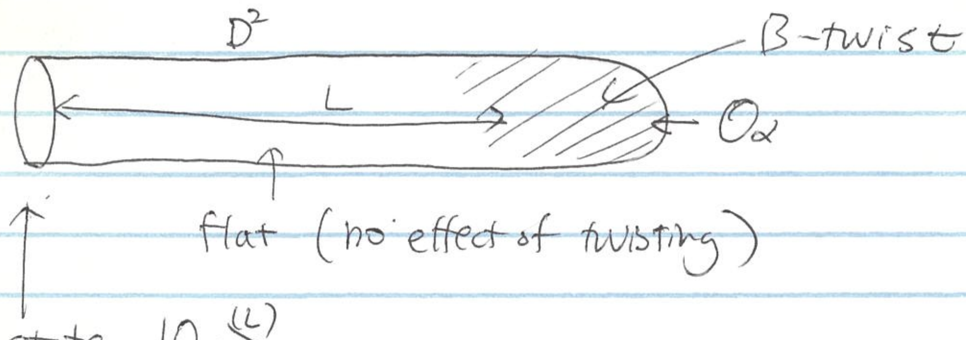
Field ↔ State Correspondence

When a (2,2) theory is B-twistable,
(A- "),
~~the~~ SUSY groundstates ~~are~~ in one to one correspondence with chiral ring elements
(twisted chiral ring)

i.e. Q_B - cohomology classes of fields
(Q_A - cohomology classes of fields).

Construction of the correspondence:

consider the Riemann surface like this:



a state $|\mathcal{O}_\alpha\rangle^{(L)}$ appears.

i.e.
$$\Psi_\alpha^{(L)}(X_1) = \int_{X_1 = X|_{\partial D^2}} \mathcal{D}_{D^2} X \, e^{-S_{D^2}(X)} \, \mathcal{O}_\alpha(0)$$

Note 1. Fields are periodic along $S^1 = \partial D^2$ (called RR sector)

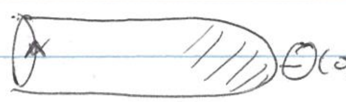
— the effect of twisting

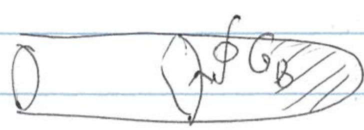
(before twist, fermions would have anti-periodic b.c. on $S^1 = \partial D^2$ (NS-NS-sector))

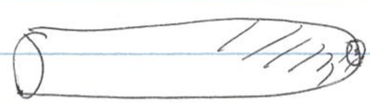
2. $Q_B | \mathcal{O}_\alpha \rangle^{(L)} = 0$, $\quad | \mathcal{O}_\alpha + [Q_B, \lambda] \rangle^{(L)} = | \mathcal{O}_\alpha \rangle^{(L)} + Q_B | \lambda \rangle^{(L)}$

Thus $\mathcal{O}_\alpha \mapsto | \mathcal{O}_\alpha \rangle^{(L)}$ defines a map $R_B \to H^k(Q_B)$

($\because$) $Q_B = \oint_{S^1} G_B$ ← closed one form in B-twisted theory (supercurrent)

$Q_B | \mathcal{O} \rangle^{(L)} = \oint_{S^1} G_B$  $\mathcal{O}(0)$

$=$  $\mathcal{O}(0)$

$=$  $\oint_0 G_B \mathcal{O}(0) = [Q_B, \mathcal{O}](0)$

$= | [Q_B, \mathcal{O}] \rangle^{(L)}$ //.

3. by topological invariance $T_{\mu\nu} = \{Q_B, G_{\mu\nu}\}$
Q_B-cohomology class does not depend on the metric on D^2, in particular L.

