Superfields

$$L = \frac{\dot{X}^2}{2} + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \frac{1}{2}(h'(X))^2 - h''(X)\bar{\psi}\psi$$

~~has~~ has supersymmetry
$$\begin{cases} \delta X = \epsilon\bar{\psi} - \bar{\epsilon}\psi \\ \delta\psi = \epsilon(i\dot{X} + h'(X)) \\ \delta\bar{\psi} = \bar{\epsilon}(-i\dot{X} + h'(X)) \end{cases}$$

(also NLSM)

How do we find such systems?

Here is one way ....

Consider a space with coordinates

$$\underset{\substack{\uparrow \\ \text{bosonic, time}}}{t}, \quad \underbrace{\theta, \quad \bar{\theta} = \theta^*}_{\text{fermionic}}$$

A Functions on this space is

$$\Phi(t,\theta,\bar{\theta}) = X(t) + \theta\psi(t) + \bar{\psi}(t)\bar{\theta} + \theta\bar{\theta}f(t)$$

$\Leftrightarrow$ Collection of four functions $(X(t), \psi(t), \bar{\psi}(t), f(t))$

	$X(t)$	$\psi(t)$	$\bar{\psi}(t)$	$f(t)$
	ev	od	od	ev
or	od	ev	ev	od

Φ .. superfield, $(X, \psi, \bar{\psi}, f)$ ... component field.

Consider
<u>odd</u> vector fields $\quad \mathcal{Q} = \frac{\partial}{\partial\theta} + i\bar{\theta}\partial_t$

$$\bar{\mathcal{Q}} = -\frac{\partial}{\partial\bar{\theta}} - i\theta\partial_t$$

$$\therefore \{\mathcal{Q},\mathcal{Q}\} = \{\bar{\mathcal{Q}},\bar{\mathcal{Q}}\} = 0, \quad \{\mathcal{Q},\bar{\mathcal{Q}}\} = -2i\partial_t$$

action of $\delta = \epsilon\bar{\mathcal{Q}} - \bar{\epsilon}\mathcal{Q}$ on $\Phi(t,\theta,\bar{\theta})$:

$$\begin{aligned}
\delta\Phi = &-i\epsilon\theta\dot{X} - i\bar{\epsilon}\bar{\theta}\dot{X} && (\text{on } X)\\
&-\bar{\epsilon}\psi - i\bar{\epsilon}\bar{\theta}\theta\dot{\psi} && (\text{on } \theta\psi)\\
&+\epsilon\bar{\psi} - i\epsilon\theta\dot{\bar{\psi}}\bar{\theta} && (\text{on } \bar{\psi}\bar{\theta})\\
&+\epsilon\theta f - \bar{\epsilon}\bar{\theta} f && (\text{on } \theta\bar{\theta} f)
\end{aligned}$$

$$\begin{aligned}
= (\epsilon\bar{\psi} - \bar{\epsilon}\psi) + \theta\epsilon(i\dot{X} - f) + \bar{\epsilon}(-i\dot{X} - f)\bar{\theta}\\
+\theta\bar{\theta}(-i\bar{\epsilon}\dot{\psi} + i\epsilon\dot{\bar{\psi}})
\end{aligned}$$

$$\Leftrightarrow \left[\begin{aligned}
\delta X &= \epsilon\bar{\psi} - \bar{\epsilon}\psi\\
\delta\psi &= \epsilon(i\dot{X} - f)\\
\delta\bar{\psi} &= \bar{\epsilon}(-i\dot{X} - f)\\
\delta f &= -i\bar{\epsilon}\dot{\psi} + i\epsilon\dot{\bar{\psi}}
\end{aligned}\right.$$

resembles the susy transf. in the example.

