SUSY ground states

a, b : branes preserving the same SUSY

(e.g. both A-branes or both B-branes)

$\mathcal{H}_{a,b}$ (Space of openstring states) has operators

Q, $Q^\dagger$, $\cancel{H}$, F (or $(-1)^F$) s.t.

$$Q^2 = 0, \quad [F, Q] = Q, \quad \{Q, Q^\dagger\} = H.$$

$$\mathcal{H}^{\cdot}_{\text{zero energy}} = \mathcal{H}^{\cdot}_{SUSY} \cong H^{\cdot}(Q\text{-complex})$$

Approximation with "smaller" $\mathcal{H}_{a,b}$ is often useful.

B-branes in NLSM

X : Kähler mfd

a, b : branes wrapped on X, supporting holomorphic vector b'dles with hermitian connections

$$\begin{array}{ccc} E_a, A_a & & E_b, A_b \\ \downarrow & & \downarrow \\ X & & X \end{array}$$

$$Q = \int_0^\pi d\sigma \left\{ g_{i\bar{j}} (\bar{\psi}_+^{\bar{j}} + \bar{\psi}_-^{\bar{j}}) \partial_t \phi^i - g_{i\bar{j}} (\bar{\psi}_+^{\bar{j}} - \bar{\psi}_-^{\bar{j}}) \partial_\sigma \phi^i \right\}$$

$$+ (\bar{\psi}_+^{\bar{j}} + \bar{\psi}_-^{\bar{j}}) A_{\bar{j}}^{(b)} \Big|_{\sigma=\pi} - (\bar{\psi}_+^{\bar{j}} + \bar{\psi}_+^{\bar{j}}) A_{\bar{j}}^{t(a)} \Big|_{\sigma=0}$$

Space of states :

$$\Omega(X) = \{ \phi : [0,\pi] \to X \mid \text{Neumann b.c.} \}$$

$$ev_\sigma : \Omega(X) \to X \qquad \phi \mapsto \phi(\sigma)$$

$$\mathcal{H} = \Omega^{0,\bullet}\left(\Omega(X) , ev_0^* E_a^* \otimes ev_\pi^* E_b \otimes \Lambda^\bullet T_{\Omega(X)} \right)$$

— "huge" !

σ- dependent modes have positive energy $\sim g\left(\substack{\text{metric} \\ \text{on } X}\right)$.

$\rightsquigarrow$ zero mode approximation : restrict to σ-indep modes:

$$\phi^I(t), \quad \psi_+^I(t) \underset{\uparrow}{=} \psi_-^I(t) = \tfrac{1}{2} \psi^I(t)$$

by b.c. $\psi_+^I = \psi_-^I$ at $\sigma = 0, \pi$.

$$\mathcal{H}^{\text{zero mode}} = \Omega^{0,\bullet}(X, E_a^* \otimes E_b)$$

$$[\bar{\phi}^{\bar{j}}, \phi^i] = i g^{i\bar{j}} , \quad \{ \psi^i, \bar{\psi}^{\bar{j}} \} = g^{i\bar{j}}$$

$$\phi^i \sim g^{i\bar{j}} \frac{\partial}{\partial \bar{z}^{\bar{j}}} , \quad \bar{\psi}^{\bar{j}} \sim d\bar{z}^{\bar{j}} \wedge , \quad \psi^i \sim g^{i\bar{j}} \iota_{\frac{\partial}{\partial \bar{z}^{\bar{j}}}}$$

$$Q^{\text{zero mode}} = d\bar{z}^{\bar{j}} \frac{\partial}{\partial \bar{z}^{\bar{j}}} + d\bar{z}^{\bar{j}} A^{(b)}_{\bar{j}} - d\bar{z}^{\bar{j}}\, {}^{t}A^{(a)}_{\bar{j}}$$

$$= \bar{\partial} + A_b^{0,1} - {}^{t}A_a^{0,1}$$

$$= \bar{\partial}_{E_a^* \otimes E_b} \quad \text{Dolbeault Operator.}$$

$$\mathcal{H}^{\text{zero mode}}_{SUSY} \cong \bigoplus_p H^{0,p}(X, E_a^* \otimes E_b) \quad \text{Dolbeault cohomology}$$

