Today : topological field theory

Supercurrent & supercharges

Conserved charges ... Nöther procedure $\epsilon_\pm, \bar{\epsilon}_\pm \to \epsilon_\pm(x), \bar{\epsilon}_\pm(x)$

$$\delta S = \int d^2x \left(\partial_\mu \epsilon_+ G_-^\mu - \partial_\mu \epsilon_- G_+^\mu - \partial_\mu \bar{\epsilon}_+ \bar{G}_-^\mu + \partial_\mu \bar{\epsilon}_- \bar{G}_+^\mu \right)$$

$G_\pm^\mu, \bar{G}_\pm^\mu$ supercurrent $\qquad \partial_\mu G_\pm^\mu = \partial_\mu \bar{G}_\pm^\mu = 0$ local conservation.

supercharges

$$Q_\pm, \bar{Q}_\pm^\mu = \int dx^1 G_\pm^0, \int dx^1 \bar{G}_\pm^0, \qquad \dot{Q}_\pm = \dot{\bar{Q}}_\pm = 0 \quad \text{Conserved}$$

e.g. LG model with $W(\phi)$

$$G_\pm^0 = (\partial_0 \pm \partial_1)\bar{\phi}\, \psi_\pm \mp i \bar{\psi}_\mp \overline{W'(\phi)}$$

$$G_\pm^1 = \mp (\partial_0 \pm \partial_1)\bar{\phi}\, \psi_\pm - i \bar{\psi}_\mp \overline{W'(\phi)}$$

$$\bar{G}_\pm^0 = \bar{\psi}_\pm (\partial_0 \pm \partial_1)\phi \pm i \psi_\mp W'(\phi)$$

$$\bar{G}_\pm^1 = \mp \bar{\psi}_\pm (\partial_0 \pm \partial_1)\phi + i \psi_\mp W'(\phi)$$

NLSM on Kähler mfd

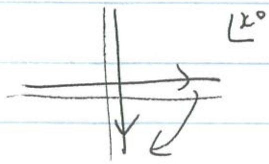
$$G_\pm^0 = g_{i\bar{j}} (\partial_0 \pm \partial_1)\bar{\phi}^{\bar{j}} \psi_\pm^i, \quad G_\pm^1 = \mp g_{i\bar{j}} (\partial_0 \pm \partial_1)\bar{\phi}^{\bar{j}} \psi_\pm^i$$

$$\bar{G}_\pm^0 = g_{i\bar{j}} \bar{\psi}_\pm^{\bar{j}} (\partial_0 \pm \partial_1)\phi^i, \quad \bar{G}_\pm^1 = \mp g_{i\bar{j}} \bar{\psi}_\pm^{\bar{j}} (\partial_0 \pm \partial_1)\phi^i$$

Exercise : Find $G_\pm^\mu, \bar{G}_\pm^\mu$ for NLSM on generalized Kähler mfd.

So far, we have been studying systems on flat Minkowski space.

Systems on Euclidean space are obtained by <u>Wick rotation</u>



$$x^0 \to -i x^2$$

<u>Operator</u> $\quad e^{-i\Delta x^0 H} \to e^{-i(-i\Delta x^2)H} = e^{-\Delta x^2 H}$ (good operator)

<u>Path integral</u> $\quad e^{iS_{Minkowski}} \to e^{-S_{Euclidean}}$

$$S_E = -i S_{Min}\Big|_{x^0 \to -ix^2}$$

<u>e.g.</u> Complex scalar ϕ : $S_{Min} = \int dx^1 dx^0 \{ \partial_0 \bar\phi \partial_0 \phi - \partial_1 \bar\phi \partial_1 \phi \}$

$$S_E = -i \int dx^1 (-i dx^2) \{ -\partial_2 \bar\phi \partial_2 \phi - \partial_1 \bar\phi \partial_2 \phi \}$$

$$= \int dx^1 dx^2 \{ |\partial_1 \phi|^2 + |\partial_2 \phi|^2 \} \geq 0.$$

<u>e.g.</u> Dirac fermion $\psi_\pm, \bar\psi_\pm$: $S_{Min} = \int dx^1 dx^0 \{ i\bar\psi_-(\partial_0 + \partial_1)\psi_- + i\bar\psi_+(\partial_0 - \partial_1)\psi_+ \}$

$$S_E = -i \int dx^1 (-i dx^2) \{ i\bar\psi_-(i\partial_2 + \partial_1)\psi_- + i\bar\psi_+(i\partial_2 - \partial_1)\psi_+ \}$$

$$= \int dx^1 dx^2 \{ 2i\bar\psi_- \partial_{\bar z} \psi_- - 2i \bar\psi_+ \partial_z \psi_+ \}$$

$$z = x^1 + i x^2.$$

Note: ~~hermiticity is lost, or~~

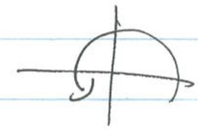
We no longer consider $\bar\psi_\pm$ as complex conj. of $\psi_\pm$.