Other differential operators $\begin{cases} D = \frac{\partial}{\partial\theta} - i\bar{\theta}\partial_t \\ \bar{D} = -\frac{\partial}{\partial\bar{\theta}} + i\theta\partial_t \end{cases}$

They anticommutes with $Q, \bar{Q}$.

$$\{D, Q\} = \{D, \bar{Q}\} = \{\bar{D}, Q\} = \{\bar{D}, \bar{Q}\} = 0.$$

$$\Rightarrow \quad \delta D\Phi = D\delta\Phi \qquad \bar{D}\delta\Phi = \delta\bar{D}\Phi.$$

Consider the action

$$S = \int dt\, d\theta\, d\bar{\theta} \left\{ \frac{1}{2} D\Phi \bar{D}\Phi - h(\Phi) \right\}$$

$$\delta S = \int dt\, d\theta\, d\bar{\theta} \left\{ \frac{1}{2} \underbrace{D\delta\Phi}_{\delta D\Phi} \bar{D}\Phi + \frac{1}{2} D\Phi \underbrace{\bar{D}\delta\Phi}_{\delta\bar{D}\Phi} - h'(\Phi)\delta\Phi \right\}$$

$$= \int dt\, d\theta\, d\bar{\theta}\, \underbrace{\delta}_{\frac{\partial}{\partial\theta},\, \frac{\partial}{\partial\bar{\theta}},\, \theta\partial_t,\, \bar{\theta}\partial_t} \left\{ \frac{1}{2} D\Phi \bar{D}\Phi - h(\Phi) \right\}$$

$= 0$ by stokes theorem. $\int dt\, \partial_t(-) = 0$

$\int d\theta\, \theta = 1$

$\int d\theta\, 1 = 0$

$\int d\theta \frac{\partial}{\partial\theta}(-) = 0$ No θ

$$D\Phi = \psi + \bar\theta f + i\theta\bar\theta\dot\psi - i\bar\theta\dot X$$

$$\bar D\Phi = \bar\psi + \theta f + i\theta\dot X - i\bar\theta\theta\dot{\bar\psi}$$

$$\frac{1}{2}\int d\theta d\bar\theta\, D\Phi\bar D\Phi = \frac{1}{2}D\Phi\bar D\Phi\Big|_{\bar\theta\theta} = \frac{1}{2}\left(\dot X^2 + i\psi\dot{\bar\psi} - i\dot\psi\bar\psi + f^2\right)$$

$$\int d\theta d\bar\theta\, h(\Phi) = h(X) + h'(X)\left(\theta\psi + \bar\psi\bar\theta + \theta\bar\theta f\right) + \frac{1}{2}h''(X)\left(\theta\psi + \bar\psi\bar\theta\right)^2\Big|_{\bar\theta\theta}$$

$$= -h'(X)f + h''(X)\bar\psi\psi$$

$$S = \int dt\left[\frac{1}{2}\dot X^2 + \frac{i}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) + \underbrace{\frac{1}{2}f^2 + h'(X)f}_{-\frac{1}{2}(h'(X))^2 + \frac{1}{2}(f + h'(X))^2} - h''(X)\bar\psi\psi\right]$$

eliminate f by EOM: $f = -h'(X)$

$$S = \int dt\left[\frac{1}{2}\dot X^2 + \frac{i}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) - \frac{1}{2}(h'(X))^2 - h''(X)\bar\psi\psi\right]$$

$$\begin{cases} \delta X = \epsilon\bar\psi - \bar\epsilon\psi \\ \delta\psi = \epsilon(i\dot X - f) \\ \delta\bar\psi = \bar\epsilon(-i\dot X - f) \end{cases}$$

The system we considered!

