Topological string (very brief)

is a topological field theory coupled to

topological gravity

theory with SUSY Q

bosonic variable ~~includ~~ metric $h_{\mu\nu}$

$$\begin{cases} \delta h_{\mu\nu} = \epsilon \chi_{\mu\nu} \\ \delta \chi_{\mu\nu} = \epsilon \mathcal{L}_C h_{\mu\nu} \\ \delta C^\mu = 0 \end{cases}$$

"Couple" ?

$$S = S_{TFT} + S_{int} + S_{TG}$$

$$\delta S_{TFT} = \int \delta h_{\mu\nu} T^{\mu\nu} = \int \epsilon \chi_{\mu\nu} \{Q, G^{\mu\nu}\}$$

Cancel.

$$S_{int} \sim \int \chi_{\mu\nu} G^{\mu\nu} \Rightarrow \delta S_{int} \sim \int \chi_{\mu\nu} \epsilon \{Q, G^{\mu\nu}\}$$

$\langle O_1(x_1) \cdots O_s(x_s) \rangle_{g = \text{genus}}$

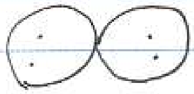
Path integral for the $(h_{\mu\nu}, \chi_{\mu\nu})$ sector reduces to integral over the complex structure of Σ

$$\{ J : T\Sigma \to T\Sigma \mid J^2 = -1 \} / \text{Diffeo fixing insertion points } x_1 \cdots x_s$$

$$= \overline{\mathcal{M}}_{g,s} \quad (\dim_{\mathbb{C}} = 3g - 3 + s)$$

compactified

compactification : include surfaces like



$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle_g = \int_{\overline{\mathcal{M}}_{g,s}} \Phi_{\mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s)}$$

$\overline{\Phi}_{\mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s)}$: volume form on $\mathcal{M}_{g,s}$ that extends to $\overline{\mathcal{M}}_{g,s}$

$$T^{1,0}_{(\Sigma, x_1 \cdots x_s)} \mathcal{M}_{g,s} = H^{0,1}(\Sigma, T_\Sigma \otimes \mathcal{O}(-x_1) \cdots \otimes \mathcal{O}(-x_s))$$

$$\in$$

$$\mu^z_{\bar z} \, d\bar z \otimes \frac{\partial}{\partial z} \qquad \mu^z_{\bar z} = \text{"} \delta J^z_{\bar z} \text{"}$$

$$\overline{\Phi}_{\mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s)}(\mu_1, \mu_2, \cdots)$$

$$= \left\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \int_\Sigma \mu_1 \cdot G \int_\Sigma \mu_2 \cdot G \cdots \right\rangle^{TFT}_\Sigma$$

$$\mu \cdot G = \mu^z_{\bar z} G_{zz} \, d\bar z \, dz$$

Note: $\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_g \neq \langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle^{TFT}_{\Sigma_g}$

except $g = 0, s = 3$ $\quad \overline{\mathcal{M}}_{0,3} = \{pt\}$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{g=0} = \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle^{TFT}_{S^2}$$

Topological String has more information than TFT.

R-symmetries

$$U(1)_V : [F_V, Q_\pm] = -Q_\pm, \quad [F_V, \bar{Q}_\pm] = \bar{Q}_\pm$$

$$U(1)_A : [F_A, Q_\pm] = \mp Q_\pm, \quad [F_A, \bar{Q}_\pm] = \pm \bar{Q}_\pm$$

For superfields

$$U(1)_V : \Phi_i(x, \theta^\pm, \bar\theta^\pm) \mapsto e^{i\alpha q_{Vi}} \Phi_i(x, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar\theta^\pm)$$

$$U(1)_A : \Phi_i(x, \theta^\pm, \bar\theta^\pm) \mapsto e^{i\beta q_{Ai}} \Phi_i(x, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar\theta^\pm)$$

Under what conditions these are symmetries ?

$$S = \int d^4\theta \; K + \left(\int d^2\theta \; W + c.c. \right) + \left(\int d^2\tilde\theta \; \widetilde{W} + c.c. \right)$$

Charge	$U(1)_V$	$U(1)_A$
$d^4\theta$	0	0
$d^2\theta$	-2	0
$d^2\tilde\theta$	0	-2

Classical
--> Condition

	$U(1)_V$	$U(1)_A$
K	0	0
W	2	0
$\widetilde{W}$	0	2

NLSM $\mathcal{L} = \int d^4\theta \, K(\Phi^i, \bar{\Phi}^{\bar{i}})$

Both $U(1)_V$ & $U(1)_A$ are classical symmetries

by $q_{V_i} = q_{A_i} = 0$

i.e.

