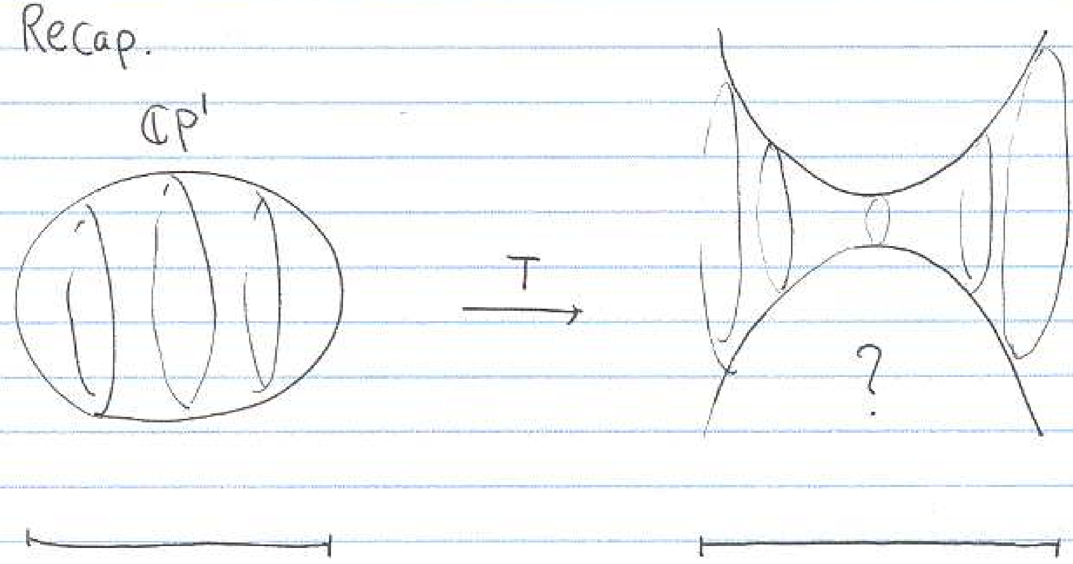
Recap.



<u>good</u>:

momentum conservation $\longleftrightarrow$ winding # conservation

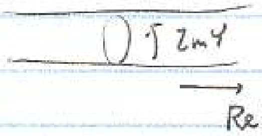
<u>bad</u>:

winding # non-conservation $\overset{?}{\longleftrightarrow}$ momentum seems to be conserved

compact target $\overset{?}{\longleftrightarrow}$ non-compact target

<u>The actual story</u>: $\exists$ superpotential in the mirror side

$$W = e^{-Y} + e^{-t+Y}$$



$-\infty < \mathrm{Re}\,Y < \infty$ by renormalization effect.

- momentum non-conserved ($Y \to Y + i\,\mathrm{const}$ is <u>NOT</u> a symmetry)
- Potential wall in the non-compact direction.

derivation:

$$\begin{tikzcd}
\underline{\text{LSM}}\ \ \Phi_i\text{ chiral} \arrow[r, "T_{\Phi_1}T_{\Phi_2}"] & \underline{\text{mirror LSM}}\ \ \overset{\text{twisted chiral}}{Y_i} \equiv Y_i + 2\pi i
\end{tikzcd}$$

$$\mathcal{L} = \int d^4\theta \left(\bar{\Phi}_1 e^V \Phi_1 + \bar{\Phi}_2 e^V \Phi_2 - \frac{\bar{\Sigma}\Sigma}{2e^2} \right) + \mathrm{Re} \int d^2\tilde{\theta} \, (-t\Sigma)$$

$$\tilde{\mathcal{L}} = \int d^4\theta \left(-K(Y_1, \bar{Y}_1, Y_2, \bar{Y}_2, \Sigma, \bar{\Sigma}, e) \right) + \mathrm{Re} \int d^2\tilde{\theta} \, \tilde{W}(Y_1, Y_2, t, \Sigma)$$

$$\tilde{W} = (Y_1 + Y_2 - t)\Sigma + e^{-Y_1} + e^{-Y_2}$$

$$\begin{tikzcd}
\text{LSM} \arrow[d, "e^2\to\infty"] & \text{mirror LSM} \arrow[d, "e^2\to\infty"] \\
\text{NLSM on } \mathbb{CP}^1,\ \mathbb{C}\text{Kähler} = t & \text{LG model}
\end{tikzcd}$$

$$Y_1 + Y_2 - t = 0, \quad \tilde{W} = e^{-Y_1} + e^{-Y_2}$$

or $\tilde{W} = e^{-Y} + e^{-t+Y}$

Remark ① modulo gauge symmetry, this T-duality is wrt $\Phi_1 \to e^{i\delta}\Phi_1$, $\Phi_2 \to e^{-i\delta}\Phi_2$ — i.e. along S^1 fibre.