The limit $L \to \infty$ projects onto a SUSY ground state

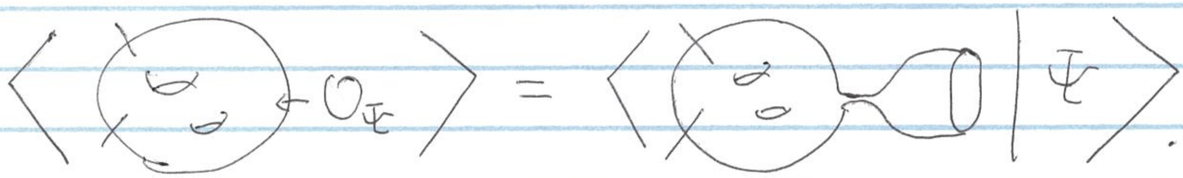
$$|O_\alpha\rangle := \lim_{L\to\infty} |O_\alpha\rangle^{(L)} \in \mathcal{H}_{SUSY}. \qquad \left(\lim_{L\to\infty} e^{-LH} = \begin{cases} 1 & \text{SUSY gnd} \\ 0 & E>0 \end{cases} \right)$$

We obtain a map

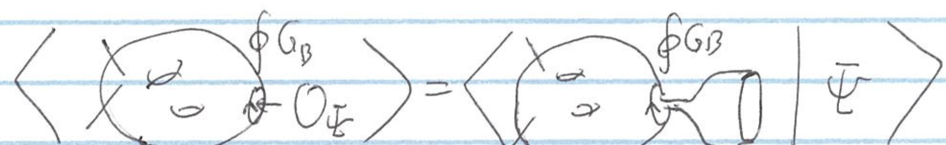
$$R_B \to \mathcal{H}_{SUSY}$$
$$O_\alpha \mapsto |O_\alpha\rangle$$

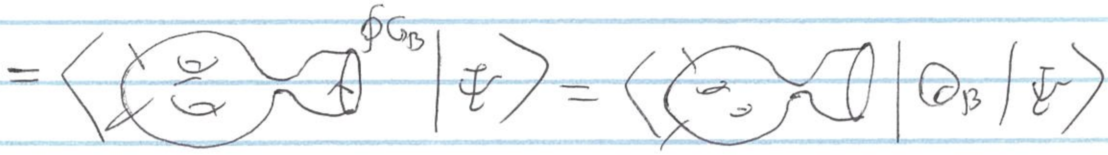
The inverse

Given a SUSY ground state Ψ, a field O_Ψ is obtained by



It is chiral,





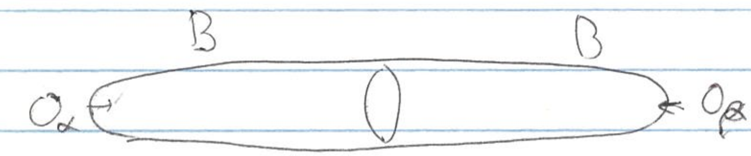
$= 0$.

That it is the inverse of $O_\alpha \mapsto |O_\alpha\rangle$ is obvious.

$|O_\alpha\rangle$ basis of $\mathcal{H}_{SUSY}$

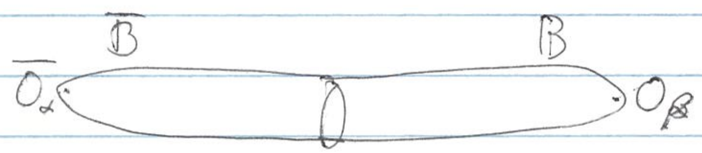
$\langle O_\beta|$ basis of $\check{\mathcal{H}}_{SUSY}$

Thus: $\eta_{\alpha\beta} = \langle O_\alpha | O_\beta \rangle$ is nondegenerate.



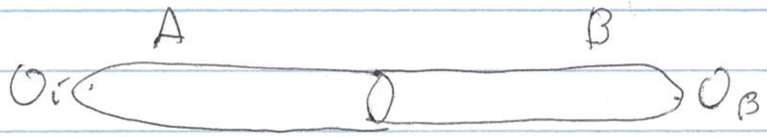
$= \eta_{\alpha\beta}$

topological metric



$= g_{\bar{\alpha}\beta}$

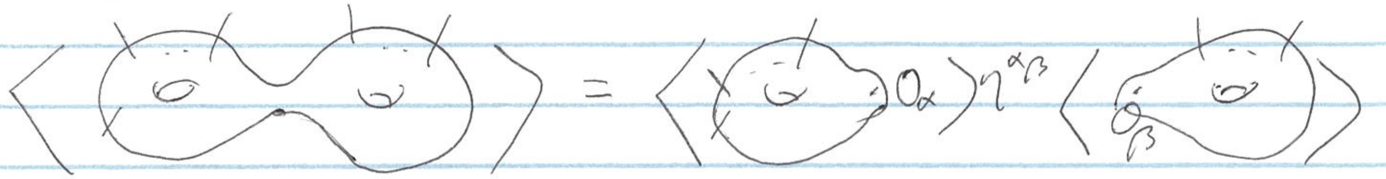
tt^* metric



$= ?$

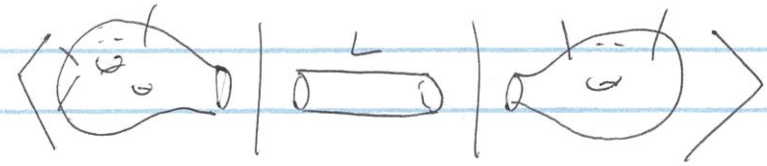
(when both twistable)