$\cancel{\delta f = -i\bar\epsilon\dot\psi + i\epsilon\dot{\bar\psi}}$

Example 2 $\quad S = \int dt\, d\theta\, d\bar\theta\ \frac{1}{2} g_{IJ}(\Phi) D\Phi^I \bar D \Phi^J$

$g_{IJ}(\Phi) D\Phi^I \bar D\Phi^J$

$= \cdots + \bar\theta\theta\, g_{IJ}(X)\left(\dot X^I \dot X^J + i\psi^I\dot{\bar\psi}^J - i\dot\psi^I\bar\psi^J + f^I f^J\right)$

$\quad + g_{IJ,K}(X)\left(\theta\psi^K + \bar\psi^K\bar\theta + \theta\bar\theta f^K\right) D\Phi^I \bar D\Phi^J$

$\quad + \frac{1}{2} g_{IJ,KL}(X)\ (\theta\psi^K + \bar\psi^K\bar\theta)(\theta\psi^L + \bar\psi^L\bar\theta)\psi^I\bar\psi^J$

$\overset{\bar\theta\theta}{=} g_{IJ}\left(\dot X^I \dot X^J + i(\bar\psi^J\dot\psi^I - \dot{\bar\psi}^J\psi^I) + f^I f^J\right)$

$\quad + f^I(g_{IJ,K} + g_{KI,J} - g_{KJ,I})\psi^K\bar\psi^J = 2f^I g_{IL}\Gamma^L_{KJ}\psi^K\bar\psi^J$

$\quad - i\dot X g_{IJ,K}\psi^K\bar\psi^J - i\dot X^J g_{IJ,K}\bar\psi^K\psi^I + g_{IJ,KL}\psi^I\bar\psi^J\psi^K\bar\psi^L$

$= g_{IJ}\left(\dot X^I\dot X^J + i(\bar\psi^J D_t\psi^I - D_t\bar\psi^J\psi^I)\right) + \cancel{f^I f^J}$

$\quad - g_{IJ,KL}\psi^I\bar\psi^J\psi^K\bar\psi^L - g_{IJ}\Gamma^I_{MK}\psi^M\bar\psi^K\Gamma^J_{NL}\psi^N\bar\psi^L$

$\quad\quad + \cancel{g_{IJ}(f^I + \Gamma^I_{MK}\psi^M\bar\psi^K)(f^J + \Gamma^J_{NL}\psi^N\bar\psi^L)}$

$= g_{IJ}\dot X\dot X^J + i g_{IJ}(\bar\psi^J D_t\psi^I - D_t\bar\psi^J\psi^I)$

$\quad + R_{IJKL}\psi^I\bar\psi^J\psi^K\bar\psi^L \quad + g_{IJ}(f^I + \cdots)(f^J + \cdots).$

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§ Superspace & Superfields

Minkowski space: commuting coordinates $x^0 = t,\ x^1 = \sigma$ or $x^\pm = x^0 \pm x^1$

We introduce anti-commuting coordinates

$$\theta^+,\ \theta^-,\ \bar\theta^+,\ \bar\theta^-$$

complex, $\bar\theta^\pm = (\theta^\pm)^*$ complex conjugates

Under Lorentz transformation

$$x^\pm \to e^{\pm\gamma} x^\pm$$
$$\theta^\pm \to e^{\pm\frac{\gamma}{2}}\theta^\pm,\quad \bar\theta^\pm \to e^{\pm\frac{\gamma}{2}}\bar\theta^\pm$$

We call this space with coordinates $(x^0, x^1, \theta^+, \theta^-, \bar\theta^+, \bar\theta^-)$

the (2, 2) superspace.

Superfields are functions on this space.

They can be expanded as

$$\begin{aligned} f(x^0, x^1, \theta^+, \theta^-, \bar\theta^+, \bar\theta^-) &= f_0(x^0, x^1) + \theta^+ f_+(x^0, x^1) + \theta^- f_-(x^0, x^1) \\ &\quad + \bar\theta^+ \tilde f_+(x^0, x^1) + \bar\theta^- \tilde f_-(x^0, x^1) \\ &\quad + \theta^+\theta^- f_{+-}(x^0, x^1) + \cdots \end{aligned}$$

(in total there can be $1 + 4 + \binom{4}{2} + \cdots + \binom{4}{4} = 2^4 = 16$ terms)