$$U(1)_V \begin{cases} \phi^i \longrightarrow \phi^i \\ \psi_\pm^i \to e^{-i\alpha} \psi_\pm^i \\ \bar{\psi}_\pm^{\bar{i}} \to e^{i\alpha} \bar{\psi}_\pm^{\bar{i}} \end{cases} \qquad U(1)_A \begin{cases} \phi^i \longrightarrow \phi^i \\ \psi_\pm^i \to e^{\mp i\beta} \psi_\pm^i \\ \bar{\psi}_\pm^{\bar{i}} \longrightarrow e^{\pm i\beta} \bar{\psi}_\pm^{\bar{i}} \end{cases}$$

LG model $\mathcal{L} = \int d^4\theta \, K(\Phi^i, \bar{\Phi}^{\bar{i}}) + \left(\int d^2\theta \, W(\Phi^i) + c.c. \right)$

$U(1)_A$ inv by $q_A^i = 0$

$U(1)_V$ inv iff $\exists\, q_V^i$ s.t. $W(e^{i\alpha q_V^i} \Phi^i) = e^{2i\alpha} W(\Phi^i)$

$\longrightarrow$ $W(\Phi^i)$ (quasi) homogeneous.

e.g. $W(\Phi) = \Phi^n$, $q_V = \frac{1}{n}$ $\Rightarrow$ $U(1)_V$ symmetry.

$W(\Phi) = \Phi^n + \Phi^m \cdots$ not homogeneous.

no $U(1)_V$ symmetry.

Some discrete subgroup may be a symmetry ~~unbroken~~ ~~anomaly free~~.

$$W = \Phi^{n+1} - \Phi$$

$$\Phi \to \lambda^{q} \Phi \qquad \lambda^{q(n+1)} = \lambda^{q} = \lambda^{2} \qquad \overset{q=2}{\lambda^{2n} = 1}$$

$$\lambda = e^{\frac{2\pi i}{2n}}$$

$U(1)_V \supset \mathbb{Z}_{2n}$ subgroup is ~~anomaly free~~ ~~unbroken~~ ~~tog~~ + a symmetry.

Anomaly Are they sym in quantum theory?

i.e. $\mathcal{D}\phi\,\mathcal{D}\psi_\pm\,\mathcal{D}\bar\psi_\pm$ also invariant?

toy model E : rank n vector bundle over Σ
with $U(n)$ connection A.

Dirac fermion with values in E :

$$S = i\int_\Sigma \left(\bar\psi_- D_{\bar z}\psi_- + \bar\psi_+ D_z \psi_+\right) d^2z$$

$$D_{\bar z} = \partial_{\bar z} + A_{\bar z}\,, \quad D_z = \partial_z + A_z$$

$$\psi_\pm \in \Gamma(\Sigma, E\otimes S_\pm)\,, \quad \bar\psi_\pm \in \Gamma(\Sigma, \bar E \otimes S_\pm) \quad \begin{cases} S_+ = \sqrt{\bar K} \\ S_- = \sqrt{K}. \end{cases}$$

$$\psi_+ = \psi_+^{(0)} + \psi_+' \qquad \bar\psi_+ = \bar\psi_+^{(0)} + \bar\psi_+'$$

$$\psi_- = \psi_-^{(0)} + \psi_-' \qquad \bar\psi_- = \bar\psi_-^{(0)} + \bar\psi_-'$$

zero mode + non-zero mode.

$$\mathcal{D}\psi_\pm\mathcal{D}\bar\psi_\pm = \underbrace{d^{n_+}\psi_+^{(0)}\, d^{n_-}\psi_-^{(0)}\, d^{\bar n_+}\psi_+^{(0)}\, d^{\bar n_-}\psi_-^{(0)}}_{\text{Not paired.}}\ \underbrace{\mathcal{D}\psi_-'\mathcal{D}\bar\psi_-'}_{\substack{\text{paired}\\ \text{by } D_{\bar z}}}\ \underbrace{\mathcal{D}\psi_+'\mathcal{D}\bar\psi_+'}_{\substack{\text{paired}\\ \text{by } D_z}}$$

