(2,2) Supersymmetry in 1+1 dimensions

(2,2) SUSY Quantum Field Theory in 1+1 dim.

is a QFT in 1+1 dim. with $\mathbb{Z}_2$ graded

Hibert space $\quad \mathcal{H} = \underset{(-1)^F=1}{\mathcal{H}^B} \oplus \underset{(-1)^F=-1}{\mathcal{H}^F}$

with even operators $\quad \mathcal{H}^B \circlearrowleft, \mathcal{H}^F \circlearrowleft$

$$\underset{\text{Hamiltonian}}{H}, \quad \underset{\text{Momentum}}{P}, \quad \underset{\text{Lorentz}}{M} \qquad \left[\begin{array}{l} H=H^\dagger, P=P^\dagger \\ M=-M^\dagger \end{array}\right.$$

& odd operators $\quad \mathcal{H}^B \rightleftarrows \mathcal{H}^F$

$$Q_+, \underset{"Q_+^\dagger}{\overline{Q}_+}, Q_-, \underset{"Q_-^\dagger}{\overline{Q}_-}$$

(Optional
& even operators $\underset{\substack{\text{Vector} \\ \text{R-charge}}}{F_V}, \underset{\substack{\text{axial} \\ \text{R-charge}}}{F_A}$ s.t. $(-1)^{F_V} = \pm(-1)^{F_A} = \pm($

$$\left[F_{A,V} = F_{A,V}^\dagger \right.$$

↳ continued

obeying

$$Q_+^2 = \bar{Q}_+^2 = Q_-^2 = \bar{Q}_-^2 = 0$$

$$\{Q_+, \bar{Q}_+\} = H + P$$

$$\{Q_-, \bar{Q}_-\} = H - P$$

$$\{\bar{Q}_+, \bar{Q}_-\} = \overset{(Z)}{0} \qquad \{Q_+, Q_-\} = \overset{(Z^*)}{0}$$

$$\{\bar{Q}_+, Q_-\} = \overset{(\tilde{Z})}{0} \qquad \{Q_+, \bar{Q}_-\} = \overset{(\tilde{Z}^*)}{0}$$

$$i[M, Q_\pm] = \mp Q_\pm, \quad i[M, \bar{Q}_\pm] = \mp \bar{Q}_\pm$$

$$[F_V, Q_\pm] = -Q_\pm, \quad [F_V, \bar{Q}_\pm] = \bar{Q}_\pm$$

$$[F_A, Q_\pm] = \mp Q_\pm, \quad [F_A, \bar{Q}_\pm] = \pm \bar{Q}_\pm$$

—— (2,2) SUSY algebra in 1+1 dimensions.

Example 1 $t = x^0$ time, $\sigma = x^1$ space) coordinates $\quad d^2x = dt\, d\sigma$

$\phi(t,\sigma)$: $\mathbb{C}$-valued scalar field $\quad \bar{\phi} = \phi^*$

$\psi_\pm(t,\sigma)$: Dirac fermion $\quad \bar{\psi}_\pm = \psi_\pm^\dagger$

$W(\phi)$: a holomorphic function of ϕ (say a polynomial

$$L = |\partial_t\phi|^2 - |\partial_\sigma\phi|^2 - |W'(\phi)|^2$$
$$+ i\bar{\psi}_-(\partial_t + \partial_\sigma)\psi_- + i\bar{\psi}_+(\partial_t - \partial_\sigma)\psi_+ - W''(\phi)\psi_+\psi_- - \overline{W''(\phi)}\bar{\psi}_-\bar{\psi}_+$$

— Called Landau-Ginzburg model with superpotential $W(\phi)$.

Symmetry of $S = \int L\, d^2x$

- spacetime translations $\longrightarrow H, P$

$$X(t,\sigma) \to X(t + \text{const}, \sigma + \text{const}) \qquad X = \phi, \psi_\pm$$

- Lorentz $\longrightarrow M$

$x^\pm = t \pm \sigma$

$$\phi(x^+, x^-) \to \phi(e^\gamma x^+, e^{-\gamma} x^-)$$

$$\psi_\pm(x^+, x^-) \to e^{\pm\frac{\gamma}{2}}\psi_\pm(e^\gamma x^+, e^{-\gamma} x^-)$$