$\psi_\pm(x^\pm) \to e^{\mp\gamma}\psi_\pm(e^{\mp 2\gamma}x^\pm,$

Lorentz group: $SO(1,1) \to SO(2)$

$$M \leftrightarrow -2x^0\frac{\partial}{\partial x^1} - 2x^1\frac{\partial}{\partial x^0} \to 2ix^2\frac{\partial}{\partial x^1} - 2ix^1\frac{\partial}{\partial x^2}$$

$$M_E := iM \leftrightarrow 2\left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right)$$



$$i[M, \psi_\pm] = \mp\psi_\pm - 2\left(x^0\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^0}\right)\psi_\pm$$

$$\Rightarrow i[M_E, \psi_\pm] = \mp i\psi_\pm + 2\left(x^1\frac{\partial}{\partial x^2} + x^2\frac{\partial}{\partial x^1}\right)\psi_\pm .$$

also $i[M_E, \overline{\psi}_\pm] = \mp i\overline{\psi}_\pm + 2\left(x^1\frac{\partial}{\partial x^2} + x^2\frac{\partial}{\partial x^1}\right)\overline{\psi}_\pm$

(Note again $\overline{\psi}_\pm \neq \psi_\pm^\dagger$
~~loss of hermiticity~~)

$\exists$ Supersymmetry after Wick rotation:

$$\delta S_E = -\int d^2x_E\left(\partial_\mu\epsilon_+ G_-^\mu - \partial_\mu\epsilon_- G_+^\mu - \partial_\mu\overline{\epsilon}_+\overline{G}_-^\mu + \partial_\mu\overline{\epsilon}_-\overline{G}_+^\mu\right)$$

For $\epsilon_\pm, \overline{\epsilon}_\pm$ constant $\delta S_E = 0$.

$$\left[\begin{array}{l} G_\pm^1, \overline{G}_\pm^1 = G_\pm^1, \overline{G}_\pm^1 \Big|_{x^0 \to -ix^2} \\ G_\pm^2, \overline{G}_\pm^2 = iG_\pm^0, i\overline{G}_\pm^0 \Big|_{x^0 \to -ix^2} \end{array}\right.$$

Systems on curved Riemann surface is obtained by covariantization. For $ds^2 = h_{\mu\nu} dx^\mu dx^\nu$

$$d^2x \to \sqrt{h}\, d^2x = h_{z\bar{z}}\, d^2z$$

$$\sum_{\mu=1,2} |\partial_\mu \phi|^2 \to h(d\phi, d\phi) = h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

$$\partial_\mu \overset{(-)}{\psi}_- \to \left(\partial_\mu + \frac{1}{2}\Gamma^z_{\mu z}\right) \overset{(-)}{\psi}_-$$

$$\partial_\mu \overset{(-)}{\psi}_+ \to \left(\partial_\mu + \frac{1}{2}\Gamma^{\bar{z}}_{\mu\bar{z}}\right) \overset{(-)}{\psi}_+$$

$\psi_-, \bar{\psi}_-$ sections of $\sqrt{K}$

$\psi_+, \bar{\psi}_+$ sections of $\sqrt{\bar{K}}$

K ... canonical bdle $\sim dz$

(For spinors, one needs to choose a spin structure (a choice of $\sqrt{K}$).)

SUSY variation makes sense.

provided

$\epsilon_-, \bar{\epsilon}_-$ sections of $\sqrt{K}$

$\epsilon_+, \bar{\epsilon}_+$ sections of $\sqrt{\bar{K}}$

e.g. LG.

$$\delta\phi = \epsilon_+\psi_- - \epsilon_-\psi_+$$

$$\delta\psi_+ = 2i\bar{\epsilon}_- \partial_{\bar{z}}\phi - \epsilon_+ \overline{W'(\phi)}$$

$$\delta\psi_- = 2i\bar{\epsilon}_+ \partial_z\phi - \epsilon_- \overline{W'(\phi)}$$

$$\delta\bar{\phi} = -\bar{\epsilon}_+\bar{\psi}_- + \bar{\epsilon}_-\bar{\psi}_+$$

$$\delta\bar{\psi}_+ = -2i\epsilon_- \partial_{\bar{z}}\bar{\phi} - \bar{\epsilon}_+ W'(\phi)$$