A superfield f is bosonic if $f\theta^\alpha = \theta^\alpha f$

fermionic if $f\theta^\alpha = -\theta^\alpha f$

Sets of differential operators

One set
$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar\theta^\pm\partial_\pm \qquad \partial_\pm = \frac{\partial}{\partial x^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$$
$$\bar{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\bar\theta^\pm} - i\theta^\pm\partial_\pm$$

They obey $\mathcal{Q}_\pm^2 = \bar{\mathcal{Q}}_\pm^2 = 0$
$$\{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm\} = -2i\partial_\pm \qquad \left(\text{like SUSY algebra}\right)$$

Another set
$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar\theta^\pm\partial_\pm$$
$$\bar{D}_\pm = -\frac{\partial}{\partial\bar\theta^\pm} + i\theta^\pm\partial_\pm$$

They obey similar rels: $D_\pm^2 = \bar{D}_\pm^2 = 0$ $\{D_\pm, \bar{D}_\pm\} = 2i\partial_\pm$

Important
$$\{\mathcal{Q}_\alpha, D_\beta\} = \{\bar{\mathcal{Q}}_\alpha, D_\beta\}$$
$$= \{\mathcal{Q}_\alpha, \bar{D}_\beta\} = \{\bar{\mathcal{Q}}_\alpha, \bar{D}_\beta\} = 0$$

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$$\left(\begin{array}{llll} \text{Last time} & F_A : & \phi \to \phi & \\ & & \psi_\pm \to e^{\mp i\beta}\psi_\pm & \text{Correction to the class} \\ W = \phi^n \Rightarrow & F_V : & \phi \to e^{\frac{2}{n} i\alpha}\phi & \\ & & \psi_\pm \to e^{\frac{2-n}{n} i\alpha}\psi_\pm = e^{\frac{2i\alpha}{n}} \cdot e^{-i\alpha}\psi_\pm & \end{array}\right)$$

R-rotations

Vector : $e^{i\alpha F_V} : f(x^\mu, \theta^\pm, \bar\theta^\pm) \mapsto e^{i\alpha q_V} f(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar\theta^\pm)$

Axial : $e^{i\beta F_A} : f(x^\mu, \theta^\pm, \bar\theta^\pm) \mapsto e^{i\beta q_A} f(x^\mu, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar\theta^\pm)$

A chiral superfield Φ is a superfield obeying

$$\bar D_\pm \Phi = 0.$$

Φ_1, Φ_2 both chiral $\Rightarrow \Phi_1\Phi_2$ chiral

Φ chiral $\Rightarrow \Phi = \phi(y^\mu) + \theta^\alpha \psi_\alpha(y^\mu) + \theta^+\theta^- F(y^\mu)$

cf Φ scalar   scalar   spinor

$y^\pm = x^\pm - i\theta^\pm\bar\theta^\pm$

An anti-chiral superfield $\bar\Phi$ obeys $D_\pm \bar\Phi = 0$

Φ chiral $\Rightarrow \bar\Phi$ antichiral.

(cplx conjugate)

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A twisted chiral superfield $\widetilde{\Phi}$ is a superfield obeying

$$\overline{D}_+\widetilde{\Phi} = D_-\widetilde{\Phi} = 0$$

$\widetilde{\Phi}_1, \widetilde{\Phi}_2$ both twisted chiral $\Rightarrow$ $\widetilde{\Phi}_1\widetilde{\Phi}_2$ twisted chiral

$$\widetilde{\Phi} = \widetilde{\phi}(\tilde{y}) + \theta^+\widetilde{\psi}_+(\tilde{y}) + \overline{\theta}^-\overline{\widetilde{\psi}}_-(\tilde{y}) + \theta^+\overline{\theta}^-\widetilde{F}(\tilde{y})$$

$$\tilde{y}^\pm = x^\pm \mp i\theta^\pm\overline{\theta}^\pm$$

$\widetilde{\Phi}$ twisted chiral $\Rightarrow$ $\overline{\widetilde{\Phi}}$ twisted antichiral $(D_+\overline{\widetilde{\Phi}} = \overline{D}_-\overline{\widetilde{\Phi}} = 0)$