· supersymmetry $\to Q_\pm, \overline{Q}_\pm$

$$\left[\begin{aligned}
\delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+ \\
\delta\psi_\pm &= \pm i\overline{\epsilon}_\mp(\partial_t \pm \partial_\sigma)\phi - \epsilon_\pm \overline{W'(\phi)} \\
\delta\overline{\phi} &= -\overline{\epsilon}_+\overline{\psi}_- + \overline{\epsilon}_-\overline{\psi}_+ \\
\delta\overline{\psi}_\pm &= \mp i\epsilon_\mp(\partial_t \pm \partial_\sigma)\overline{\phi} - \overline{\epsilon}_\pm W'(\phi)
\end{aligned}\right.$$

$$\begin{aligned}
[\delta^{(1)}, \delta^{(2)}]X = \; & i(\epsilon_-^{(1)}\overline{\epsilon}_-^{(2)} - \epsilon_-^{(2)}\overline{\epsilon}_-^{(1)})(\partial_t + \partial_\sigma)X \\
& + i(\epsilon_+^{(1)}\overline{\epsilon}_+^{(2)} - \epsilon_+^{(2)}\overline{\epsilon}_+^{(1)})(\partial_t - \partial_\sigma)X .
\end{aligned}$$

$\rightsquigarrow (\underset{\epsilon_+,\overline{\epsilon}_+}{\underset{\uparrow}{2}}, \underset{\epsilon_-,\overline{\epsilon}_-}{\underset{\uparrow}{2}})$ supersymmetry.

· phase rotation (R-symmetry)

$$\left.\begin{aligned}
\phi &\to \phi \\
\psi_\pm &\to e^{\mp i\beta}\psi_\pm
\end{aligned}\right) \to F_A \qquad \text{(Axial R-sym)}$$

If $W(\phi)$ is a monomial, say $W = \phi^n$,

$$\left.\begin{aligned}
\phi &\to e^{\frac{2i\alpha}{n}}\phi \\
\psi_\pm &\to e^{-i\alpha}e^{\frac{2i\alpha}{n}}\psi_\pm
\end{aligned}\right) \to F_V \qquad \text{(vector-R-sym)}.$$

~~→ We will see more examples later, by systematic construction.~~

Example 2 (M, g) Compact Kähler manifold

$\phi(t,\sigma)$: M-valued Scalar field

$\psi_\pm(t,\sigma)$: $\phi^* T^{1,0}_{\phi(t,\sigma)} M$ - valued Dirac fermion

Or
$$\begin{cases} \phi : \mathbb{R}^2 \to M \quad \text{map} \\ \psi_\pm : \text{a section of } \underset{\substack{\uparrow \\ \text{spin bundle.}}}{S_\pm} \otimes \phi^* T^{1,0} M \end{cases}$$

$$L = g_{i\bar{j}}\left(\partial_t \phi^i \partial_t \bar{\phi}^{\bar{j}} - \partial_\sigma \phi^i \partial_\sigma \bar{\phi}^{\bar{j}}\right)$$
$$+ i g_{i\bar{j}} \bar{\psi}_-^{\bar{j}} (D_t + D_\sigma) \psi_-^i + i g_{i\bar{j}} \bar{\psi}_+^{\bar{j}} (D_t - D_\sigma) \psi_+^i$$
$$+ R_{i\bar{j}k\bar{l}} \psi_+^i \psi_-^k \bar{\psi}_-^{\bar{j}} \bar{\psi}_+^{\bar{l}}$$

$$D_\mu \psi^i = \partial_\mu \psi^i + \partial_\mu \phi^j \Gamma^i_{jk}$$

$$\begin{cases} \delta\phi^i = \epsilon_+ \psi_-^i - \epsilon_- \psi_+^i \\ \delta\psi_\pm^i = \pm i \bar{\epsilon}_\mp (\partial_t \pm \partial_\sigma)\phi^i + \epsilon_\pm \Gamma^i_{jk} \psi_+^j \psi_-^k \\ \delta\bar{\phi}^{\bar{i}} = -\bar{\epsilon}_+ \bar{\psi}_-^{\bar{i}} + \bar{\epsilon}_- \bar{\psi}_+^{\bar{i}} \\ \delta\bar{\psi}_\pm^{\bar{i}} = \mp i \epsilon_\mp (\partial_t \pm \partial_\sigma)\bar{\phi}^{\bar{i}} + \bar{\epsilon}_\pm \overline{\Gamma^i_{jk} \psi_+^j \psi_-^k} \end{cases}$$

NLSM with target M.

we will see systematic construction of these models

Define

$$\begin{cases} Q_A := \bar{Q}_+ + Q_- \\ Q_A^\dagger = Q_+ + \bar{Q}_- \end{cases} \qquad \begin{cases} Q_B := \bar{Q}_+ + \bar{Q}_- \\ Q_B^\dagger = Q_+ + Q_- \end{cases}$$

$(Q, F) = (Q_A, F_A)$ or (Q_B, F_V) obeys the follo

$$\left.\begin{aligned} Q^2 &= 0 \\ \{Q, Q^\dagger\} &= 2H \\ [F, Q] &= Q \end{aligned}\right] \text{ SUSY algebra !}$$

Compactify the space direction, say $\sigma \equiv \sigma + 2\pi$,
& impose periodic boundary condition on fields

$$X(t,\sigma) = X(t, \sigma+2\pi).$$

$\Rightarrow$ The system can be regarded as SUSY QM
with infinite degrees of freedom.

general properties apply

① $E \geqslant 0$, $E=0 \Leftrightarrow Q = Q^\dagger = 0$

Stronger : $H = \{Q_+, \bar{Q}_+\} + \{Q_-, \bar{Q}_-\}$

$\therefore E=0 \Leftrightarrow Q_+ = \bar{Q}_+ = Q_- = \bar{Q}_- = 0$

$\Rightarrow$ SUSY ground states : annihilated by all four supercharges.