② I have only shown the case with $\left\{ \begin{array}{l} \text{single } U(1) \text{ gauge.} \\ \text{single } \Phi \end{array} \right.$

$$\mathcal{L} = \int d^4\theta \left(\bar{\Phi} e^V \Phi - \frac{1}{2e^2}|\Sigma|^2 \right) + \mathrm{Re}\int d^2\theta \, (-t\Sigma) \xrightarrow{\Phi \to Y \equiv Y + 2\pi i} \tilde{W} = (\underbrace{Y}_{\Phi\text{ charged}} - t)\Sigma + \underbrace{e^{-Y}}_{\text{Vortex insted}}$$

I haven't shown why $\underset{\substack{\uparrow \\ \text{non-perturbative correction}}}{\text{(it)}}$ is just the sum $e^{-Y_1} + e^{-Y_2}$

The trick

LSM has one $U(1)$ gauge sym $\Phi_1 \to e^{i\alpha}\Phi_1$, $\Phi_2 \to e^{i\alpha}\Phi_2$

& one $U(1)$ global sym $\Phi_1 \to e^{i\delta}\Phi_1$, $\Phi_2 \to e^{-i\delta}\Phi_2$

promote this global sym to gauge sym.

$$\xrightarrow[\text{change of } U(1)\times U(1) \text{ basis}]{} \left(U(1) \text{ with one } \underset{\Phi_1}{\overset{\Phi}{\shortparallel}} \right) \otimes \left(U(1) \text{ with one } \underset{\Phi_2}{\overset{\Phi}{\shortparallel}} \right)$$

$$\mathcal{L}_{\text{promoted}} = \int d^4\theta \left(\bar{\Phi}_1 e^{V_1} \Phi_1 + \bar{\Phi}_2 e^{V_2} \Phi_2 - \sum_{i,j} \frac{1}{e_{i,j}^2} \bar{\Sigma}_i \Sigma_j \right) + \mathrm{Re} \int d^2\tilde{\theta} \left(-t_1 \Sigma_1 - t_2 \Sigma_2 \right)$$

If $\frac{1}{e_{i,j}^2} \bar{\Sigma}_i \Sigma_j = \frac{1}{e_1^2} |\Sigma_1|^2 + \frac{1}{e_2^2} |\Sigma_2|^2$ … two decoupled system

Dualize $\Phi_1 \to Y_1$, $\Phi_2 \to Y_2$

$$\tilde{\mathcal{L}}_{\text{promoted}} = \tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_2$$

$$= \sum_{i=1}^{2} \left[\int d^4\theta \left(-R(Y_i, \bar{Y}_i, \Sigma_i, \bar{\Sigma}_i, e_i) \right) + \mathrm{Re} \int d^2\tilde{\theta} \left[(Y_i - t_i)\Sigma_i + e^{-Y_i} \right] \right]$$

The original system is recovered by demoting the extra $U(1)$ gauge symmetry to global $U(1)$

Done by $\sum_{i,j} \frac{1}{e_{i,j}^2} \bar{\Sigma}_i \Sigma_j = \frac{1}{e^2} \left| \frac{1}{2} \sum_{i=1}^{2} \Sigma_i \right|^2 + \frac{1}{\epsilon e^2} |\Sigma_1 - \Sigma_2|^2$

$\epsilon \to 0$: constraint $\Sigma_1 = \Sigma_2$ $(=: \Sigma)$

& $t_1+t_2=t$. ⇒ original system.

In the mirror side

$$\widetilde{L} = \int d^4\tilde{\theta}\left(-K(Y_1,\overline{Y}_1,Y_2,\overline{Y}_2,\Sigma,\overline{\Sigma},e)\right)$$

$$+\ \mathrm{Re}\int d^2\tilde{\theta}\ \underbrace{\left[(Y_1-t_1)\Sigma_1+e^{-Y_1}+(Y_2-t_2)\Sigma_2+e^{-Y_2}\right]\Big|_{\substack{\Sigma_1=\Sigma_2=\Sigma\\ t_1+t_2=t}}}_{(Y_1+Y_2-t)\Sigma+e^{-Y_1}+e^{-Y_2}.}$$

<u>Note</u>
- Changing the detail of $\frac{1}{e_{ij}^2}\overline{\Sigma}_i\Sigma_j$ cannot change the twisted superpotential
- But $K(Y_i,\overline{Y}_i,\Sigma,\overline{\Sigma},e)$ uncontrolable, except that it is close to the naive one in the region where the non-perturbative effect is weak — the region where $Y_1, Y_2 \sim \frac{t}{2}$ with t large.