$U(1)_V$: $\psi_\pm \to e^{-i\alpha}\psi_\pm, \bar\psi_\pm \to e^{i\alpha}\bar\psi_\pm$

$$\mathcal{D}\psi_\pm\mathcal{D}\bar\psi_\pm \to e^{i\alpha(n_+ + n_- - \bar n_+ - \bar n_-)}\mathcal{D}\psi_\pm\mathcal{D}\bar\psi_\pm$$

inv under $\psi_- \to e^{i\alpha_-}\psi_-$

any phase rotation. $\psi_+ \to e^{i\alpha_+}\psi_+$

$U(1)_A$: $\psi_\pm \to e^{\mp i\beta}\psi_\pm, \bar\psi_\pm \to e^{\pm i\beta}\bar\psi_\pm$

$$\mathcal{D}\psi_\pm\mathcal{D}\bar\psi_\pm \longrightarrow e^{i\beta(n_+ - n_- - \bar n_+ + \bar n_-)}\mathcal{D}\psi_\pm\mathcal{D}\bar\psi_\pm$$

ψ_- zero modes : $\operatorname{Ker} D_{\bar z} : \Gamma(E\otimes\sqrt{K}) \to \Omega^{0,1}(E\otimes\sqrt{K})$

$= H^0(\Sigma, E\otimes\sqrt{K})$

$\overline{\psi}_-$ zero modes $H^0(\Sigma, E^\vee\otimes\sqrt{K}) \overset{\text{dual}}{\leftrightarrow} H^1(\Sigma, E\otimes\sqrt{K})$

ψ_+ zero modes : $\operatorname{Ker} D_z : \Gamma(E\otimes\sqrt{K}) \to \Omega^{1,0}(E\otimes\sqrt{K})$

$= \operatorname{Ker} D_{\bar z}^\dagger : \Omega^{0,1}(E\otimes\sqrt{K}) \to \Gamma(E\otimes\sqrt{K})$

$= H^{0,1}(\Sigma, E\otimes\sqrt{K}) = H^1(\Sigma, E\otimes\sqrt{K})$

$\overline{\psi}_+$ zero modes : $H^1(\Sigma, E^\vee\otimes\sqrt{K}) \overset{\text{dual}}{\longleftrightarrow} H^0(\Sigma, E\otimes\sqrt{K})$

~~∴ under "$U(1)_V$" : $\psi_\pm \to e^{-i\alpha}\psi_\pm$, $\overline{\psi}_\pm \to e^{i\alpha}\overline{\psi}_\pm$~~

~~$d^{n_++n_-+\overline{n}_++\overline{n}_-}\varphi(\omega) \to$~~

$$\overline{n}_+ - n_+ = n_- - \overline{n}_- = \dim H^0(E\otimes\sqrt{K}) - \dim H^1(\Sigma, E\otimes\sqrt{K})$$

$$= \int_\Sigma \left(\frac{i}{2\pi}F_A\right) = \int_\Sigma c_1(E)$$ Atiyah-Singer index formula.

$U(1)_V$: $\mathcal{D}\psi_\pm\mathcal{D}\overline{\psi}_\pm$ — invariant

$U(1)_A$: $\mathcal{D}\psi_\pm\mathcal{D}\overline{\psi}_\pm \to e^{-2i\beta\int_\Sigma c_1(E)}\mathcal{D}\psi_\pm\mathcal{D}\overline{\psi}_\pm$.

anomalous !

($e^{i\beta}$ s.t. $e^{-2i\beta\int_\Sigma c_1(E)} = 1$ is a symmetry $\rightsquigarrow$ $\mathbb{Z}_{2\int_\Sigma c_1(E)}$.

local version: J_A ... current corresponding to $U(1)_A$

$$\int_\Sigma dJ_A = -2 \int_\Sigma \mathrm{tr}\left(\frac{i}{2\pi} F_A\right)$$

actually

$$\rightsquigarrow \quad dJ_A = -2\, \mathrm{tr}\left(\frac{i}{2\pi} F_A\right) = -2\, c_1(E;A)$$

local version.

- NLSM with target X (Kähler)

$$E = \phi^* T^{1,0} X = \phi^* T_X \ , \quad c_1(\phi^* T_X) = \phi^* c_1(T_X)$$

$c_1(T_X) = c_1(X)$ …. first Chern-class of X

… $\frac{1}{2\pi} \mathrm{Tr} R \propto$ Ricci tensor

- $U(1)_A$ anomaly free $\Leftrightarrow$ $c_1(X) = 0$ Calabi-Yau.