⊙ $E > 0$ boson $\underset{}{\overset{1:1}{\rightleftarrows}}$ fermion

⊙ $\mathcal{H}_0^B = H^B(Q)$, $\mathcal{H}_0^F = H^F(Q)$

or $\mathcal{H}_0^p = H^p(Q)$

or, if both F_A, F_V are conserved

$$\underset{\substack{\uparrow \\ F=p,\, F'=q}}{\mathcal{H}_0^{q,p}} = H^p(\mathcal{H}|_{F'=q}, Q)$$

⊙ Witten index $Tr(-1)^F = Tr_{\mathcal{H}}(-1)^F e^{-\beta H}$

$= \dim \mathcal{H}_0^B - \dim \mathcal{H}_0^F$

$= \sum_p (-1)^p \dim \mathcal{H}_0^p$ (Euler characteristic)

$= \mathrm{index}(Q + Q^\dagger)$

Chiral Ring

So far, we have been considering Q-cohomology of states

Now, let's consider Q-cohomology of fields, or local ops

A field $\mathcal{O}(x) = \mathcal{O}(t,\sigma)$ is called a chiral field when it (anti-)commutes with Q_B:

$$[Q_B, \mathcal{O}(x)] = Q_B \mathcal{O}(x) - (-1)^{|\mathcal{O}|} \mathcal{O}(x) Q_B = 0.$$

☆ If $\mathcal{O}(x)$ is chiral

$$\frac{\partial}{\partial x^{\pm}} \mathcal{O}(x) = \frac{i}{2}[H \pm P, \mathcal{O}(x)] \overset{\text{susy alg } \&\, [Q_B, \mathcal{O}]=0}{=} \frac{i}{2}[Q_B, [Q_\pm, \mathcal{O}(x)]] \quad \text{... exact}$$

Cohomology class of $\mathcal{O}(x)$ does not change under translation.

☆ If $\mathcal{O}_1(x)$, $\mathcal{O}_2(y)$ chiral, so is $\mathcal{O}_1(x)\mathcal{O}_2(y)$

$$[Q_B, \mathcal{O}_1(x)\mathcal{O}_2(y)] = \underbrace{[Q_B, \mathcal{O}_1(x)]}_{=0}\mathcal{O}_2(y) + (-1)^{|\mathcal{O}_1|}\mathcal{O}_1(x)\underbrace{[Q_B, \mathcal{O}_2(y)]}_{=0}$$

$$= 0$$

Since cohomology class does not depend on x nor y,

One can define $[\mathcal{O}_1 \mathcal{O}_2](y) = \lim_{x \to y} [\mathcal{O}_1(x)\mathcal{O}_2(y)]$.

$\mathcal{R}_B = \{ Q_B\text{-cohomology class of fields} \}$

form a ring, called <u>chiral ring</u>

(or B-chiral ring)

Similarly

$\mathcal{R}_A = \{ Q_A\text{-cohomology class of fields} \}$

form a ring, <u>twisted chiral ring</u>

(or A-chiral ring)

<u>Example 1</u> LG model with $W(\phi)$ polynomial.

δ_{Q_B}-variation corresponds to $\bar{\epsilon}_+ = -\bar{\epsilon}_- = \bar{\epsilon}$, $\epsilon_+ = \epsilon_- = 0$

$$\begin{cases} \delta_B \phi = 0 & \delta_B \bar{\phi} = -\bar{\epsilon}(\bar{\psi}_- + \bar{\psi}_+) \\ \delta_B \psi_\pm = -i\bar{\epsilon}(\partial_t \pm \partial_\sigma)\phi, & \delta_B \bar{\psi}_\pm = \mp \bar{\epsilon}\, W'(\phi) \end{cases}$$

$$\mathcal{R}_B = \{ \text{Polynomials of } \phi \} / W'(\phi) = 0.$$

$$= \mathbb{C}[\phi]/(W'(\phi)) \quad \cdots \text{Jacobi ring.}$$

Example 2 NLSM on a Kähler mfd (M, g)