Supersymmetric Actions

Actions of superfields $S[f_1, \cdots, f_N]$

invariant under $f_i \to f_i + \delta f_i$

$$\delta f_i = \epsilon_+ Q_- f_i - \epsilon_- Q_+ f_i - \overline{\epsilon}_+ \overline{Q}_- f_i + \overline{\epsilon}_- \overline{Q}_+ f_i .$$

Note $\begin{cases} \delta(f_1 f_2) = \delta f_1 f_2 + f_1 \delta f_2 \\ \delta \overset{(-)}{D}_\pm f = \overset{(-)}{D}_\pm \delta f \end{cases}$

under $f_i \to f_i + \delta f_i$

$$K(f_i, D_\pm f_i, \overline{D}_\pm f_i, \cdots) \to K(f_i, D_\pm f_i, \cdots) + \delta K(f_i, D_\pm f_i, \cdots)$$

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So, if we consider

$$\int d^2x\, d^4\theta\ K(f_i, D_\pm f_i, \bar{D}_\pm f_i, \cdots) \qquad (d^4\theta = d\theta^+ d\theta^- d\bar{\theta}^- d\bar{\theta}^+)$$

$$\delta(\ \text{''}\ ) = \int d^2x\, d^4\theta\ \underbrace{(\epsilon_+ Q_- - \epsilon_- Q_+ - \bar{\epsilon}_+ \bar{Q}_- + \bar{\epsilon}_- \bar{Q}_+)K}_{\text{total derivative}}$$

$$\equiv 0$$

Such a ~~term~~ functional is called <u>D-term</u> invariant.

$\Phi_1, \cdots, \Phi_N$ <u>chiral</u> superfield. $W(\cdots)$ hol. function

$$\bar{D}_\pm W(\Phi_1, \cdots, \Phi_N) = 0 \qquad \text{chiral.}$$

Consider

$$\int d^2x\, d^2\theta\ W(\Phi_1, \cdots, \Phi_N) = \int d^2x\, d\theta^- d\theta^+\ W(\Phi_1, \cdots, \Phi_N)\Big|_{\bar{\theta}^+ = \bar{\theta}^- = 0}$$

$$\delta \int d^2x\, d^2\theta\, W(\Phi_1, \cdots, \Phi_N) = \int d^2x\, d\theta^- d\theta^+ (\epsilon_+ Q_- - \epsilon_- Q_+ - \bar{\epsilon}_+ \bar{Q}_- + \bar{\epsilon}_- \bar{Q}_+) W\Big|_{\bar{\theta}^\pm = \bar{\theta}^+ = 0} = 0$$

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm \partial_\pm \xrightarrow{\bar{\theta}^\pm = 0} \frac{\partial}{\partial\theta^\pm} \quad \text{stokes}$$

$$\bar{Q}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm \partial_\pm = \underbrace{\bar{D}_\pm}_{0 \text{ on } W} - 2i\theta^\pm \underbrace{\partial_\pm}_{\text{stokes}}$$

$\int d^2\theta\, W(\Phi_1, \cdots, \Phi_N)$ is called <u>F-term</u>.

$\widetilde{\Phi}_1, \cdots, \widetilde{\Phi}_N$ <u>twisted chiral</u> <u>twisted F-term</u> is

$$\int d^2\widetilde{\theta}\, W(\widetilde{\Phi}_1, \cdots, \widetilde{\Phi}_N) = \int d\bar{\theta}^- d\theta^+\, W(\widetilde{\Phi}_1, \cdots, \widetilde{\Phi}_N)\Big|_{\bar{\theta}^+ = \theta^- = 0}$$

$\delta \int d^2\widetilde{\theta}\, W(\widetilde{\Phi}_1, \cdots, \widetilde{\Phi}_N) = 0$ in the similar way.

