C})$$

$$\mathcal{H}_0^p = H^p_{DR}(M)$$

$$\mathrm{Tr}(-1)^F = \sum_{p=0}^{n} (-1)^p H^p(M) = \chi(M)$$ Euler # of M

Example 1

$$Q = \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + h'(x) & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & -\frac{d}{dx} + h'(x) \\ 0 & 0 \end{pmatrix}$$

$$\Psi(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

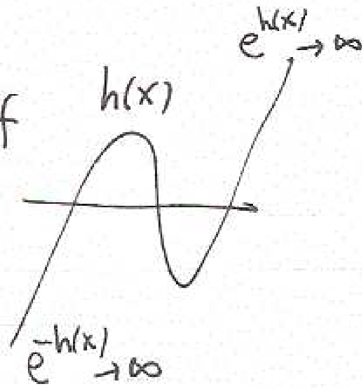
SUSY ground states $Q\Psi = Q^\dagger\Psi = 0$

$$\Leftrightarrow \begin{cases} \left(\frac{d}{dx} + h'(x)\right) f_1(x) = 0 \\ \left(-\frac{d}{dx} + h'(x)\right) f_2(x) = 0 \end{cases}$$

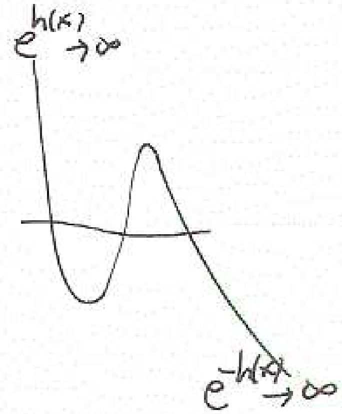
$$\Leftrightarrow \Psi = \begin{pmatrix} c_1 e^{-h(x)} \\ c_2 e^{h(x)} \end{pmatrix}$$

but it must be normalizable.

If



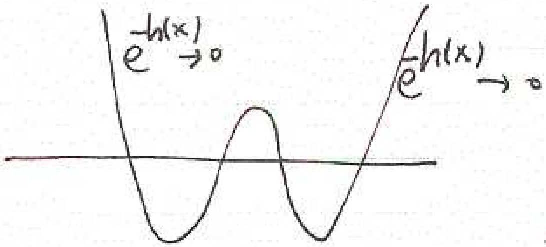
or



$e^{h(x)}$ nor $e^{-h(x)}$ normalizable

$\mathcal{H}_0^{B,F} = 0$, $\mathrm{Tr}(-1)^F = 0$

If

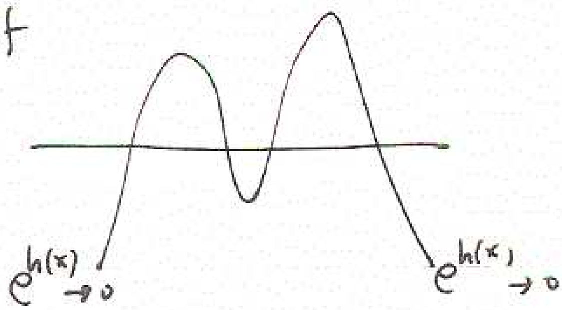


$e^{-h(x)}$ normalizable

$$\mathcal{H}_0^B = \mathbb{C}\begin{pmatrix} e^{-h(x)} \\ 0 \end{pmatrix}, \quad \mathrm{Tr}(-1)^F = 1$$

$$\mathcal{H}_0^F = 0$$

If



$e^{h(x)}$ normalizable

$$\mathcal{H}_0^B = 0$$

$$\mathcal{H}_0^F = \mathbb{C}\begin{pmatrix} 0 \\ e^{h(x)} \end{pmatrix}, \quad \mathrm{Tr}(-1)^F = -$$

How do we find SUSY QM?

Usually: we start with a classical system

Variables, Lagrangian

& then quantize.

Example 1

$X(t)$: real bosonic field

$\psi(t)$: complex fermionic field $(\bar{\psi} := \psi^*)$

$$L = \frac{\dot{x}^2}{2} + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \frac{1}{2}(h'(x))^2 - h''(x)\bar{\psi}\psi.$$

$S = \int L\,dt$ is invariant under

$$(\star)\quad \begin{cases} \delta X = \epsilon\bar{\psi} - \bar{\epsilon}\psi \\ \delta\psi = \epsilon(i\dot{X} + h'(x)) \\ \delta\bar{\psi} = \bar{\epsilon}(-i\dot{X} + h'(x)) \end{cases}$$

i.e. $\delta S = \int \delta L\,dt = \int \partial_t(\cdots)\,dt = 0$

$(\star)$ is a symmetry of the classical system

The system is also time-translation sym $\delta X = \dot{X}$, $\delta\psi = \dot{\psi}$

fermion # sym $\psi \to e^{i\alpha}\psi$, $X \to X$.

for $\begin{matrix} \delta_1 \leftrightarrow \epsilon_1 \\ \delta_2 \leftrightarrow \epsilon_2 \end{matrix} \Big]$ $[\delta_1, \delta_2] X = 2i(\epsilon_1 \bar{\epsilon}_2 - \epsilon_2 \bar{\epsilon}_1) \dot{X}$

$[\delta_1, \delta_2] \psi = 2i(\quad " \quad) \dot{\psi}$

$\qquad\qquad\quad \bar{\psi} \qquad\qquad\qquad \dot{\bar{\psi}}$

(if we use EOM)

δ squares to time translation $\leftrightarrow \{Q, Q^\dagger\} = 4H$

-- supersymmetry.