— it is that of flat cylinder.

(of cirumference "very small")

<u>actually 0!</u>

Sausage model ⟷ finite circumference

generalization

$U(1)^k$ $\Phi_1, \cdots \Phi_N$ $\quad\xleftrightarrow{T}\quad$ $U(1)^k, Y_1, \cdots, Y_N$

charge $Q_1^a, \cdots, Q_N^a$ $\qquad \widetilde{W} = \sum_{a=1}^{k} \left(\sum_{i=1}^{N} Q_i^a Y_i - t^a\right)\Sigma_a + e^{-Y_1} + \cdots + e^{-Y_N}$

$\downarrow e^2 \to \infty \qquad\qquad \downarrow e^2 \to \infty$

NLSM on toric mfd $\mathbb{C}^N /\!/ (\mathbb{C}^\times)^k$ $\qquad$ LG model on $(\mathbb{C}^\times)^{N-k}$

$\mathbb{C}$ Kähler $= (t^1, \cdots, t^k)$ $\qquad \widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_N}$, $\sum_{i=1}^{N} Q_i^a Y_i = t^a$

Examples

① $\underline{\mathbb{C}P^{N-1}}$ $\longleftrightarrow$ $\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_{N-1}} + e^{-t+Y_1+\cdots+Y_{N-1}}$ affine Toda

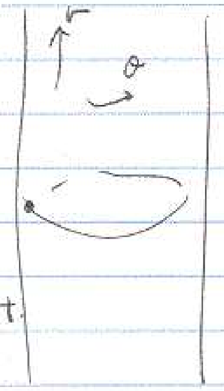
② resolved conifold

$U(1)$ $\quad \Phi_1, \Phi_2, \Phi_3, \Phi_4$

charge $\quad 1 \quad 1 \quad -1 \quad -1$ $\qquad \Sigma$ charge $= 1+1-1-1 = 0$.

$\rightarrow U(1)_A$ anomaly free

$t = r - i\theta$ genuine parameter



t=0 is the singular pt.

$$M_{vac} = \{ |\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 = r \} / U(1)$$

$r>0$, $r<0$ = total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$

$r>0$: (ϕ_1, ϕ_2) base (ϕ_3, ϕ_4) fibre

$r<0$: (ϕ_3, ϕ_4) base (ϕ_1, ϕ_2) fibre.

This is so called the resolved conifold

$x = \phi_1 \phi_3$, $y = \phi_2 \phi_4$, $z = \phi_1 \phi_4$, $w = \phi_2 \phi_3$ -- gauge inv.

$$\underline{xy = zw}$$

As a hypersurface in $\mathbb{C}^4$, it is singular at

$x = y = z = w = 0$.

In Mvac, this point is resolved to $\mathbb{CP}^1$. $\left(\begin{array}{l} r > 0 : \phi_3 = \phi_4 = 0 \\ r < 0 : \phi_1 = \phi_2 = 0 \end{array} \right)$

$r_{\gg 0} \leftrightarrow r \ll 0$. called <u>flop transition</u>

<u>The mirror</u> $Y_1 + Y_2 - Y_3 - Y_4 = t$

$$\widetilde{W} = e^{-Y_1} + e^{-Y_2} + e^{-Y_3} + e^{-Y_4}$$

Solve constraints by $Y_1 = Y_0 + \Theta_1$

$Y_2 = Y_0 + \Theta_2$

$Y_3 = Y_0 + \Theta_1 + \Theta_2$

$Y_4 = Y_0 + t$

$$\widetilde{W} = e^{-Y_0} \left(e^{-\Theta_1} + e^{-\Theta_2} + e^{-\Theta_1 - \Theta_2} + e^{t} \right)$$

LG model on $(\mathbb{C}^*)^3 = \{ (Y_0, \Theta_1, \Theta_2) \}$

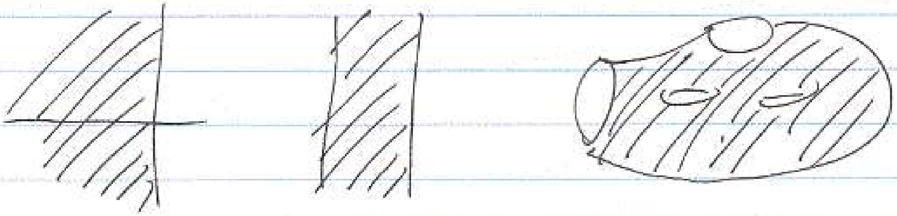
There is another description of the mirror.