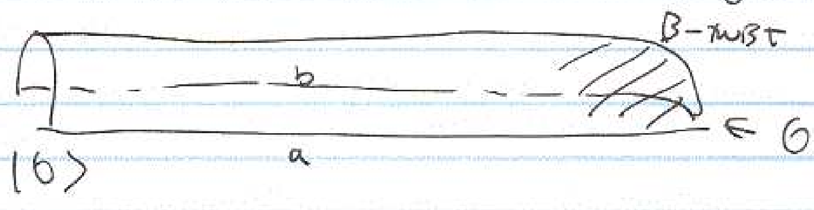
$$\left(= \bigoplus_p \mathrm{Ext}^p(\mathcal{E}_a, \mathcal{E}_b) \right) \text{ the form that generalizes to coherent sheaves}$$

<u>Remark</u> $\mathcal{H}_{SUSY} \cong \mathcal{H}^{\text{zero mode}}_{SUSY}$ looks obvious

but no strong proof, except in the case ① & ②

① X: Calabi-Yau

SUSY ground states $\longleftrightarrow$ chiral ring



$$\to \mathcal{H}_{SUSY} \cong \mathcal{H}^{\text{zero-mode}}_{SUSY}$$

② E_a, E_b "exceptional pair"

$$H^{0,p}(X, E_a^* \otimes E_b) \neq 0 \text{ only at one value of } p$$

$\Rightarrow$ no danger of "$\underbrace{\text{pair}}_{\text{boson-fermion}}$ lifting to $E>0$"

at finite volume (i.e. away from zero mode approx.)

(~~or~~ also if $H^{0,p}(X, E_a^* \otimes E_b) \neq 0$ only at even p (or only at odd p))

$$\rightsquigarrow \quad \mathcal{H}_{SUSY} \cong \mathcal{H}_{SUSY}^{\text{zero energy}}$$

Witten index

$$\operatorname*{Tr}_{a,b}(-1)^F = \sum_i (-1)^i \dim \mathcal{H}^i_{SUSY}$$

$$= \sum_i (-1)^i \dim H^{0,i}(X, E_a^* \otimes E_b)$$

$$= \chi(E_a, E_b)$$

$$\overset{\text{RR, or AS}}{=} \int_X ch(E_a^*)\, ch(E_b)\, Td(X).$$

assume noncompact CY.

B-brane in LG X = target, W = superpotential

$Z_a, Z_b \subset X$ cplx submflds

$W \equiv W_a$ on Z_a, $W \equiv W_b$ on Z_b.

$$Q = \int_0^\pi d\sigma \left\{ g_{i\bar{j}} (\overline{\psi}_+^{\bar{j}} + \overline{\psi}_-^{\bar{j}}) \partial_t \phi^i - g_{i\bar{j}} (\overline{\psi}_+^{\bar{j}} - \overline{\psi}_-^{\bar{j}}) \partial_\sigma \phi^i \right.$$
$$\left. + (\psi_-^i - \psi_+^i) \partial_i W \right\}$$

($A_a = A_b = 0$ for simplicity)

Canonical commutation relation

$$\{ \psi_\pm^i(\sigma), \overline{\psi}_\pm^{\bar{j}}(\sigma') \} = g^{i\bar{j}} \delta(\sigma - \sigma'), \ \ldots$$

$$\Rightarrow Q^2 = -\int_0^\pi d\sigma \, \partial_\sigma \phi^i \partial_i W = -W_b + W_a$$

$\therefore$ Unless $W_b = W_a$ (ie. Z_a, Z_b in the same level set)

$Q^2 \neq 0$ Fails to make a complex!