A-ring Q_A-variation $\Leftrightarrow$ $\epsilon_+ = \bar{\epsilon}_- = \bar{\epsilon}$, $\bar{\epsilon}_+ = \epsilon_- = 0$

$$\begin{cases} \delta_A \phi^i = \bar{\epsilon} \psi_-^i & \delta_A \bar{\phi}^{\bar{i}} = \epsilon \bar{\psi}_+^{\bar{i}} \\ \delta_A \psi_-^i = 0 & \delta_A \bar{\psi}_+^{\bar{i}} = 0 \\ \delta_A \psi_+^i = \bar{\epsilon}\left(i(\partial_t + \partial_\sigma)\phi^i + \Gamma^i_{jk} \psi_+^j \psi_-^k\right), & \delta_A \bar{\psi}_-^{\bar{i}} = \epsilon\left(i(\partial_t - \partial_\sigma)\bar{\phi}^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \bar{\psi}_-^{\bar{j}} \bar{\psi}_+^{\bar{k}}\right) \end{cases}$$

$\phi^i \sim z^i$, $\bar{\phi}^{\bar{i}} \sim \bar{z}^{\bar{i}}$ $\psi_-^i \sim dz^i$, $\bar{\psi}_+^{\bar{i}} \sim d\bar{z}^{\bar{i}}$ $\delta_A \sim d$

$$\omega = \sum \omega_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \cdots dz^{i_p} d\bar{z}^{\bar{j}_1} \cdots d\bar{z}^{\bar{j}_q} \in \Omega^{p,q}(M)$$

$$\Leftrightarrow \quad \mathcal{O}_\omega = \sum \omega_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \psi_-^{i_1} \cdots \psi_-^{i_p} \bar{\psi}_+^{\bar{j}_1} \cdots \bar{\psi}_+^{\bar{j}_q}$$

$$\delta_A \mathcal{O}_\omega = \mathcal{O}_{d\omega}$$

$$\mathcal{R}_A = \left\{ \begin{array}{c} \text{closed differential} \\ \text{forms on } M \end{array} \right\} \Big/ \text{d-exact forms} = H_{DR}(M)$$

product Naively $\mathcal{O}_{\omega_1} \cdot \mathcal{O}_{\omega_2} = \mathcal{O}_{\omega_1 \wedge \omega_2}$ [cohomology ring of M]

But actually, there is a deformation

$\rightarrow$ quantum cohomology ring.

<u>B-ring</u> Q_B-variation $\Leftrightarrow$ $\bar{\epsilon}_+ = -\bar{\epsilon}_- = \bar{\epsilon}$, $\epsilon_+ = \epsilon_- = 0$.

define $\quad \eta^{\bar{i}} = -(\bar{\psi}_-^{\bar{i}} + \bar{\psi}_+^{\bar{i}})$

$$\theta_i = g_{i\bar{j}}(\bar{\psi}_-^{\bar{j}} - \bar{\psi}_-^{\bar{j}})$$

$$\left[\begin{array}{ll} \delta_B \phi^i = 0 & \delta_B \bar{\phi}^{\bar{i}} = \bar{\epsilon}\eta^{\bar{i}} \\ \delta_B \theta_i = 0 & \delta_B \eta^{\bar{i}} = 0 \\ \delta_B \psi_\pm^i = i\bar{\epsilon}(\partial_t \pm \partial_\sigma)\phi^i & \end{array}\right.$$

$$\phi^i \sim z^i,\ \bar{\phi}^{\bar{i}} \sim \bar{z}^{\bar{i}},\ \theta_i \sim \frac{\partial}{\partial z^i},\ \eta^{\bar{i}} \sim d\bar{z}^{\bar{i}},\ \delta_B \sim \bar{\partial}$$

$$\omega = \omega_{\bar{i}_1 \cdots \bar{i}_p}^{j_1 \cdots j_q}\, d\bar{z}^{\bar{i}_1} \cdots d\bar{z}^{\bar{i}_p} \frac{\partial}{\partial z^{j_1}} \cdots \frac{\partial}{\partial z^{j_q}}$$

$$\leftrightarrow \quad O_\omega = \omega_{\bar{i}_1 \cdots \bar{i}_p}^{j_1 \cdots j_q}\, \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p}\, \theta_{j_1} \cdots \theta_{j_q} \qquad \delta_B O_\omega = O_{\bar{\partial}\omega}$$

$$0 \to \Omega^{0,0}(M, \Lambda^q T_M) \xrightarrow{\bar{\partial}} \Omega^{0,1}(M, \Lambda^q T_M) \to \cdots \xrightarrow{\bar{\partial}} \Omega^{0,n}(M, \Lambda^q T_M) \to 0$$

$$R_B = \bigoplus_{p,q=0}^{n} H^{0,p}(M, \Lambda^q T_M)$$

<u>product</u> Naively wedge product

real story : $\left[\begin{array}{l} M = CY \ : \text{Yes, it's wedge product} \\ M \neq CY \ : \text{Not Known.} \end{array}\right.$