Derivation of

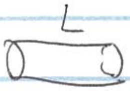


in the B-twisted model. $\eta^{\alpha\beta}$ inverse of $\eta_{\alpha\beta} = \langle O_\alpha \bigcirc O_\beta \rangle$

LHS =

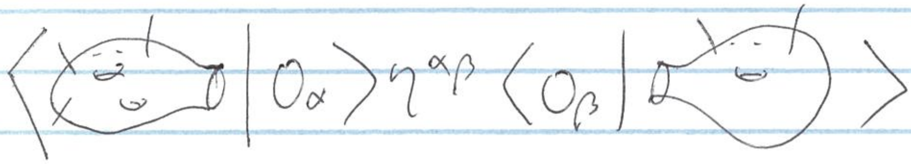


$\lim_{L\to\infty}$

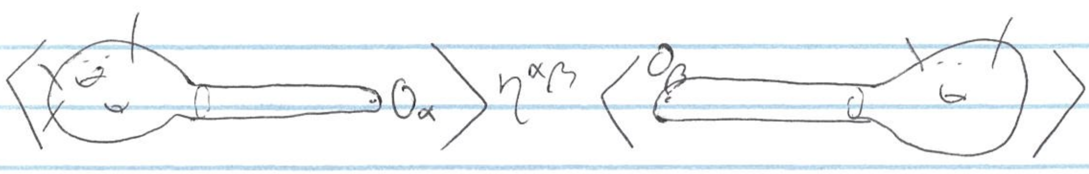


= projection to the ground states $= |O_\alpha\rangle \eta^{\alpha\beta} \langle O_\beta|$

=



=

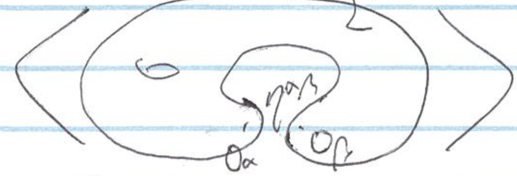


= RHS.

Also

$$= \int \mathcal{D}_{S^1}X_1 \mathcal{D}_{S^1}X_2 \; Z_\Sigma(X_1, X_2)\, \Psi_\alpha(X_2)\, \eta^{\alpha\beta}\, \Psi_\beta^\dagger(X_1)$$

=



$= \mathrm{Tr}\left((-1)^F Z_\Sigma\right)$

or $\int \mathcal{D}_{S^1}X_1 \mathcal{D}_{S^1}X_2 \, (-1)^{|\alpha||\beta|}\, \Psi_\beta^\dagger(X_1)\, Z_\Sigma(X_1, X_2)\, \Psi_\alpha(X_2)$

~~Decoupling Theorem~~ Parameter dependence

Recall that SUSY Lagrangian can be written as

$$L = \int d^4\theta \, K(\Phi_i, \bar{\Phi}_i, \tilde{\Phi}_p, \bar{\tilde{\Phi}}_p ; \xi_A)$$
$$+ \left[\int d^2\theta \, W(\Phi_i ; t_a) + c.c. \right]$$
$$+ \left[\int d^2\tilde{\theta} \, \tilde{W}(\tilde{\Phi}_p ; \tilde{t}_\alpha) + c.c. \right]$$

Φ_i : chiral superfield

$\tilde{\Phi}_p$: twisted chiral superfield

$\xi_A, t_a, \tilde{t}_\alpha$ ... parameters

We call t_a -- chiral parameters

$\tilde{t}_\alpha$ -- twisted chiral parameters

decoupling thm

Correlation functions of A-twisted model (when possible) depends holomorphically on twisted chiral parameters and are indep of chiral parameters.