$W_a = W_b$ assumed.
zeromode approx:

$$\mathcal{H}_{a,b}^{\text{zero mode}} = \bigoplus_{p,q} \Omega^{0,p}(Z_a \cap Z_b, \Lambda^q (N_{Z_a} \cap N_{Z_b}))$$

$$Q^{\text{zero}} = \bar{\partial} + \partial W \cdot$$

contraction on $\Lambda^\bullet(N_{Z_a} \cap N_{Z_b})$

- $Z_a \cap Z_b = \phi \Rightarrow H^\cdot(Q) = 0$

- $Z_a = Z_b = \text{point } \{p\}$ $\quad N_{\{p\}} = \mathbb{C}^n \quad (\dim X = n)$

$$\mathcal{H}^{zero}_{p,p} = \Omega^{0,0}(\{p\}, \Lambda^\cdot \mathbb{C}^n) = \Lambda^\cdot \mathbb{C}^n$$

$$Q^{zero} = \partial W(p) \cdot$$

$$0 \leftarrow \mathbb{C} \xleftarrow{\partial W(p)\cdot} \mathbb{C}^n \xleftarrow{\partial W(p)\cdot} \Lambda^2 \mathbb{C}^n \leftarrow \cdots \xleftarrow{\partial W(p)\cdot} \Lambda^n \mathbb{C}^n \leftarrow 0$$

$\underline{p \in Crit(W)}$ $\quad \partial W(p) = 0 \quad \therefore \underline{H^\cdot(Q^{zero}) = \Lambda^\cdot \mathbb{C}^n \text{ itself}}$

$\underline{p \notin Crit(W)}$ $\quad \partial W(p) \neq 0 \Rightarrow \underline{H^\cdot(Q^{zero}) = 0}$.

physically, if $\partial W(p) \neq 0$ energy is
obviously ~~zero at~~ non-zero
by potential term $|\partial W(p)|^2$

<u>B-branes via factorization of W</u>

a, b branes associated with

$$W = f_a g_a - c_a \quad , \quad W = f_b g_b - c_b$$

$$Q = Q_{bulk} + Q_{bdy}$$

$$Q_{bdy} = \eta_b f_b(\phi)\big|_\pi + \bar\eta_b g_b(\phi)\big|_\pi - \eta_a f_a(\phi)\big|_0 - \bar\eta_a g_a(\phi)\big|_0$$

$$Q^2 = \underbrace{Q_{bulk}^2}_{-W(\pi)+W(0)} + \underbrace{\{Q_{bulk}, Q_{bdy}\}}_{=0} + \underbrace{Q_{bdy}^2}_{\underbrace{f_b g_b}_{W(\pi)+c_b}\big|_\pi - \underbrace{f_a g_a}_{W(0)+c_a}\big|_0}$$

$$= c_b - c_a$$

$\therefore$ $c_b = c_a \Rightarrow$ $\underline{Q \text{ defines a complex}}$.

Q_{bdy} acts on the representation of $\eta_b, \bar\eta_b$, $\eta_a, \bar\eta_a$ algebra

$$\mathrm{Hom}(\mathbb{C}_a^2, \mathbb{C}_b^2) = \{ |0\rangle_b, \bar\eta_b|0\rangle_b \} \otimes \{ {}_a\langle 0|, {}_a\langle 0|\eta_a \} \text{ -- } 2\times 2 \text{ matrix}$$

$$Q_{bdy}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & f_b(\phi) \\ g_b(\phi) & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\begin{pmatrix} 0 & f_a(\phi) \\ g_a(\phi) & 0 \end{pmatrix}$$

Zero mode:

$$Q = \bar\partial + Q_{bdy} \quad \text{on} \quad \Omega^{0,\bullet}(X, \mathrm{Hom}^{\overset{\text{even or odd}}{\downarrow}\bullet}(\mathbb{C}_a^2, \mathbb{C}_b^2))$$

$\rightsquigarrow$ generalizes to $f_b, g_b : n_b \times n_b$

$f_a, g_a : n_a \times n_a$

$$\mathcal{H}^{zero} = \Omega^{0,\bullet}(X, \mathrm{Hom}(\mathbb{C}^{2n_a}, \mathbb{C}^{2n_b}))$$

$$Q = \bar\partial + Q_{bdy}$$

X: Stein space $\left(\bar\partial\alpha = 0 \Rightarrow \alpha = \bar\partial\beta \right)$
except α: 0-form.