<u>Example</u>

Φ a bosonic & scalar chiral superfield

$$\Phi = \phi(y) + \theta^\alpha \psi_\alpha(y) + \theta^+\theta^- F(y)$$

$$\int d^2x\, d^4\theta\, \bar{\Phi}\Phi = \int d^2x \Big(|\partial_0\phi|^2 - |\partial_1\phi|^2 + i\bar{\psi}_-(\partial_0+\partial_1)\psi_- + i\bar{\psi}_+(\partial_0-\partial_1)\psi_+ + |F|^2 \Big)$$

$$\int d^2x\, d^2\theta\, W(\Phi) = W'(\phi)F - W''(\phi)\psi_+\psi_-$$

$$\overline{\int d^2x\, d^2\theta\, W(\Phi)} = \overline{W'(\phi)}\,\bar{F} - \overline{W''(\phi)}\,\bar{\psi}_-\bar{\psi}_+$$

$$\text{Sum} = \int d^2x \Big(|\partial_0\phi| - |\partial_1\phi|^2 - |W'(\phi)|^2 + i\bar{\psi}_-(\partial_0+\partial_1)\psi_- + i\bar{\psi}_+(\partial_0-\partial_1)\psi_+ - W''(\phi)\psi_+\psi_- - \overline{W''(\phi)}\,\bar{\psi}_-\bar{\psi}_+ + |F + \overline{W}'(\phi)|^2 \Big)$$

F : no kinetic term. $\xrightarrow{\text{int out}}$ disappears.

We are left with the action for the system we considered first.

11/7 → 11/12

How to read the SUSY variation of the Component fields?

$$f = f_0 + \theta^\alpha f_\alpha + \bar\theta^\alpha \tilde f_\alpha + \cdots$$

$$\delta f = g_0 + \theta^\alpha g_\alpha + \bar\theta^\alpha \tilde g_\alpha + \cdots$$

$\Rightarrow$ $\delta f_0 = g_0$, $\delta f_\alpha = g_\alpha$, $\vdots$

Non-trivial if constrained.

Constrained case: Φ chiral superfield

$\delta\Phi$ also has the same form since $\bar D_\pm \delta\Phi = \delta \bar D_\pm \Phi = 0$

i.e. $\delta\Phi$ chiral.

$$\Phi = \phi(y) + \theta^\alpha \psi_\alpha(y) + \theta^+\theta^- F(y)$$

$$\Rightarrow \quad \delta\phi = \epsilon_+\psi_- - \epsilon_-\psi_+$$

$$\delta\psi_\pm = \pm 2i\bar\epsilon_\mp \partial_\pm \phi + \epsilon_\pm F$$

$$\delta F = -2i\bar\epsilon_+\partial_-\psi_+ - 2i\bar\epsilon_-\partial_+\psi_-$$

Use "EOM" $F = -\overline{W}'(\bar\phi)$. This δ agrees with the first one.

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Noether procedure $\epsilon_\pm, \bar{\epsilon}_\pm \to \epsilon_\pm(x), \bar{\epsilon}_\pm(x)$

$$\delta S = \int d^2x \Big(\partial_\mu \epsilon_+ G_-^\mu - \partial_\mu \epsilon_- G_+^\mu - \partial_\mu \bar{\epsilon}_+ \bar{G}_-^\mu + \partial_\mu \bar{\epsilon}_- \bar{G}_+^\mu \Big)$$

$$G_\pm^0 = (\partial_0 \pm \partial_1)\bar{\phi}\, \psi_\pm \mp i \bar{\psi}_\mp \overline{W'(\phi)}$$

$$G_\pm^1 = \mp(\partial_0 \pm \partial_1)\bar{\phi}\, \psi_\pm - i \bar{\psi}_\mp \overline{W'(\phi)}$$