Correlation function of B-twisted model (when possible) depend holomorphically on chiral parameters and are indep. of twisted chiral parameters.

(physically -- no mixing of chiral & twisted chiral parts)

Let us show this (say, the latter part)

We want to show that correlators of B-twisted model are independent of $\bar{t}_a$, $\tilde{t}_\alpha$, $\bar{\tilde{t}}_\alpha$.

Namely
$$\left\langle \int d^2x \int d^2\bar\theta \frac{\partial}{\partial \bar t_a} \overline{W}(\bar\Phi_i, \bar t_a)\ O_1 O_2 \cdots \right\rangle_\Sigma = 0 \quad ①$$

$$\left\langle \int d^2x \int d^2\tilde\theta \frac{\partial}{\partial \tilde t_\alpha} \widetilde{W}(\tilde\Phi_p, \tilde t_\alpha)\ O_1 O_2 \cdots \right\rangle_\Sigma = 0 \quad ②$$

$$\left\langle \int d^2x \int d^2\bar{\tilde\theta} \frac{\partial}{\partial \bar{\tilde t}_\alpha} \overline{\widetilde{W}}(\bar{\tilde\Phi}_p, \bar{\tilde t}_\alpha)\ O_1 O_2 \cdots \right\rangle_\Sigma = 0 \quad ③$$

One can show that $① = [\overline{Q}_+, [\overline{Q}_-, \partial_{\bar a} \overline{W}(\bar\phi_i, \bar t_a)]]$

$$② = [\overline{Q}_+, [Q_+, \partial_\alpha \widetilde{W}(\phi_p, \tilde t_\alpha)]]$$

$$③ = [\overline{Q}_+, [Q_-, \partial_{\bar\alpha} \overline{\widetilde{W}}(\bar\phi_p, \bar{\tilde t}_\alpha)]]$$

They are all of the form $[Q_B, \text{---}]$ up to total derivative.

Thus they all vanish.

proof of the expressions ~~for~~ ①, ② ③ :

Here I prove the basic one

(this should have been done when we discussed superfields sorry!)

$$\int d^2\theta \, W(\Phi) = [Q_-, [Q_+, W(\phi)]]$$

(Others follow by complex conjugation or $Q_- \leftrightarrow \bar{Q}_-$).

Recall that the chiral superfield $W(\Phi)$ or simply Φ can be written as

$$\Phi = \phi(y) + \theta^+ \psi_+(y) + \theta^- \psi_-(y) + \theta^+\theta^- F(y)$$

$$y^\pm = x^\pm - i\theta^\pm \bar{\theta}^\pm .$$

wrt $(y, \theta^\pm, \bar{\theta}^\pm)$, $Q_\pm = \frac{\partial}{\partial \theta^\pm}$, $\bar{Q}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} - 2i\theta^\pm \frac{\partial}{\partial y^\pm}$

$[Q_\pm, \Phi] = Q_\pm \Phi$

$\Rightarrow$ $[Q_\pm, \phi(y)] - \theta^+ [Q_\pm, \psi_+(y)] - \theta^- [Q_\pm, \psi_-(y)] + \theta^+\theta^- [Q_\pm, F(y)]$

$= \psi_\pm(y) \pm \theta^\mp F(y)$

$\psi_\pm = [Q_\pm, \phi]$, $F = -[Q_+, \psi_-] = [Q_-, \psi_+]$

$= [Q_-, [Q_+, \phi]]$

$\int d^2\theta \, \Phi = F = [Q_-, [Q_+, \phi]]$. Same for $\Phi \to W(\Phi)$.

Note also

$$[\bar{Q}_\pm, \bar{\Phi}] = \bar{\mathcal{D}}_\pm \bar{\Phi}$$

$$= -2i\theta^\pm \partial_\pm \phi(y) - 2i\theta^\pm \theta^\mp \partial_\pm \psi_\mp(y)$$

$$\Rightarrow [\bar{Q}_\pm, \phi] = 0, \quad [\bar{Q}_\pm, \psi_\pm] = 2i\partial_\pm \phi$$

$$[\bar{Q}_\pm, \psi_\mp] = 0$$

$$[\bar{Q}_\pm, F] = \mp 2i \partial_\pm \psi_\mp .$$

- lowest component of a chiral superfield is chiral.
- $\psi_\pm = [Q_\pm, \phi]$, $F = [Q_-, [Q_+, \phi]]$ — descent eqn.