$Q\alpha = 0 \qquad \alpha = \alpha^0 + \alpha^1 + \cdots + \alpha^n \qquad \underline{\alpha^i \in \Omega^{0,i}}$

$$\Rightarrow \bar\partial\alpha^0 + Q_{bdy}\alpha^1 = 0$$
$$\bar\partial\alpha^1 + Q_{bdy}\alpha^2 = 0$$
$$\vdots$$
$$\bar\partial\alpha^{n-1} + Q_{bdy}\alpha^n = 0$$
$$\bar\partial\alpha^n = 0 \quad (\text{trivially})$$

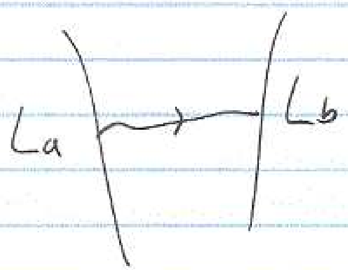
Stein $\Rightarrow \alpha^n = \bar\partial\beta^{n-1}$

$$0 = \bar\partial\alpha^{n-1} + Q_{bdy}\bar\partial\beta^{n-1} = \bar\partial(\alpha^{n-1} - Q_{bdy}\beta^{n-1})$$
$$\Rightarrow \alpha^{n-1} - Q_{bdy}\beta^{n-1} = \bar\partial\beta^{n-2}$$
$$0 = \bar\partial\alpha^{n-2} + Q_{bdy}\bar\partial\beta^{n-2} = \bar\partial(\alpha^{n-2} - Q_{bdy}\beta^{n-2})$$
$$\vdots$$
$$\Rightarrow \alpha^1 - Q_{bdy}\beta^1 = \bar\partial\beta^0$$
$$0 = \bar\partial\alpha^0 + Q_{bdy}\bar\partial\beta^0 = \bar\partial(\alpha^0 - Q_{bdy}\beta^0)$$
$$\Rightarrow \alpha^0 - Q_{bdy}\beta^0 = \alpha^0_{hol}$$

$$\alpha = \alpha^0 + \alpha^1 + \cdots + \alpha^n = (\alpha^0_{hol} + Q_{bdy}\beta^0) + (\bar\partial\beta^0 + Q_{bdy}\beta^1) + \cdots$$
$$+ \cdots + (\bar\partial\beta^{n-2} + Q_{bdy}\beta^{n-1}) + \bar\partial\beta^{n-1}$$
$$= \alpha^0_{hol} + Q(\beta^0 + \cdots + \beta^{n-1}) \quad \therefore \alpha \cong \alpha^0_{hol}$$

A-branes in NLSM X : Kähler

$L_a, L_b \subset X$ Lagrangian submfds



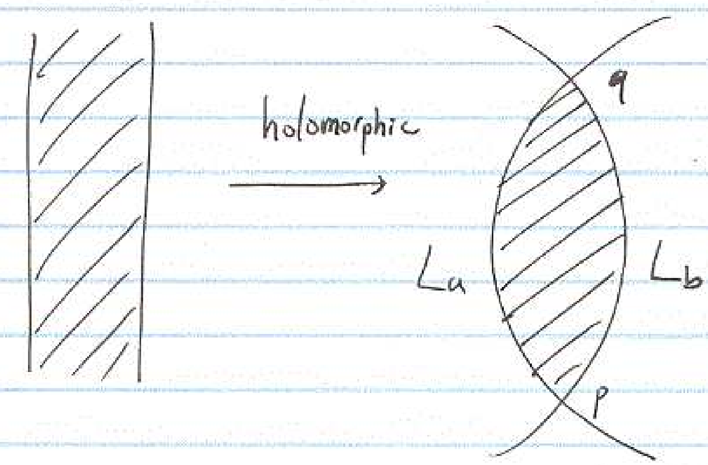
Classically

Zero energy $\Longleftrightarrow$ constant map to $L_a \cap L_b$.

$$\mathcal{H}^{\text{classical}}_{\text{SUSY}} = \bigoplus_{p \in L_a \cap L_b} \mathbb{C}[p]$$

In quantum theory, the energy of some states may be lifted by <u>tunnelling effects</u>

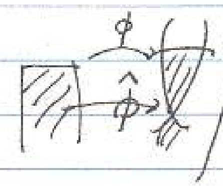
Tunnelling configuration :

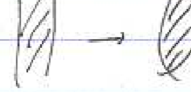


Q action in $\mathcal{H}^{\text{classical}}_{SUSY}$ is obtained by summing up such holomorphic strips. $\rightsquigarrow$ (Floer "complex")

$$Q[p] = \sum_{\substack{q \in L_a \cap L_b \\ \mu(q) = \mu(p)+1}} \left(\sum_{\substack{\phi:\ \text{holomorphic} \\ \text{strip } p \to q}} e^{-\int \phi^*(\omega - iB)} \right) [q]$$

$\mu(p)$ = "regularized" index (# of negative eigenvalue) of the operator ∂_σ acting on $T_p \Omega(L_a, L_b)$ tangent vectors to the pathspace.