- $c_1(X) \in N H^2(X, \mathbb{Z})$ sometimes

$$\Rightarrow \int_\Sigma \phi^* c_1(x) \in N\mathbb{Z} \quad \Rightarrow \quad e^{-2i\beta \int_\Sigma \phi^* c_1(x)} = e^{-2i\beta N k_\phi}$$

$$= 1^{\forall \phi} \text{ iff } \quad 2N\beta \in 2\pi\mathbb{Z} \qquad \text{i.e. } e^{i\beta} = e^{\frac{2\pi i}{2N} k}$$

$\mathbb{Z}_{2N} \subset U(1)_A$ -- anomaly free subgroup.

e.g. $X = \mathbb{CP}^{N-1}$

$c_1(X) = NH$ $\quad$ H… generator of $H^1(X, \mathbb{Z})$.

$\mathbb{Z}_{2N}$ is anomaly free.

Summary	$U(1)_V$	$U(1)_A$
CY NLSM	Yes	Yes
NLSM on X $c_1(X) \neq 0$	Yes	No
LG with inhomog. W	No	Yes
LG with homogeneous W	Yes	Yes

$$\int_\Sigma c_1(\phi^* T_X)\,\underbrace{Td(T\Sigma)}_{1+\frac{1}{2}c_1(T\Sigma)+\cdots}$$

After twisting

A-twist $\psi_- \cdots \phi^* T_X \xrightarrow{\text{zero}} H^0(\phi^* T_X)$

$\bar\psi_- \cdots \phi^* T_X^\vee \otimes K \xrightarrow{\text{zero}} H^0(\phi^* T_X^\vee \otimes K) \overset{\text{dual}}{\longleftrightarrow} H^1(\phi^* T_X)$

$\psi_+ \cdots \phi^* T_X \otimes \bar K \xrightarrow{\text{zero}} H^1(\phi^* T_X)$

$\bar\psi_+ \cdots \phi^* T_X^\vee \otimes \xrightarrow{\text{zero}} H^1(\phi^* T_X^\vee \otimes K) \overset{\text{dual}}{\longleftrightarrow} H^0(\phi^* T_X)$

$$n_- - \bar n_- = \bar n_+ - n_+ = \int_\Sigma c_1(\phi^* T_X) + \mathrm{rank}\,\phi^* T_X\,(1-g)$$

$$= \int_\Sigma \phi^* c_1(X) + \dim X\,(1-g)$$

(even for X : CY, $U(1)_A$ is anomalous.)

Selection rule

$\int d\psi \cdot 1 = 0$

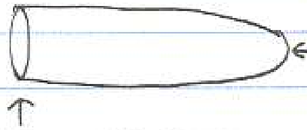
$\langle O_1 \cdots O_s \rangle_\Sigma \neq 0$ only when

$O_1, \cdots, O_s$ includes the fermion zero modes.

$\Rightarrow$ $U(1)_A$ charge of $O_1 \cdots O_s$

$$= 2\int_\Sigma \phi^* c_1(X) + 2\dim X\,(1-g)$$

State ↔ Operator Correspondence



$\mathcal{O}_\omega \qquad \omega \in H^{\bullet}_{\bar\partial}(X)$

$\Psi_{\Sigma_\omega} = |\mathcal{O}_\omega\rangle$

What is the R-charge of ~~Ψ~~ $|\mathcal{O}_\omega\rangle$?

- $Q_V = \oint_{S^1} J_V \qquad dJ_V = 0$ conserved.

$$Q_V |\mathcal{O}_\omega\rangle = |\oint J_V \mathcal{O}_\omega\rangle$$

$$\omega \in H^{p,q}(X) \Rightarrow \mathcal{O}_\omega = \omega_{\bar i_1 \cdots i_p \bar j_1 \cdots \bar j_q} \psi_-^{i_1} \cdots \psi_-^{i_p} \bar\psi_+^{\bar j_1} \cdots \bar\psi_+^{\bar j_q}$$

$U(1)_V$ charge $\underline{-p+q}$ $\quad Q_V|\mathcal{O}_\omega\rangle = (-p+q)|\mathcal{O}_\omega\rangle$