  ∴ One can reconstruct a chiral superfield by a chiral field.

— ~~descent eqn~~

Example

For a non-linear sigma model on a Kähler mfd

① chiral parameters are Complex structure parameters

② twisted chiral parameters are "complexified Kähler class parameters".

Kähler class $[\omega] \in H^2(X, \mathbb{R})$ $\omega = \frac{i}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$

Complexified Kähler class $[\omega] - i[B] \in H^2(X, \mathbb{C})$

flat B-field $(dB = 0)$

— This is not obvious by the patch-by-patch description

$$\mathcal{L} = \int d^4\theta\, K(\Phi^i, \bar{\Phi}^{\bar{i}}).$$

① sounds natural since Cplx str enters into the relation of chiral fields

$\Phi^i = f^i(\Phi'^i, t_a)$ between different patches

② looks mysterious, but one can show this by looking at A-topological correlators.

§ Localization Principle

Consider a system of n-bosonic & m-fermionic variables $X^1, \cdots, X^n, \psi^1, \cdots, \psi^m$.

with supersymmetry

$$\delta\left(d^nX\, d^m\psi\, e^{-S(X,\psi)} \right) = 0.$$

Suppose $(\delta\psi^1, \cdots, \delta\psi^m) \neq (0, \cdots, 0)$ ~~$\delta\psi^i \neq 0$~~ at any (X, ψ).

<u>In some case</u>, one can find a change of variables

$$\tilde{X}^i = f^i(X, \psi) \qquad i = 1 \cdots n$$

$$\tilde{\psi}^j = g^j(X, \psi) \qquad j = 1 \cdots m$$

s.t.

$$\begin{cases} \delta\tilde{X}^i = 0 \\ \delta\tilde{\psi}^1 = \epsilon \\ \delta\tilde{\psi}^j = 0 \quad j \neq 1 \end{cases}$$

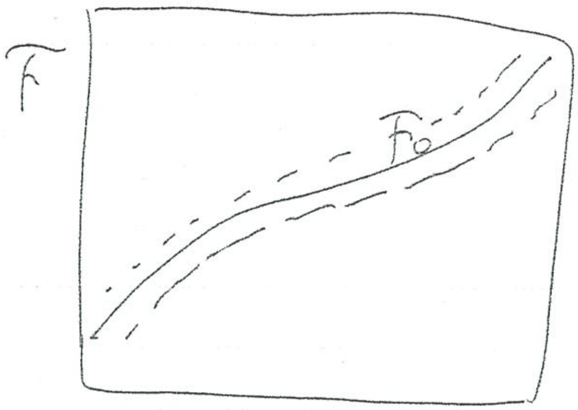
Then, by the invariance of the weighted measure

$$d^mX\, d^m\psi\, e^{-S(X,\psi)} = J(\tilde{X}^i, \tilde{\psi}^{j\neq 1})\, d^n\tilde{X}\, d^m\tilde{\psi}\, e^{-\tilde{S}(\tilde{X}, \tilde{\psi}^{j\neq 1})}$$

$$\Rightarrow Z = \int d^nx\, d\tilde{\psi}\, e^{-S(X,\psi)} = \int d\tilde{\psi}^1 \cdot 1 \cdot \int d^n\tilde{X}\, d^{m-1}\tilde{\psi}\, J\, e^{-\tilde{S}}$$

$$= 0.$$

In general, $\delta\psi^i = 0 \ \forall i$ in some locus $\mathcal{F}_0$



$$Z = \int_{\mathcal{F}} e^{-S} = \int_{\mathcal{F}_0} e^{-S} + \underbrace{\int_{\mathcal{F} \setminus \mathcal{F}_0} e^{-S}}_{=0}$$

One can focus on the locus $\mathcal{F}_0$ where $\delta\psi^i = 0 \ \forall i$.

This is the <u>Localization Principle</u>

Under which case, $\exists$ such change of variable?

$\boxed{\begin{array}{l} \text{We need } \delta\delta(\text{fields}) = 0. \\ \text{In fact } \delta^2 = 0 \text{ is sufficient.} \end{array}}$

<u>proof</u> If $(\delta\psi^1, \cdots, \delta\psi^m) \neq (0, \cdots, 0)$ at any (X, ψ),

Choose the direction of $\delta(\cdots)$ as one fermionic coordinate & take the invariant ones as the rest of the coordinate. (Rectification).