Symmetry $\leftrightarrow$ conserved charge

- Let ϵ depend on time $\epsilon(t)$:

$$\delta S = \int \left(-i\dot{\epsilon} Q - i\dot{\bar{\epsilon}} \overline{Q}\right) dt$$

$$Q = \bar{\psi}\left(i\dot{X} + h'(x)\right)$$

$$\overline{Q} = \psi\left(-i\dot{X} + h'(x)\right) = Q^\dagger$$

- time translation $\leftrightarrow$ Hamiltonian (Energy) $= \frac{\dot{X}^2}{2} + \frac{1}{2}(h'(x))^2 + h''(x)\frac{\bar{\psi}\psi - \psi\bar{\psi}}{2}$
- fermion # $\leftrightarrow$ $F = \bar{\psi}\psi$.

Quantization $\left\langle \begin{array}{l} \text{path integral} \\ \text{operator formalism.} \Leftarrow \end{array} \right.$ $p = \hat{\dot{x}}$

$$[x,p] = i \ , \ [x,x] = [p,p] = 0 \qquad [x,\psi] = 0 \text{ etc.}$$

$$\{\psi,\bar{\psi}\} = 1 \qquad \{\psi,\psi\} = \{\bar{\psi},\bar{\psi}\} = 0$$

representation : $\left.\begin{array}{l} x = x \times \\ p = -i\frac{d}{dx} \end{array}\right\}$ on $L^2(\mathbb{R},\mathbb{C})$

$$\psi|0\rangle = 0 \ , \quad \bar{\psi}|0\rangle \neq 0 \ , \quad \psi\bar{\psi}|0\rangle = |0\rangle \ ,$$

$$\mathbb{C}^2 = \{|0\rangle, \bar{\psi}|0\rangle\} \qquad \psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\therefore \mathcal{H} = L^2(\mathbb{R};\mathbb{C})|0\rangle \oplus L^2(\mathbb{R};\mathbb{C})\bar{\psi}|0\rangle$$

$$\left\{\begin{array}{l} Q = \bar{\psi}(ip + h'(x)) = \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + h'(x) & 0 \end{pmatrix} \\ Q^\dagger = \psi(-ip + h'(x)) = \begin{pmatrix} 0 & -\frac{d}{dx} + h'(x) \\ 0 & 0 \end{pmatrix} \\ H = \frac{p^2}{2} + \frac{1}{2}(h'(x))^2 + h''(x)\underbrace{\frac{\bar{\psi}\psi - \psi\bar{\psi}}{2}}_{\text{choice here.}} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ F = \bar{\psi}\psi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right.$$

$\rightsquigarrow$ SUSY algebra.

Example 2 (M, g) compact oriented Riemannian mfd

$X^I(t)$ bosonic variable valued in $\left(\begin{smallmatrix}\text{local}\\ \text{coordinates of}\end{smallmatrix}\right) M$

$\psi^I(t)$ fermionic tangent vector of M at $X(t)$.

or $\quad X : \mathbb{R} = \{t\} \longrightarrow M \quad$ map

$\quad \psi$: a section of $X^* TM$ over $\mathbb{R} = \{t\}$

$$L = \frac{1}{2} g_{IJ}(X)\dot{X}^I \dot{X}^J + i g_{IJ}(X)\overline{\psi}^I D_t \psi^J - \frac{1}{2} R_{IJKL}\psi^I \overline{\psi}^J \psi^K \overline{\psi}^L$$

$$D_t \psi^J = \partial_t \psi^J + \dot{X}^I \Gamma^J_{IK} \psi^K$$

Levi-Civita connection

R_{IJKL} Riemann curvature

Supersymmetry

$$\begin{cases} \delta X^I = \epsilon \overline{\psi}^I - \overline{\epsilon}\psi^I \\ \delta \overline{\psi}^I = \overline{\epsilon}(-i\dot{X}^I - \Gamma^I_{JK}\overline{\psi}^J \psi^K) \\ \delta \psi^I = \epsilon(i\dot{X}^I - \Gamma^I_{JK}\overline{\psi}^J \psi^K) \end{cases}$$

Conserved charge (supercharge)

$$\delta S = -i \int (\dot{\epsilon} Q + \dot{\bar{\epsilon}} \bar{Q})\, dt$$

$$Q = g_{IJ} \bar{\psi}^I i \dot{X}^J$$

$$\bar{Q} = g_{IJ} \psi^I i \dot{X}^J$$

Quantization

$$p_I = \frac{\partial L}{\partial \dot{X}^I} = g_{IJ} \dot{X}^J$$

$$[X^I, p_J] = i\delta^I_J \qquad [X^I, X^J] = [p_I, p_J] = 0$$

$$\{\psi^I, \bar{\psi}^J\} = g^{IJ} \qquad \{\psi^I, \psi^J\} = \{\bar{\psi}^I, \bar{\psi}^J\} = 0$$

$$Q = \bar{\psi}^I i p_I \,, \quad \bar{Q} = -i p_I \psi^I \,, \quad H = \frac{1}{2}\{Q, \bar{Q}\}$$

representation

$$\mathcal{H} = \Omega^{\cdot}(M)$$

$$\left.\begin{array}{l} \cancel{X}^I \longleftrightarrow X^I \times \\ p_J \longleftrightarrow -i \frac{\partial}{\partial X^J} \\ \bar{\psi}^J \longleftrightarrow dX^I \wedge \\ \psi^I \longleftrightarrow g^{IJ}\, i_{\left(\frac{\partial}{\partial X^J}\right)} \end{array}\right\} \quad \begin{array}{l} Q = i\bar{\psi}^I p_I = dX^I \wedge \frac{\partial}{\partial X^I} = \\ \bar{Q} = Q^\dagger = d^\dagger \\ H = \frac{1}{2}\{Q, Q^\dagger\} = \frac{1}{2}\Delta \end{array}$$