- $Q_A = \oint_{S^1} J_A \qquad dJ_A = -2\Big(c_1(\phi^* T_X) + \frac{1}{2}\overset{\text{rank}\,\phi^* T_X}{\vee} c_1(T_\Sigma)\Big)$

anomalous

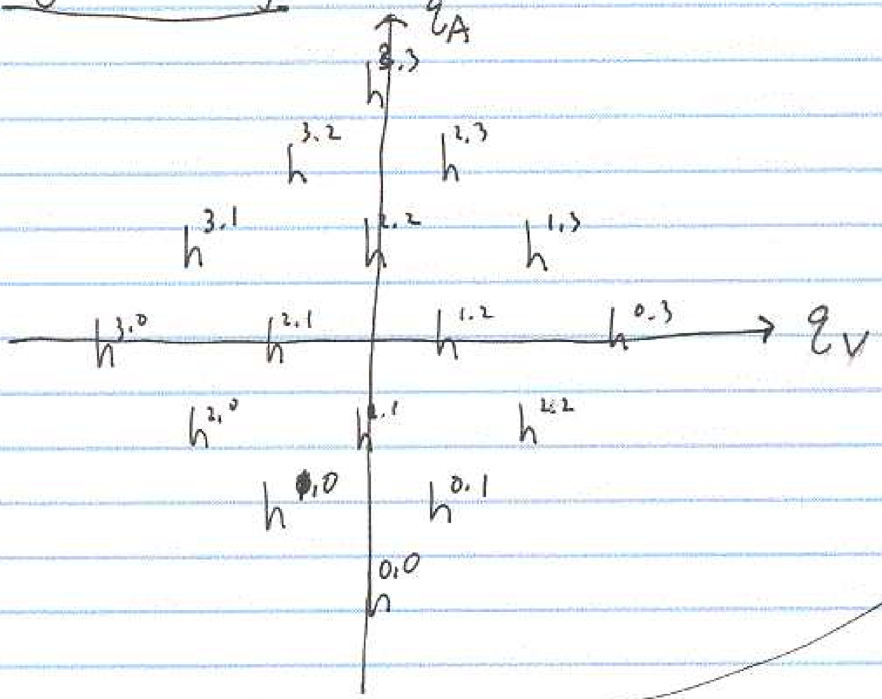
$$Q_A|\mathcal{O}_\omega\rangle = \underbrace{|\oint J_A \mathcal{O}_\omega\rangle}_{\overset{\shortparallel}{(p+q)|\mathcal{O}_\omega\rangle}} - 2\int \Big(c_1(\phi^* T_X) + \underset{\dim X}{\frac{1}{2} c_1(T_\Sigma)}\Big)|\mathcal{O}_\omega\rangle$$

X : CY, $c_1(T_X) = 0$.

$\int c_1(T_\Sigma) = 1$

$$Q_A|\mathcal{O}_\omega\rangle = (p+q-\dim X)|\mathcal{O}_\omega\rangle$$

Hodge diamond (say $\dim X = 3$)



Summary

	q_V	q_A
$H^{p,q}(X)$	$-p+q$	$p+q-\dim X$

Mirror Symmetry

(2,2) theories "1" & "2" are said to be

mirror to each other when (1) & (2) holds

(1) They are equivalent as QFT.

i.e. states, operators, correlation fcns

all matches

(2) If $\Psi : \mathcal{H}_1 \to \mathcal{H}_2$ is the isom of states,

it transforms $Q_-^{(1)} \to \overline{Q}_-^{(2)}$

$\overline{Q}_-^{(1)} \to Q_-^{(2)}$

& when $\exists$ R-syms $F_A^{(1)} \to F_V^{(2)}$

$F_V^{(1)} \to F_A^{(2)}$

all other $\overset{(2,2)}{\text{generators}}$ invariant.

Consequence

"1" $\longleftrightarrow$ "2"

chiral $\longleftrightarrow$ twisted chiral

twisted chiral $\longleftrightarrow$ chiral.

A-ring $\longleftrightarrow$ B-ring

B-ring $\longleftrightarrow$ A-ring.

A-twistable $\longleftrightarrow$ B-twistable

(F_V conserved integral) (F_A conserved integral)

B- $\longleftrightarrow$ A-

i.e. both NLSM on CY $\Longrightarrow$

complex str. moduli $\longleftrightarrow$ Kähler class moduli

Kähler class moduli $\longleftrightarrow$ Cplx str moduli

(F_V, F_A) diamond $\longleftrightarrow$ (F_A, F_V) diamond

$\vdots$

Example

$\omega \in H^2(X, \mathbb{Z})$

ω: ~~Kähler class~~

1: $X = \mathbb{CP}^1$ NLSM.

twisted chiral: $t\omega \in H^2(X, \mathbb{C}) \cong \mathbb{C}\ t$

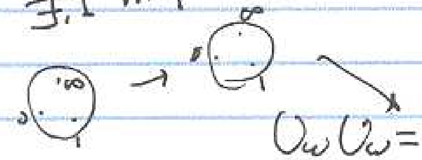
2: LG model $W = e^{-Y} + e^{-t+Y}$ $e^{-Y} \in \mathbb{C}^*$