$$\delta\bar{\psi}_- = -2i\epsilon_+ \partial_z\bar{\phi} - \bar{\epsilon}_- W'(\phi)$$

$$\delta S_E = -\int \sqrt{h}\, d^2x \left(\nabla_\mu^{\sqrt{\bar{K}}} \epsilon_+ G^\mu_- - \nabla_\mu^{\sqrt{K}} \epsilon_- G^\mu_+ - \nabla_\mu^{\sqrt{K}} \bar{\epsilon}_+ \bar{G}^\mu_- + \nabla_\mu^{\sqrt{K}} \bar{\epsilon}_+ \bar{G}^\mu_+ \right)$$

Now, if it is curved, $R(h) \neq 0$,

There is no covariantly constant spinors!

$\delta S_E \neq 0$ supersymmetry is lost!

~~There~~ A way to recover : <u>Twisting</u>

It is to make the variational parameter scalars

So that $\exists$ constant parameters.

$$e^{iM_E} : Q_\pm \to e^{\mp i\gamma} Q_\pm , \quad \bar{Q}_\pm \to e^{\mp i\gamma} \bar{Q}_\pm .$$

recall R-symmetries

$$e^{i\alpha F_V} : Q_\pm \to e^{-i\alpha} Q_\pm , \quad \bar{Q}_\pm \to e^{i\alpha} \bar{Q}_\pm$$

$$e^{i\beta F_A} : Q_\pm \to e^{\mp i\beta} Q_\pm , \quad \bar{Q}_\pm \to e^{\pm i\beta} \bar{Q}_\pm$$

Idea: ~~Combine~~ dress M_E with appropriate R-symmetry

so that some of supercharges ~~are~~ invariant.

There are essentially two ways $<$ A twist, B twist

A-twist $M_E + F_V$ as new M_E'.

Q_- & $\overline{Q}_+$ are scalars $\left(\begin{array}{c}\text{in particular} \\ Q_A = \overline{Q}_+ + Q_-\end{array}\right)$

If some field f is originally a section of $K^{\frac{m}{2}}$ $\left(\sqrt{K} = K^{\frac{1}{2}},\ \overline{K} = K^{-1}\right)$ & has charge $F_V = q$

after twisting f is a section of $K^{\frac{m+q}{2}}$

$\longrightarrow$ covariantized action is different!

e.g. ~~NLS~~ sigma model

	Originally	F_V	after twist		
ψ_+	$\sqrt{\overline{K}} = K^{-\frac{1}{2}}$	-1	K^{-1}	(antihol 1-form)	$d\bar{z}$
ψ_-	$\sqrt{K} = K^{\frac{1}{2}}$	-1	K^{0}	scalar	
$\overline{\psi}_+$	$K^{-\frac{1}{2}}$	1	K^{0}	scalar	
$\overline{\psi}_-$	$K^{\frac{1}{2}}$	1	K^{1}	(holo 1-form)	dz

B-twist $M_E + F_A$ as new M_E

$\overline{Q}_+$ & $\overline{Q}_-$ are scalars $\left(\begin{array}{c}\text{in particular} \\ Q_B = \overline{Q}_+ + \overline{Q}_-\end{array}\right)$

e.g. LG model

	originally	F_A	after twist	
ψ_+	$K^{-\frac{1}{2}}$	-1	K^{-1}	antihol 1-form
ψ_-	$K^{\frac{1}{2}}$	1	K^{1}	hol 1-form
$\overline{\psi}_+$	$K^{-\frac{1}{2}}$	1	K^{0}	scalars
$\overline{\psi}_-$	$K^{\frac{1}{2}}$	-1	K^{0}	scalar

Important: To be able to twist, ~~F_V or F_A~~

- F_V or F_A must be symmetry
- Charges of F_V or F_A must be integers.

	A twistable ?	B-twistable ?
NLSM on generic Kähler mfd	YES	NO (F_A anomalous)
NLSM on CY manifold	YES	YES
LG model with generic superpotential	NO	YES
LG model with homogeneous superpotential	usually NO ~~in generally~~ (F_V conserved but charges are usually fractional) cured by "orbifold".	YES

Correlation Functions (Σ, h) 2d mfld with metric

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle_{\Sigma, h} = \int \mathcal{D}_{\Sigma, h} X \, e^{-S_{\Sigma, h}(X)} \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s)$$

Consider a B-twisted theory $Q = Q_B$

$$☆ \cdot \langle [Q, \mathcal{O}(x)] \rangle_{\Sigma, h} = 0$$

.. called <u>Ward identity</u>

... Consider a finite dimensional model

$$\langle \mathcal{O} \rangle = \int_{\mathcal{F}} d\mu \, e^{-S} \mathcal{O}$$

$(\mathcal{F}, d\mu \, e^{-S})$ ... specifies a "theory".