To find it, we make a brief digression.

Digression: D-brane (brief. more detail later)

So far, we have been considering worldsheet without boundary ($\mathbb{R}^2$, $\mathbb{R}\times S^1$, Σ closed Riem surface)

But we can also consider the worldsheet with boundary



To define a worldsheet theory in such a case, we need to specify the boundary condition.

This is called D-brane.

Example: NLSM with target X.

$\gamma \subset X$ submanifold

B.C: $\partial\Sigma$ is mapped to γ

(+ suitable detail $\left\{ \begin{array}{l} \text{Neumann b.c. in tangent direction} \\ \text{b.c. for fermions} \end{array} \right.$)

— called D-brane wrapped on γ

When the theory has <u>(2,2) SUSY</u>,

we are interested in B.C.'s preserving $\frac{1}{2}$ of them.

Preserve $Q_A = \overline{Q}_+ + Q_-\,,\ \overline{Q}_A = Q_+ + \overline{Q}_-$ ··· A-brane

Preserve $Q_B = \overline{Q}_+ + \overline{Q}_-\,,\ \overline{Q}_B = Q_+ + Q_-$ ··· B-brane.

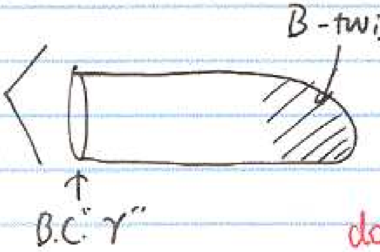
For SUSY σ-model on a Kähler target $X \left\langle \begin{array}{l} \omega : \text{Symplectic} \\ J : \text{cplx} \end{array} \right.$

D-brane wrapped on $\gamma \subset (X, \omega)$ Lagrangian $\Rightarrow$ A-brane

D-brane wrapped on $\gamma \subset (X, J)$ cplx submfd $\Rightarrow$ B-brane.

- Consider a (2,2) theory which is B-twistable.

Pick an A-brane "γ". $\left(\begin{array}{l} \text{Lagrangian submfd} \subset CY \in \text{NLSM} \\ \text{Lagrangian} \subset (\mathrm{Im}\,W)^{-1}(*) \in \text{LG model} \end{array} \right)$

$Z(\gamma) = \Big\langle$  $\Big\rangle =$

$$\begin{cases} \int_\gamma \Omega & (\text{NLSM on } CY) \\ \int_\gamma e^{-W}\Omega & (\text{LG model}) \end{cases}$$

Period integral

weighted period integral

doesn't depend on twisted chiral parameter (e.g. Kähler)

Ω ··· holomorphic volume form.

spacetime

$Z(\gamma)$ characterize the mass of the D-brane γ. (End digression).

Back to the mirror of the resolved conifold.

$$\widetilde{W} = e^{-Y_0}\left(e^{-\Theta_1} + e^{-\Theta_2} + e^{-\Theta_1-\Theta_2} + e^{t}\right)$$

Consider the LG model with two variables (U, V) with

$$\Delta\widetilde{W} = U \cdot V$$

This is massive and is empty in the IR limit.

Thus the theory with superpotential $\widetilde{W} + \Delta\widetilde{W}$ determines the same IR fixed point as $\widetilde{W}$.

Consider an A-brane γ in this theory.

$$Z(\gamma) = \int_\gamma e^{-\widetilde{W} - \Delta\widetilde{W}}\, dY_0\, d\Theta_1\, d\Theta_2\, dU\, dV$$

Change of variable: $U = e^{-Y_0} u$, $V = v$

$$Z(\gamma) = \int_\gamma e^{-e^{-Y_0}\left(e^{-\Theta_1} + e^{-\Theta_2} + e^{-\Theta_1-\Theta_2} + e^{t} + uv\right)} e^{-Y_0} dY_0\, d\Theta_1 d\Theta_2\, du\, dv$$

$$= \int_\gamma \delta\left(e^{-\Theta_1} + e^{-\Theta_2} + e^{-\Theta_1-\Theta_2} + e^{t} + uv\right) d\Theta_1 d\Theta_2\, du\, dv$$

This is a period integral (of some cycle) in

$$Y = \{ e^{-\Theta_1} + e^{-\Theta_2} + e^{-\Theta_1 - \Theta_2} + e^{t} + uv = 0 \} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^2$$

wrt $\Omega = d\Theta_1 \, d\Theta_2 \frac{du}{u}$

This implies that the mirror can be described as the NLSM (no superpotential!) on Y.