(Morse index of the "function" $\int \hat{\phi}^* \omega$ )

$\Delta\mu$ = index of the fermion Dirac operator in the background ϕ :  .

$Q^2 = 0$? <u>Not always !</u>

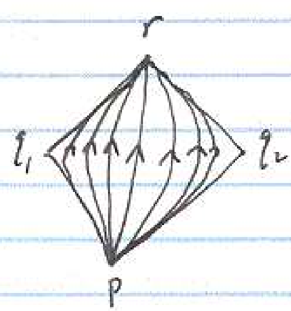
Usual proof of $Q^2 = 0$ in Morse-Witten complex:

$$\to C^i \xrightarrow{Q} C^{i+1} \xrightarrow{Q} C^{i+2} \to \cdots \qquad C^i = \bigoplus_{\mu(p)=i} [p]$$

$$\mathcal{M}_{p,q} = \{ \text{gradient flow } p \to q \} / \mathbb{R} (= \text{translation})$$

$\mu(r) = \mu(p) + 2$

$$\partial \mathcal{M}_{p,r} \overset{(\star)}{=} \bigcup_{\substack{q \\ \mu(q) = \mu(p)+1}} \underbrace{\mathcal{M}_{p,q} \times \mathcal{M}_{q,r}}_{\text{broken flows}}$$

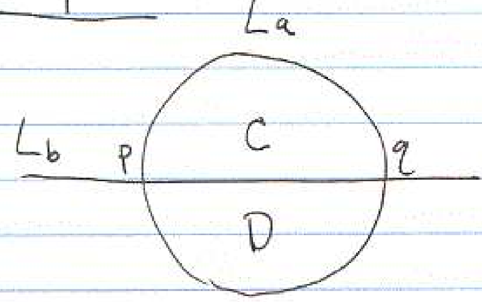


$$0 = \# \partial \mathcal{M}_{p,q} = \sum_{\substack{q \\ \mu(q) = \mu(p)+1}} \# \mathcal{M}_{p,q} \cdot \# \mathcal{M}_{q,r}$$

$$\Leftrightarrow \quad 0 = Q \circ Q .$$

However, for open string theory $(\star)$ may fail !