Then
$$\begin{cases} \delta \hat{X}^i = 0 \\ \delta \hat{\psi}^1 = \epsilon\, u(\hat{X}, \hat{\psi}) \qquad u(\hat{X}, \hat{\psi}) = 1 + \cdots \neq 0. \\ \delta \hat{\psi}^j = 0 \quad j \neq 1. \end{cases}$$

If $\delta^2 = 0$, then $u \dfrac{\partial u}{\partial \psi_1} = 0 \overset{u \neq 0}{\Longrightarrow} \dfrac{\partial u}{\partial \psi_1} = 0$

$\therefore\ u = u(\hat{X}, \hat{\psi}^{j \neq 1})$

Define $\tilde{\psi}^1 = u^{-1}(\hat{X}, \hat{\psi}^{j \neq 1})\, \hat{\psi}^1$ then $\delta \tilde{\psi}^1 = \epsilon$. //

Thus, for a supersymmetry with $\delta^2 = 0$ Localization Principle applies.

Computation of topological correlators

Basic tool — Localization.

A-twisted NLSM on a Kähler mfd X

$$\left(\delta_A \phi^i = \bar\epsilon \psi_-^i \ , \ \delta_A \phi^{\bar i} = \bar\epsilon \bar\psi_+^{\bar i} \right)$$

$$\delta_A \psi_-^i = 0 \ , \ \delta_A \bar\psi_+^{\bar i} = 0$$

$$\delta_A \psi_+^i = \bar\epsilon \left(i(\partial_t + \partial_\sigma)\phi^i + \Gamma^i_{jk} \psi_+^j \psi_-^k \right)$$

$$\delta_A \bar\psi_-^{\bar i} = \epsilon \left(i(\partial_t - \partial_\sigma)\bar\phi^{\bar i} + \Gamma^{\bar i}_{\bar j \bar k} \bar\psi_-^{\bar k} \bar\psi_+^{\bar j} \right)$$

$$\partial_t + \partial_\sigma \xrightarrow{\text{Wick}} i\partial_2 + \partial_1 = 2\frac{\partial}{\partial \bar z} \qquad z = x^1 + i x^2$$

$$\partial_t - \partial_\sigma \longrightarrow -2\frac{\partial}{\partial z}$$

RHS $= 0$ at $\psi_+ = \bar\psi_- = 0$

$$\Leftarrow \quad \partial_{\bar z} \phi^i = 0$$

$\phi : \Sigma \longrightarrow X$ holomorphic map.

(bosonic part of) Path integral weight $= e^{-S_b}$ ~~$\big|_{\phi\text{ hol}}$~~

$$S_b = \int_\Sigma g_{i\bar{j}}\left(\partial_z\phi^i\partial_{\bar{z}}\bar{\phi}^{\bar{j}} + \partial_{\bar{z}}\phi^i\partial_z\bar{\phi}^{\bar{j}}\right)d^2z + i\int_\Sigma \phi^* B$$

$$= \int_\Sigma\left(2g_{i\bar{j}}\,\partial_{\bar{z}}\phi^i\partial_z\bar{\phi}^{\bar{j}} + g_{i\bar{j}}\left(\partial_z\phi^i\partial_{\bar{z}}\bar{\phi}^{\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\bar{\phi}^{\bar{j}}\right)\right)d^2z$$
$$-i\int_\Sigma \phi^* B$$

$$= \underbrace{2\int_\Sigma g_{i\bar{j}}\,\partial_{\bar{z}}\phi^i\partial_z\bar{\phi}^{\bar{j}}\,d^2z}_{\geq 0} + \underbrace{\int_\Sigma \phi^*(\omega - iB)}_{\text{(locally) constant}}$$

$=0$ iff holomorphic   or topological

At $\phi:\Sigma\to X$ hol., $e^{-S_b} = e^{-\int_\Sigma \phi^*(\omega-iB)} = e^{-\int_{\phi_*[\Sigma]}(\omega - iB)}$.

The ∞-dim integral reduces to integration over the moduli space of holomorphic maps.