Y's defining eqn can be written as

$$(e^{-\Theta_1} + 1)(e^{-\Theta_2} + 1) + uv = 1 - e^{t}$$

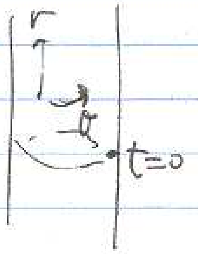
At $t = 0$ (where the original theory is singular), Y develops the conifold singularity $\left(\text{at } \begin{matrix} e^{-\Theta_1} = e^{-\Theta_2} = -1 \\ u = v = 0 \end{matrix} \right)$

Away from $t = 0$, it is deformed to a smooth space.

Thus, we find the mirror relation

resolved conifold $\longleftrightarrow$ deformed conifold

$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ in $(\mathbb{C}^*)^2 \times \mathbb{C}^2$



Example ③ A_k - singularity.

LSM : $U(1)^k$

$$\begin{array}{ccccccccc}
\Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \cdots & \Phi_k & \Phi_{k+1} & \Phi_{k+2} \\
1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & \cdots & & \\
\vdots & & . & . & 1 & -2 & 1 & \\
0 & 0 & 0 & -0 & - & 1 & -2 & 1
\end{array}$$

gauge invariants

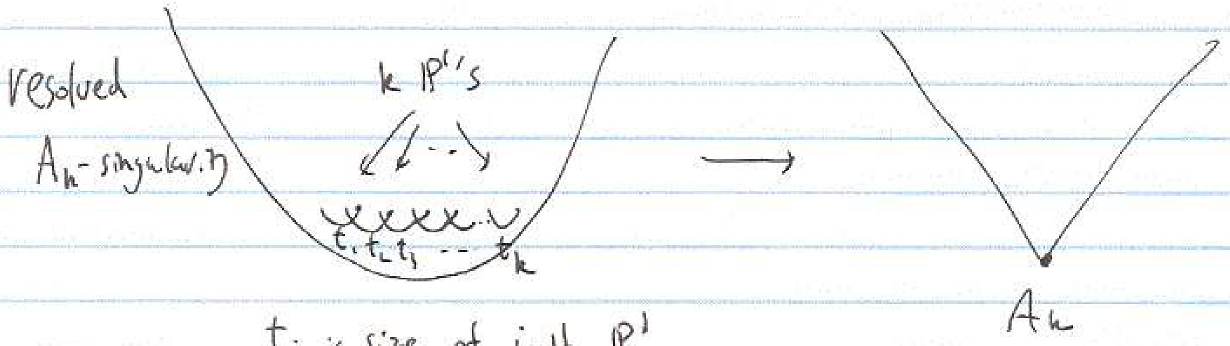
$$x = \Phi_1^{k+1} \Phi_2^{k} \Phi_3^{k-1} \cdots \Phi_k^{2} \Phi_{k+1}$$

$$y = \Phi_2 \Phi_3^{2} \cdots \cdots \Phi_{k+1}^{k} \Phi_{k+2}^{k+1}$$

$$z = \Phi_1 \Phi_2 \cdots \Phi_{k+1} \Phi_{k+2}$$

They obey $xy = z^{k+2}$ … A_k-singularity

$$M_{vac} \longrightarrow \{ xy = z^{k+2} \} \subset \mathbb{C}^3$$



t_i : size of i-th $\mathbb{P}^1$.

same as $\mathbb{C}^2/\mathbb{Z}_{k+1}$

The mirror: $\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_{k+2}}$

$$\begin{aligned} Y_1 - 2Y_2 + Y_3 &= t_1 \\ Y_2 - 2Y_3 + Y_4 &= t_2 \\ &\ddots \\ Y_k - 2Y_{k+1} + Y_{k+2} &= t_k \end{aligned}$$