$$\bar{G}_\pm^0 = \bar{\psi}_\pm (\partial_0 \pm \partial_1)\phi \pm i \psi_\mp W'(\phi)$$

$$\bar{G}_\pm^1 = \mp \bar{\psi}_\pm (\partial_0 \pm \partial_1)\phi + i \psi_\mp W'(\phi)$$

$$Q_\pm = \int d\sigma\, G_\pm^0 = \int d\sigma \left\{ (\partial_0 \pm \partial_1)\bar{\phi}\, \psi_\pm \mp i \bar{\psi}_\mp \overline{W'(\phi)} \right\}$$

$$\bar{Q}_\pm = \int d\sigma\, \bar{G}_\pm^0 = \int d\sigma \left\{ \bar{\psi}_\pm (\partial_0 \pm \partial_1)\phi \pm i \psi_\mp W'(\phi) \right\}$$

$$\{ Q_\pm, \bar{Q}_\pm \} = H \pm P \quad \ldots \text{etc}$$

(1,1) superspace.

$$\theta^+ = i\theta_1^+ \qquad \theta_1^+ \text{ real}$$

$$\theta^- = i\theta_1^- \qquad \theta_1^- \text{ real.}$$

$$Q'_\pm := Q_\pm + \overline{Q}_\pm \ , \quad D'_\pm := D_\pm + \overline{D}_\pm$$ preserves this subspace

$$Q'_\pm = -i\frac{\partial}{\partial\theta_1^\pm} + 2\theta_1^\pm\partial_\pm$$

$$D'_\pm = -i\frac{\partial}{\partial\theta_1^\pm} - 2\theta_1^\pm\partial_\pm$$

$$\left[\begin{array}{l} \{Q'_\pm, Q'_\pm\} = -4i\partial_\pm, \quad \{Q'_+, Q'_-\} = 0 \\ \{D'_\alpha, Q'_\beta\} = 0 . \end{array}\right.$$

One can consider superfield defined on this (1,1) superspace.

$$\Phi^I = \phi^I + i\theta_1^+\psi_+^I + i\theta_1^-\psi_-^I + i\theta_1^+\theta_1^- f^I$$

$$S = \int d^2x\, d\theta_1^+ d\theta_1^- \left(\frac{1}{2}\left(\underset{\substack{\uparrow \\ \text{metric} \\ \text{symmetric}}}{g_{IJ}(\Phi)} + \underset{\substack{\uparrow \\ \text{antisym} \\ \text{metric}}}{B_{IJ}(\Phi)} \right) D'_-\Phi^I D'_+\Phi^J \right)$$

$= \cdots$ eliminate aux. field f^I

$$= \int d^2x \Big[\frac{1}{2} g_{IJ} \left(\partial_t \phi^I \partial_t \phi^J - \partial_\sigma \phi^I \partial_\sigma \phi^J \right)$$

$$+ \frac{1}{2} B_{IJ} \left(\partial_t \phi^I \partial_\sigma \phi^J - \partial_\sigma \phi^I \partial_t \phi^J \right)$$

$$+ \frac{i}{2} g_{IJ} \psi_-^I \left(\nabla_t^{(-)} + \nabla_\sigma^{(-)} \right) \psi_-^J$$

$$+ \frac{i}{2} g_{IJ} \psi_+^I \left(\nabla_t^{(+)} - \nabla_\sigma^{(+)} \right) \psi_+^J$$

$$+ \frac{1}{4} R^{(-)}_{IJKL} \psi_+^I \psi_+^J \psi_-^K \psi_-^L \Big].$$

$$\nabla_\mu^{(\pm)} \psi^I = \partial_\mu \psi^I + \partial_\mu X^J \Big(\underset{\substack{\uparrow \\ \text{LeviCivita}}}{\Gamma^I_{JK}} \pm \frac{1}{2} g^{IM} \underbrace{H_{JMK}}_{H = dB} \Big) \psi^K$$