example

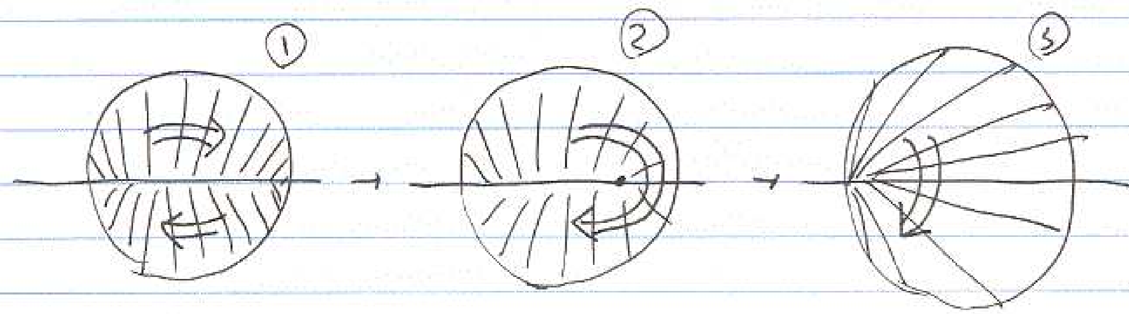


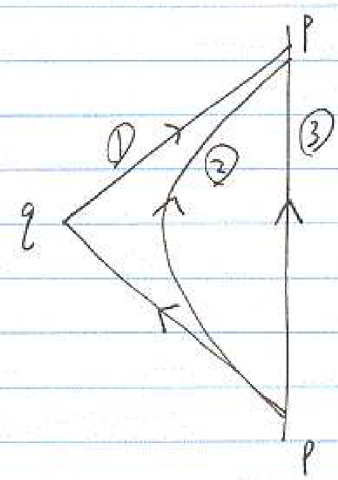
$$Q[p] = e^{-A(C)}[q]$$

$$Q[q] = e^{-A(D)}[p]$$

$$Q \circ Q[p] = e^{-(A(C)+A(D))}[p] \neq 0$$

Consider $\mathcal{M}_{p,p}$





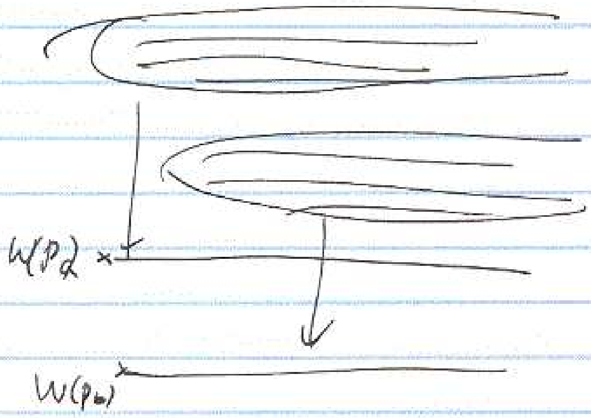
bubble of holomorphic disc

Failure in $Q^2 = 0$ here $\xleftrightarrow{\text{mirror}}$ $Q^2 = \Delta W$

A-branes in LG

$$L_a, L_b \subset X \qquad \mathrm{Im}\, W = \mathrm{Im}\, W_a,\ \mathrm{Im}\, W_b$$

e.g. $p_a, p_b \in \mathrm{Crit}(W)$



$$L_{a,b} = \text{Stable mfd of } p_{a,b}$$

Usually $L_a \cap L_b = \phi$.

$$Q = \int_0^\pi d\sigma \Big(g_{i\bar j} \bar\psi_+^{\bar j} \partial_t \phi^i + g_{i\bar j} \psi_-^i \partial_t \bar\phi^{\bar j} - g_{i\bar j} \psi_-^i \partial_\sigma \bar\phi^{\bar j} + g_{i\bar j} \bar\psi_+^{\bar j} \partial_\sigma \phi^i$$
$$+ i \psi_-^i \partial_i W + i \bar\psi_+^{\bar i} \partial_{\bar i} \overline{W} \Big)$$

$$\{Q, Q^\dagger\} = H + \underbrace{\Delta\, \mathrm{Im}\, W}_{\mathrm{Im}\, W_b - \mathrm{Im}\, W_a}$$

Shifted by a const.

Classical SUSY ground states

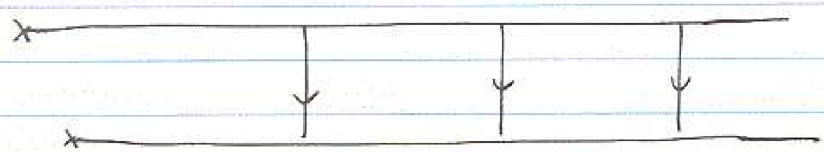
$\Leftrightarrow$ minimizes RHS

$$\Leftrightarrow \quad \partial_\sigma \phi^i = -i g^{i\bar j} \partial_{\bar j} \overline{W}$$
$$\partial_\sigma \bar\phi^{\bar j} = i g^{i\bar j} \partial_i W$$

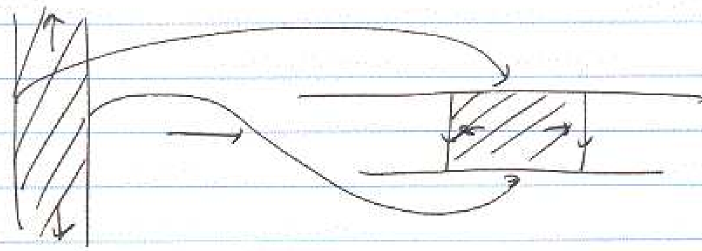
i.e. gradient flow ~~along~~ of $-\mathrm{Im}\,W$ from $\overset{\text{at }\sigma=0}{L_a}$ to $\overset{\text{at }\sigma=\pi}{L_b}$

* No solution if $\mathrm{Im}\,W_a < \mathrm{Im}\,W_b$.