Solved by
$$\begin{aligned} Y_2 &= Y_1 + \Theta \\ Y_3 &= t_1 + Y_1 + 2\Theta \\ Y_4 &= t_2 + 2t_1 + Y_1 + 3\Theta \\ &\vdots \\ Y_{k+2} &= t_k + 2t_{k-1} + \cdots + kt_1 + Y_1 + (k+1)\Theta \end{aligned}$$

$$\widetilde{W} = e^{-Y_1}\Big(\underbrace{1 + e^{-\Theta} + e^{-t_1-2\Theta} + e^{-t_2-2t_1-3\Theta} + \cdots + e^{-t_k-\cdots-kt_1-(k+1)\Theta}}_{P_{k+1}(e^{-\Theta}, e^{-t_1}, \dots, e^{-t_k})}\Big)$$

As in the resolved conifold case, $\exists$ alternative description:

NLSM on $Y = \{ P_{k+1}(e^{-\Theta}, e^{-t_1}, \dots, e^{-t_k}) + uv = 0 \} \subset \mathbb{C}^* \times \mathbb{C}^2$

$\uparrow$ deformation of $c(e^{-\Theta}+1)^{k+1} + uv = 0$

$$\text{resolved } A_k \xleftrightarrow{\text{mirror}} \text{deformed } A_k$$

in $\mathbb{C}^* \times \mathbb{C}^2$

Example ④ $\mathcal{O}(-d) \to \mathbb{C}P^{N-1}$ or $\mathbb{C}^N/\mathbb{Z}_d$

$U(1)$	P	$\Phi_1,$	$\cdots,$	Φ_N
	$-d$	1	$\cdots$	1

No superpotential

$$\begin{cases} r \gg 0 : & \mathcal{O}(-d) \to \mathbb{C}P^{N-1} \\ r \ll 0 : & \mathbb{C}^N/\mathbb{Z}_d \text{ orbifold} \end{cases}$$

$d = N$: non compact CY, $t = r - i\theta$ free parameter

Singular at $e^t = (-N)^N$

$d < N$: $r \gg 0 \xrightarrow{\text{flow}} r \ll 0$

$d > N$: $r \ll 0 \xrightarrow{\text{flow}} r \gg 0$

Mirror $\widetilde{W} = e^{-Y_1} + e^{-Y_2} + \cdots + e^{-Y_N} + e^{-Y_P}$

$$Y_1 + Y_2 + \cdots + Y_N - dY_P = t$$

Solved by $\begin{cases} Y_i = d \cdot Z_i & i = 1 \cdots N \\ Y_P = -\frac{t}{d} + Z_1 + \cdots + Z_N \end{cases}$

$(Y) \leftarrow (Z)$ $1 : (\mathbb{Z}_d)^{N-1}$: $e^{-Z_i} \to \omega_i e^{-Z_i}$

$\omega_i^d = 1$, $\omega_1 \cdots \omega_N = 1$

$$\widetilde{W} = \frac{(e^{-Z_1})^d + \cdots + (e^{-Z_N})^d + e^{\frac{t}{d}} e^{-Z_1} \cdots e^{-Z_N}}{(\mathbb{Z}_d)^{N-1}}$$

LG orbifold.

Hypersurface in $\mathbb{C}P^{N-1}$

$U(1)$ $\quad \Phi_1, \cdots, \Phi_N, P$ $\quad$ with $W = P\,G(\Phi_1, \ldots, \Phi_N)$

$\qquad\quad 1 \;\cdots\; 1 \quad -d$ $\qquad\qquad\qquad \uparrow$ degree d polynomial

$W \supset P\,\Phi_1^{\star} \cdots \Phi_N^{\star}$

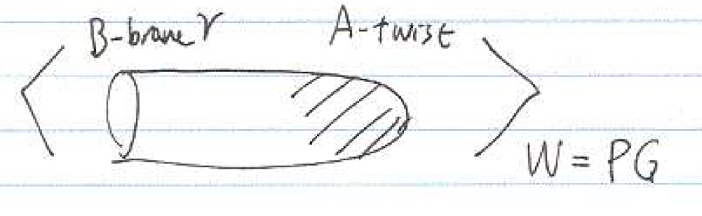
breaks symmetry of phase rotation $\Phi_i \to e^{i\alpha_i}\Phi_i$ $\longleftrightarrow$ winding # conservation of $Y_1, \cdots, Y_N, Y_P$ must be broken.