$T : \mathcal{F} \to \mathcal{F}$ $\overset{\text{orientation pres.}}{\vee}$ diffeo is a <u>symmetry</u> of the theory

iff $T^*(d\mu \, e^{-S}) = d\mu \, e^{-S}$

$\left[\begin{array}{l} T \text{ s.t. } T^* S = S \text{ is a classical symmetry.} \\ \text{Such a thing is anomalous if} \\ \qquad T^* d\mu \neq d\mu \end{array} \right]$

$$T : \text{symmetry} \Rightarrow \langle T^* \mathcal{O} \rangle = \int_{\mathcal{F}} d\mu \, e^{-S} T^* \mathcal{O} = \int_{\mathcal{F}} T^*(d\mu \, e^{-S} \mathcal{O})$$

$$= \int_{\mathcal{F}} d\mu\, e^{-S}\, \mathcal{O} = \langle \mathcal{O} \rangle$$

$$\therefore \langle T^* \mathcal{O} \rangle = \langle \mathcal{O} \rangle$$

T_t : one-parameter group of symmetries

$\cong e^{iQt}$ or $T_t^* \mathcal{O} = e^{iQt} \mathcal{O} e^{-iQt}$

take $\frac{d}{dt}\Big|_{t=0}$ of $\langle T_t^* \mathcal{O} \rangle = \langle \mathcal{O} \rangle$

$$\Rightarrow \langle [Q, \mathcal{O}] \rangle = 0 .$$

This is the Ward identity.

☆ $\langle [Q, \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s)] \rangle_{\Sigma, h} = 0$ is also true.

☆ If $[Q, \mathcal{O}_i(x_i)] = 0$ $i = 2, \cdots, s$, (i.e. chiral)

$$\langle [Q, \mathcal{O}_1(x_1)] \mathcal{O}_2(x_2) \cdots \mathcal{O}_s(x_s) \rangle_{\Sigma, h} = 0.$$

in particular $\langle \partial_\mu \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \mathcal{O}_s(x_s) \rangle_{\Sigma, h} = 0$

$\therefore$ If $\mathcal{O}_i$ all chiral

$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle_{\Sigma, h}$ does not depend on x_i's.

$\langle \cdots \rangle_{\Sigma, h}$ defines a multilinear form on chiral ring.

☆ In many cases,

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\}$$

- $\sim$ local version of $H \pm P = \{Q, Q_\pm\}$

- $h \to h + \delta h$

$$\delta \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle_{\Sigma, h} = \text{Const} \oint \left\langle \int d^2z \sqrt{h}\, \delta h^{\mu\nu} T_{\mu\nu}(z)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \right\rangle_{\Sigma, h}$$

$\therefore \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle_{\Sigma, h}$ for $\mathcal{O}_i$ all chiral

does not depend on $h_{\mu\nu}$.

It depends only on the topology of Σ

$\longrightarrow$ topological field theory.

☆ $\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_s(x_s) \rangle_\Sigma = \langle (\mathcal{O}_1 \mathcal{O}_2)(x_2)\, \mathcal{O}_3(x_3) \cdots \mathcal{O}_s(x_s) \rangle_\Sigma$

product in chiral ring.

Choose a basis of chiral ring $\{\mathcal{O}_\alpha\}_{\alpha \in A}$

$$\eta_{\alpha\beta} := \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle_{S^2} = \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathbb{1} \rangle_{S^2}$$

$\mathcal{O}_\alpha \mathcal{O}_\beta = \sum_\gamma \mathcal{O}_\gamma C^\gamma_{\alpha\beta}$