Correspondence: $R_A^{(1)} \longleftrightarrow R_B^{(2)}$

$1 \longleftrightarrow 1$

$\exists.1$ degree 1 map

$O_\omega \longleftrightarrow e^{-Y}$ $-e^{-Y} + e^{-t+Y} = 0$



$O_\omega O_\omega = e^{-t} \longleftrightarrow e^{-Y} \cdot e^{-Y} \overset{d}{=} e^{-t}$

Note LG model $W = \frac{1}{3}X^3 - e^{-t}X$ also has

$1, X, X^2 = e^{-t}$

But it's not mirror to "1".

(not equiv to "2").

(see difference in topological string amplitude also D-branes)

Another example

quintic hypersurface $\subset \mathbb{CP}^4$

$$X_1 = \{ (z_1, \cdots, z_5) \neq 0 \mid \underset{\substack{\uparrow \\ \text{degree 5 polynomial}}}{G(z_1, \cdots, z_5)} = 0 \} / \mathbb{C}^* \quad z_i \to \lambda z_i$$

(e.g. Fermat : $z_1^5 + \cdots + z_5^5 = 0$)

$$H^{1,1}(X_1) \cong \mathbb{C} \ni T$$

$$H^{2,1}(X_1) \cong \mathbb{C}^{101}$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

~~mirror~~

mirror quintic

X_2 = resolution of

$$\{ (z_1, \cdots, z_5) \neq 0 \mid z_1^5 + \cdots + z_5^5 + e^{t/5} z_1 \cdots z_5 = 0 \} / \mathbb{C}^* \times \mathbb{Z}_5^3$$

$$\mathbb{Z}_5^3 \times \mathbb{C}^* : z_i \to \lambda z_i$$

$$z_i \to \omega_i z_i \quad \omega_i^5 = 1, \ \omega_1 \cdots \omega_5 = 1$$

$$H^{1,1}(X_2) \cong \mathbb{C}^{101}$$

$$H^{2,1}(X_1) = \mathbb{C}$$

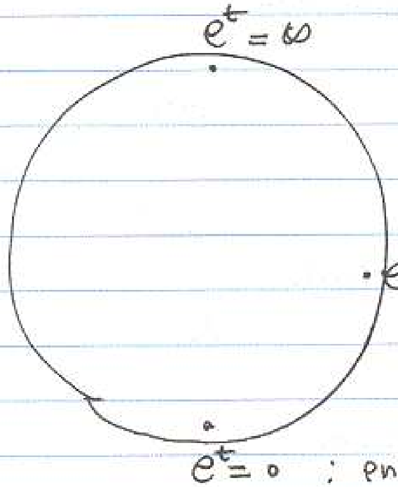
~~Correspondence~~:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$$\underset{\substack{\uparrow \\ \text{long distance limit}}}{\text{IR limit}} \text{ of NLSM on } X_1 \xleftrightarrow{\text{mirror}} \text{IR limit of NLSM on } X_2$$

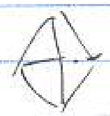
$$e^{-T} \approx e^{-t} + O(e^{-2t}) \quad \text{at } T, t \to +\infty.$$

The Complex structure moduli of mirror quintic (X_2)



$e^t = \infty$ degenerate : $z_1 \cdots z_5 = 0$

union of 5 $\mathbb{CP}^3$'s



$\cdot\, e^t = (-5)^5$ Conifold singularity

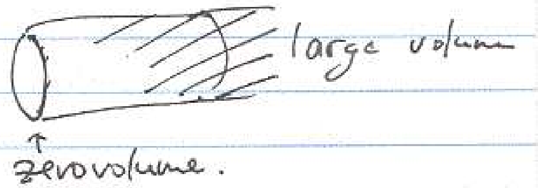
$e^t = 0$: enhanced $\mathbb{Z}_5$ symmetry $\omega_1 \cdots \omega_5 \in \mathbb{Z}_5$

(not required to be 1)

This <u>MUST</u> be the moduli space of Complexfied Kähler class (X_1)

Classically, it is $H^{1,1}(X_1)/2\pi H^2(X_1,\mathbb{Z})$ $\left(B \equiv B + 2\pi H^2(X_1,\mathbb{Z})\right)$

$\cong \mathbb{R}_+ \times S^1$



Mirror symmetry states that it <u>must</u> be extended to "negative Kähler class" !