Claim $X_i = e^{-Z_i}$ are good variables (allowed to take $X_i = 0$)

So the mirror is LG orbifold:

$$\widetilde{W} = X_1^d + \cdots + X_N^d + e^{t/d} X_1 \cdots X_N \quad /(\mathbb{Z}_d)^{N-1}$$

Explanation



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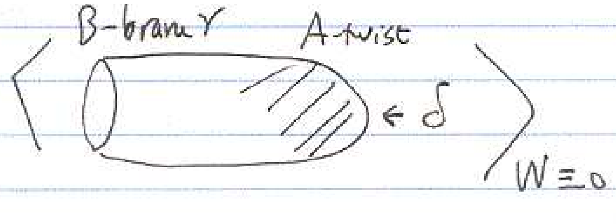
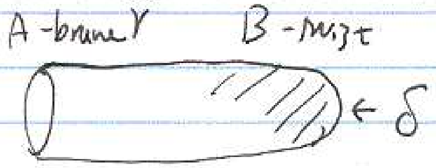


$\delta = d \cdot \Sigma$

$\overset{\text{mirror}}{=} \Big\langle$



$\Big\rangle$ mirror LSM (of $W \equiv 0$ theory)

$$= \int_{\tilde{\gamma}} e^{-\Sigma(Y_1+\cdots+Y_N-dY_p-t)-e^{-Y_1}-\cdots-e^{-Y_N}-e^{-Y_p}} \overbrace{d\cdot\Sigma}\, d\Sigma\, dY_1\cdots dY_N\, dY_p$$

$$= d\cdot\frac{\partial}{\partial t}\int_{\tilde{\gamma}} e^{-\Sigma(Y_1+\cdots+Y_N-dY_p-t)-e^{-Y_1}-\cdots-e^{-Y_p}}\, d\Sigma\, dY_1\cdots dY_N\, dY_p$$

$$= d\cdot\frac{\partial}{\partial t}\int_{\tilde{\gamma}'} e^{-((e^{-z_1})^d+\cdots+(e^{-z_N})^d+e^{\frac{t}{d}}e^{-z_1}\cdots e^{-z_N})}\frac{dz_1\cdots dz_N}{(\mathbb{Z}_d)^{N-1}}$$

$$= e^{\frac{t}{d}}\int_{\tilde{\gamma}'} e^{-[(e^{-z_1})^d+\cdots+(e^{-z_N})^d+e^{\frac{t}{d}}e^{-z_1}\cdots e^{-z_N}]}\frac{e^{-z_1}dz_1\cdots e^{-z_N}dz_N}{(\mathbb{Z}_d)^{N-1}}$$

$$=(-1)^N e^{\frac{t}{d}}\int_{\tilde{\gamma}'} e^{-[X_1^d+\cdots+X_N^d+e^{\frac{t}{d}}X_1\cdots X_N]}\frac{dX_1\cdots dX_N}{(\mathbb{Z}_d)^{N-1}}$$

$\Longrightarrow$ Claim

~~explanation of (*)~~

Alternatively, for $d \leq N$:

$$Z(\gamma) = \int_{\tilde{\gamma}} d\Sigma\, dY_1 - dY_p \frac{\partial}{\partial Y_p}\left(e^{-\Sigma(Y_1+\cdots+Y_N-dY_p-t)}\right)e^{-e^{-Y_1}-\cdots-e^{-Y_p}}$$

$$= -\int_{\tilde{\gamma}} d\Sigma\, dY_1\cdots dY_N\, e^{-Y_p}dY_p\, e^{-\Sigma(Y_1+\cdots+Y_N-dY_p-t)}e^{-e^{-Y_1}-\cdots-e^{-Y_p}}$$

$$= \int dY_1 \cdots dY_N \, de^{-Y_P} \, \delta(Y_1 + \cdots + Y_N - dY_P - t) \, e^{-e^{-Y_1} - \cdots - e^{-Y_N} - e^{-Y_P}}$$

$$e^{-Y_P} = \tilde{P}$$

$$e^{-Y_i} = \tilde{P} U_i \qquad i = 1, \cdots, d$$

$$e^{-Y_j} = U_j \qquad j = d+1, \cdots, N$$

$$Z(r) = \int \prod_{i=1}^{N} \frac{dU_i}{U_i} \, d\tilde{P} \, \delta\Big(\log \prod_{i=1}^{N} U_i + t\Big) \, e^{-\tilde{P}\left(\sum_{i=1}^{d} U_i + 1\right) - \sum_{i=d+1}^{N} U_i}$$

$$= \int \prod_{i=1}^{N} \frac{dU_i}{U_i} \, \delta\Big(\log \prod_{i=1}^{N} U_i + t\Big) \, \delta\Big(\sum_{i=1}^{d} U_i + 1\Big) \, e^{-\sum_{i=d+1}^{N} U_i}$$

Period of $\left\{ \begin{array}{l} \prod_{i=1}^{N} U_i = e^{-t} \\ U_1 + \cdots + U_d + 1 = 0 \end{array} \right\} \subset (\mathbb{C}^\times)^N$

$$\tilde{W} = U_{d+1} + \cdots + U_N$$

This can be (partially) compactified.