* Possibly many solutions if $\mathrm{Im}\,W_a > \mathrm{Im}\,W_b$



Again one may consider Morse-Witten complex (or Floer)

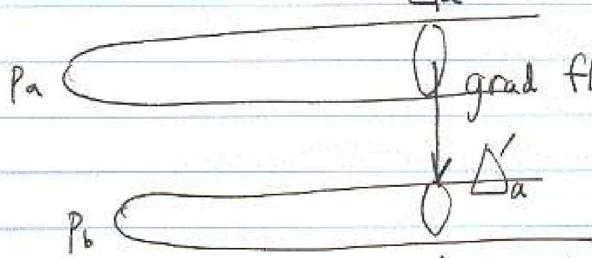


However $\exists$ trick.

$$\text{grad flow} \Rightarrow \Delta\sigma = \left| \int_{\text{path}} \frac{d\,\mathrm{Im}\,W}{|\partial W|^2} \right|^2$$

We need the flows with $\Delta\sigma = \pi$.

$\Delta_a = L_a \cap W^{-1}(*)$



$\Delta_b = L_b \cap W^{-1}(*')$

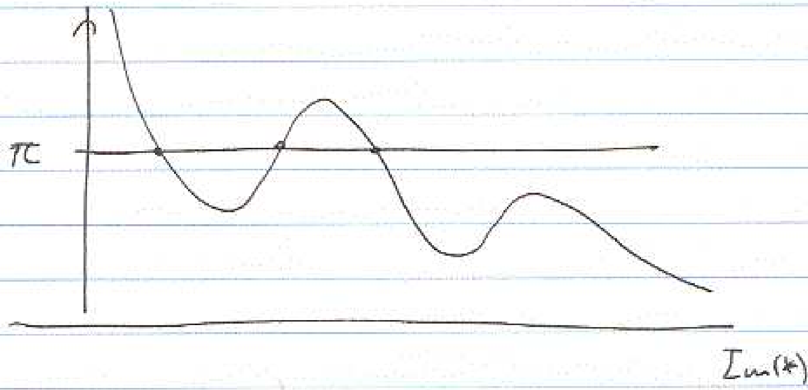
$$\begin{aligned}\#\ \text{grad flow}\ L_a \to L_b \quad &(\text{not necessarily } \Delta\sigma = \pi) \\ = \#(\Delta_a' \cap \Delta_b) &=: I_{ab}\end{aligned}$$

Move Im(*)

$\Rightarrow$ $|I_{ab}|$ families of (not nec. $\Delta\sigma = \pi$) grad flows

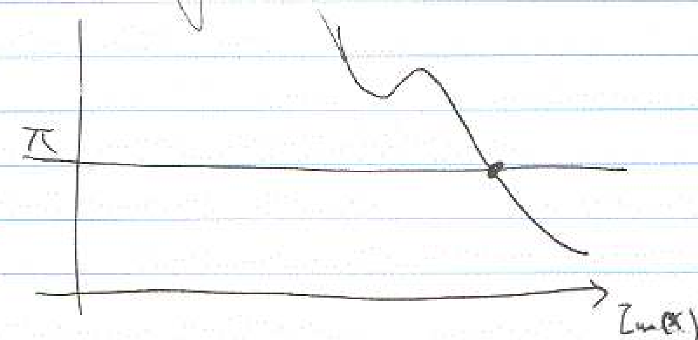
For each of them graph of $\left| \int \frac{\partial \operatorname{Im} W}{|\partial W|^2} \right|$

should be like



But ~~#~~ $\mathcal{H}_{SUSY}$ indep of $W \to e^t W$

$t \to$ large



$$\mathcal{H}_{SUSY} = \mathbb{C}^{\#(\Delta'_a \cap \Delta_b)}$$

or $\#(\Delta'_a \cap \Delta_b) = \#(\gamma_a^- \cap \gamma_b^+)$