structure constant

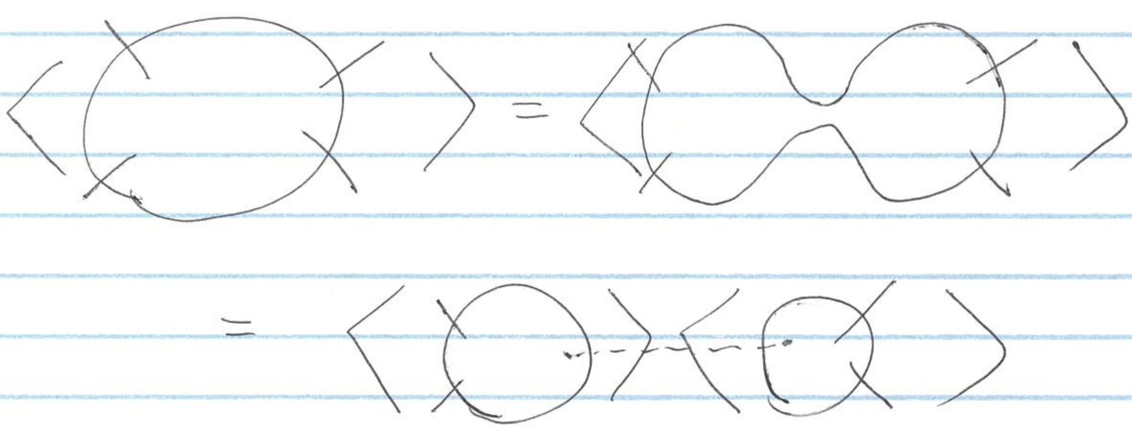
$\Rightarrow \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \rangle_{S^2} = \eta_{\alpha\delta} C^\delta_{\beta\gamma}$

In general $\eta_{\alpha\beta}$ is a non-deg. form (later)

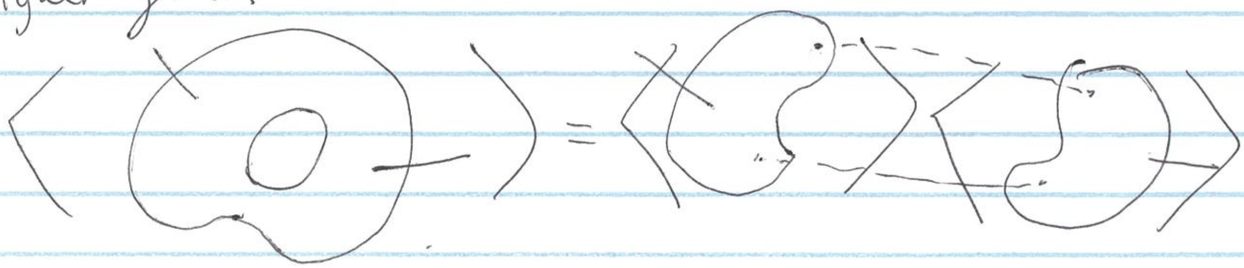
$C^\delta_{\beta\gamma} = \eta^{\delta\alpha} \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \rangle$ $\quad$ $\eta^{\delta\alpha}$ .. inverse of $\eta_{\alpha\beta}$

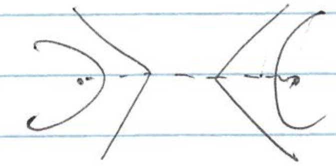
Chiral ring is determined by S^2 3pt fcns

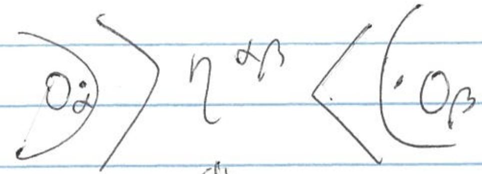
Also, higher points



higher genus



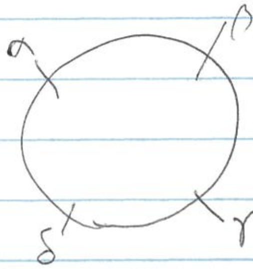
here 

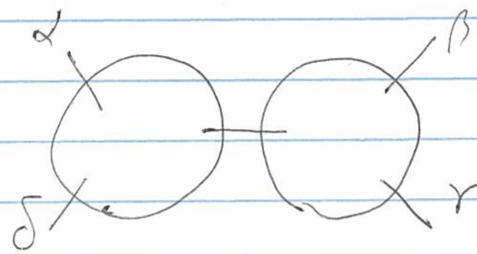
$=$ 

inverse of $\eta_{\alpha\beta}$

(later)

e.g.

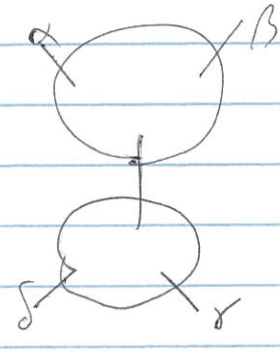


$=$ 

$$C^{\delta}_{\alpha\kappa} C^{\kappa}_{\beta\gamma}$$

$$\|$$

$$C^{\delta}_{\kappa\gamma} C^{\kappa}_{\alpha\beta}$$

$=$ 

$$\Longleftrightarrow \quad O_\alpha (O_\beta O_\gamma) = (O_\alpha O_\beta) O_\gamma$$

associativity of chiral ring product!