$R^{(-)}$ … curvature of $\nabla^{(-)}$

<u>Note</u> $\frac{1}{2} \int_\Sigma d^2x \, B_{IJ} \left(\partial_t \phi^I \partial_\sigma \phi^J - \partial_\sigma \phi^I \partial_t \phi^J \right)$

$$= \int_\Sigma \phi^* B$$

$$= \underbrace{\int_B \hat{\phi}^* H}_{\text{WZ term}}$$

$$\begin{array}{l} B \xrightarrow{\hat{\phi}} M \\ \cup \quad \nearrow \phi \\ \Sigma = \partial B \end{array}$$

$\mathcal{F}$: (2,2) superfield

$$\longrightarrow \ \mathcal{F}|_{(1,1)} = \mathcal{F}\left(\theta^+ = i\theta_1^+ ,\ \theta^- = i\theta_1^- ,\ \bar\theta^+ = -i\theta_1^+ ,\ \bar\theta^- = -i\theta_1^- \right)$$

(1,1) reduction.

$$\int d^4\theta\, \mathcal{F} = \frac{1}{4}\int d^2\theta_1 \left[(D_+ - \bar D_+)(D_- - \bar D_-)\mathcal{F} \right]_{(1,1)} .$$

Note Φ chiral ~~or twisted chiral~~

$$\Rightarrow \left[D_\pm \bar\Phi \right]_{(1,1)} = D'_\pm \bar\Phi_{(1,1)} \ , \left[\bar D_\pm \bar{\bar\Phi} \right]_{(1,1)} = D'_\pm \bar\Phi_{(1,1)}$$

$\tilde\Phi$ twisted chiral

$$\Rightarrow (D_+ \tilde\Phi)_{(1,1)} = D'_+ \tilde\Phi_{(1,1)} ,\ (\bar D_- \tilde\Phi)_{(1,1)} = D'_- \tilde\Phi_{(1,1)} ,\ \cdots .$$

$$\int d^4\theta\, K(\Phi^i, \bar\Phi^{\bar i}, \tilde\Phi^p, \bar{\tilde\Phi}^{\bar p})$$

$$= \cdots = \frac{1}{2}\int d^2\theta_1 \left[(g_{IJ} + B_{IJ})\, \Phi^I_{(1,1)}\, \Phi^J_{(1,1)} \right]_{(1,1)}$$

$$g_{IJ} = \left(\begin{array}{cc|cc} & K_{i\bar j} & & \\ K_{\bar j i} & & & \\ \hline & & & -K_{p\bar q} \\ & & -K_{\bar p q} & \end{array} \right) ,\quad B_{IJ} = \left(\begin{array}{cc|cc} & & & K_{i\bar p} \\ & & K_{\bar i p} & \\ \hline & -K_{p\bar j} & & \\ -K_{\bar p i} & & & \end{array} \right) .$$

$\cdots$ an example of generalized Kähler mfd.

Gates Hull Roček.

In general, ~~for~~

$$S = \frac{1}{2}\int d^2\theta_1 \left(g_{IJ} + B_{IJ}\right) D^1_- \Phi^I D^1_+ \Phi^J$$

~~to~~ has (2,2) supersymmetry iff

M has two complex structures J_+, J_-

s.t. $\nabla^{(+)} J_+ = 0 \quad \nabla^{(-)} J_- = 0$.

~~The ex~~ — generalized Kähler manifold —

The extra SUSY:

$$\delta_\eta \Phi^I = i\eta^1_+ D^1_- \Phi^K J_{+}{}^{I}{}_{K} + i\eta^1_- D_+ \Phi^K J_{-}{}^{I}{}_{K}$$

In the previous example

$$J_+ = \left(\begin{array}{cc|cc} i & & & \\ & -i & & \\ \hline & & i & \\ & & & -i \end{array}\right) \qquad J_- = \left(\begin{array}{cc|cc} -i & & & \\ & i & & \\ \hline & & i & \\ & & & -i \end{array}\right)$$