<u>d=N</u> $\qquad U_i = e^{-\frac{t}{N}} \dfrac{X_i^N}{X_1 \cdots X_N} \qquad 1 : (\mathbb{C}^\times) \times (\mathbb{Z}_N)^{N-2}$

$$Z(r) = \int \frac{1}{\mathrm{vol}(\mathbb{C}^\times)\,\text{[illegible]}^{N-2}} \prod_{i=1}^{N} \frac{dX_i}{X_i} \, \delta\left(\frac{X_1^N + \cdots + X_N^N + e^{t/N} X_1 \cdots X_N}{X_1 \cdots X_N}\right)$$

$$= \int \prod_{i=1}^{N-1} dX_i \, \delta\left(\widetilde{G}(X_1, \cdots, X_{N-1}, 1) \right) \qquad \widetilde{G} = X_1^N + \cdots + X_N^N + e^{\frac{t}{N}} X_1 \cdots X_N$$

$$= \int \prod_{i=1}^{N-2} dX_i \Big/ \frac{\partial \widetilde{G}}{\partial z_{N-1}} \Big|_{G=0} = \int \Omega$$

Perid m

$$\left\{ X_1^N + \cdots + X_N^N + e^{\frac{t}{N}} X_1 \cdots X_N = 0 \right\} \Big/ (\mathbb{C}^\times) \times \mathbb{Z}_N^{N-2}$$

$d < N$

$$U_i = e^{-\frac{t}{d}} \frac{X_i^d}{X_1 \cdots X_N} \qquad i = 1 \cdots d$$

$$1 : (\mathbb{C}^\times) \times (\mathbb{Z}_d)^{N-2}$$

$$U_j = X_j^d \qquad j = d+1, \cdots, N$$

$$Z = \int \Omega_{d-2} \wedge dX_{d+1} \cdots dX_N \, e^{-(X_{d+1}^d + \cdots X_N^d)}$$

Period of LG ~~model~~ orbifold $/ (\mathbb{Z}_d)^{N-2}$

$$\widehat{W} = X_{d+1}^d + \cdots + X_N^d$$

on

$$\begin{tikzcd}
\left\{ X_1^d + \cdots + X_d^d + e^{\frac{t}{d}} X_1 \cdots X_d \cdot X_{d+1} \cdots X_N = 0 \right\} \Big/ \mathbb{C}^\times \arrow[d, "CY^{d-2}\text{-fold.}"] \\
\mathbb{C}^{N-d}
\end{tikzcd}$$

explanation of (☆)

$\Big|$ 

$\leftarrow \delta \Big\rangle_{W=0}$ … normalizable SUSY ground state

with $q_A = -\underbrace{N}_{\overset{\shortparallel}{\dim(\mathcal{O}(-d)\to\mathbb{CP}^{N-1})}} + \underbrace{2}_{\Rightarrow\, q_A(\delta)}$

$\Big\downarrow\ W = \varepsilon PG$

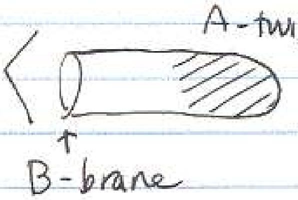
This continues to be a SUSY ground state

(of course normalizable)

with $q_A = -(N-2)$.

∃ unique such state : $\Big|$ 

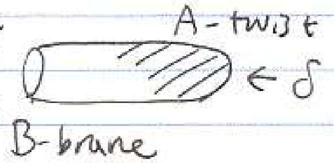
$\Big\rangle_{W=\varepsilon PG}$

$\Big\langle$ 

$\Big\rangle_{W=\varepsilon PG}$ is indep. of chiral parameters.

in particular ε-indepenent.

$\Big\|\ \varepsilon \to 0$

$\Big\langle$ 

$\Big\rangle_{W=0}$

//