$\beta\in H_2(X,\mathbb{Z})$

$$\mathcal{M}_\Sigma(X,\beta) = \left\{\phi:\Sigma\to X \;\middle|\; \begin{array}{l}\text{holomorphic}\\ \phi_*[\Sigma]=\beta\end{array}\right\}$$

$\beta = 0$ $\mathcal{M}_\Sigma(X,\beta) = X$

$$\mathcal{O}_i \longleftrightarrow \omega_i \in H^*_{DR}(X)$$

$$\langle \mathcal{O}_1^{(x_1)} \cdots \mathcal{O}_s^{(x_s)} \rangle_\Sigma = \sum_{\beta \in H_2(X;\mathbb{Z})} e^{-\int_\beta (\omega - iB)} \int_{\mathcal{M}_\Sigma(X,\beta)} ev_1^* \omega_1 \wedge \cdots \wedge ev_s^* \omega_s$$

$\beta = 0 : \int_X \omega_1 \wedge \cdots \wedge \omega_s$

$$ev_i : \mathcal{M}_\Sigma(X,\beta) \longrightarrow X$$
$$[\phi : \Sigma \to X] \mapsto \phi(x_i)$$

if $D_i \subset X$ Poincaré dual of ω_i

$\omega_i = \delta_{D_i}$ -- δ-function supported on D_i

$$\int_{\mathcal{M}_\Sigma(X,\beta)} ev_1^* \delta_{D_1} \wedge \cdots \wedge ev_s^* \delta_{D_s} = \# \left\{ \phi : \Sigma \to X \;\middle|\; \begin{array}{l} \text{holomorphic} \\ \phi(x_i) \in D_i \ \forall i \\ \phi_*(\Sigma) = \beta \end{array} \right\}$$

$$= n_{\beta, D_1 \cdots D_s}$$

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle_\Sigma = \sum_{\beta \in H_2(X;\mathbb{Z})} n_{\beta, D_1 \cdots D_s} \, e^{-\int_\beta (\omega - iB)}$$

e.g. $\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle_{\mathbb{CP}^1} = \sum_{\beta \in H_2(X;\mathbb{Z})} n_{\beta; D_1 \cdots D_s} \, e^{-\int_\beta (\omega - iB)}$

$\longrightarrow$ quantum cohomology ring.

$\beta = 0$: classical part.

Important : they depends on the class

$$[\omega - iB] \in H^2(X, \mathbb{C})$$

They must be holomorphic twisted chiral parameters of the system.

— We see it more explicitly in Linear Sigma Model.

B-twisted NLSM on a CY mfd X

$$\left(\delta_B \phi^i = 0 , \qquad \delta_B \bar{\phi}^{\bar{i}} = \epsilon \bar{\eta}^{\bar{i}} \right)$$

$\bar{\eta}^{\bar{i}} = -(\bar{\psi}_-^{\bar{i}} + \bar{\psi}_+^{\bar{i}})$

$\theta_i = g_{i\bar{j}}(\bar{\psi}_-^{\bar{j}} - \bar{\psi}_+^{\bar{j}})$

$$\delta_B \theta_i = 0 \qquad \delta_B \bar{\eta}^{\bar{i}} = 0$$

$$\delta_B \psi_\pm^i = -i\bar{\epsilon}(\partial_t \pm \partial_\sigma)\phi^i$$

$$\pm 2\frac{\partial}{\partial \bar{z}} , \ -2\frac{\partial}{\partial z}$$

RHS $= 0$ at $\partial_z \phi^i = \partial_{\bar{z}} \phi^i = 0$

i.e. $\phi : \Sigma \to X$ constant map
(map to a point)

The path integral reduces to
integration on the moduli space of constant map
which is X !

Also $\displaystyle S_b \big|_{\delta\psi=0} = \int \underbrace{g_{i\bar{j}}\, \partial^\mu \phi^i \partial_\mu \bar{\phi}^{\bar{j}}}_{=0} d^2z - i \int \underbrace{\phi^* B}_{=0} = 0.$

The result:

$$\mathcal{O}_i \leftrightarrow \omega_i \in H^{0,p_i}(X, \Lambda^{q_i} T_X)$$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_{\mathbb{CP}^1} = \int_X (\omega_1 \wedge \cdots \wedge \omega_s, \Omega) \wedge \Omega$$

X: CY 3-fold $\quad \omega_i \in H^1(X, T_X)$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{\mathbb{CP}^1} = \int_X \omega_1^i \wedge \omega_2^j \wedge \omega_3^k \, \Omega_{ijk} \wedge \Omega$$