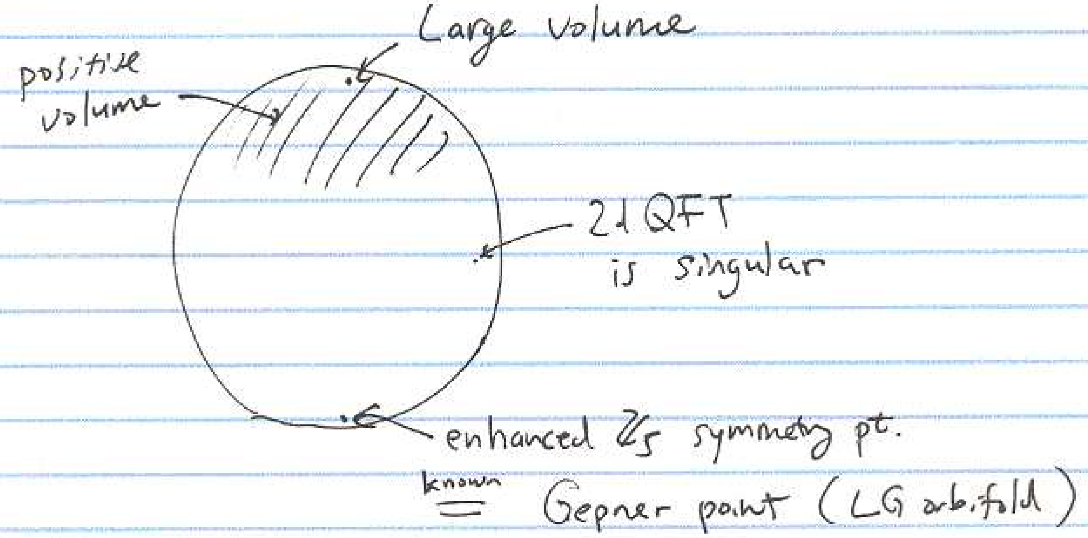
Linear Sigma Model

The complexified Kähler class moduli of quintic



This cannot be seen by just looking at NLSM (which would only tell about small quantum correction in the large volume regime).

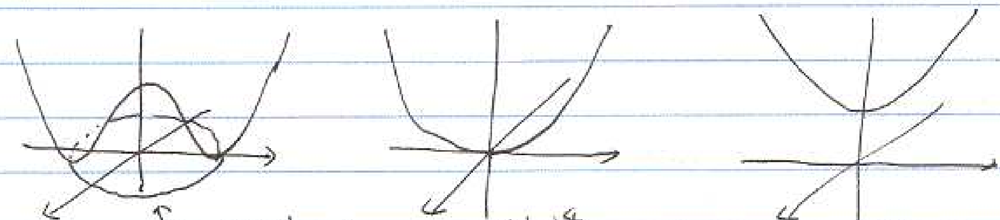
Linear Sigma Models (LSM) provide a realization of the whole family of theories, and can tell us about this global moduli space.

Basic idea

Consider real scalar fields $\phi_1, \cdots, \phi_N$ with action

$$S = \int d^2x \left\{ \frac{1}{2} \sum_{i=1}^{N} \left((\partial_0 \phi_i)^2 - (\partial_1 \phi_i)^2 \right) - \frac{\lambda^2}{2} \left(\sum_{i=1}^{N} (\phi_i)^2 - r \right)^2 \right\}$$

Potential $\quad r>0 \qquad\qquad r=0 \qquad\qquad r<0$



zero at $\{(\phi)^2 = r\} = S_r^{N-1}$ $\qquad \sim (\phi)^4 \qquad \sim \frac{\lambda^2 r^2}{2} + |r|\lambda^2(\phi)^2$

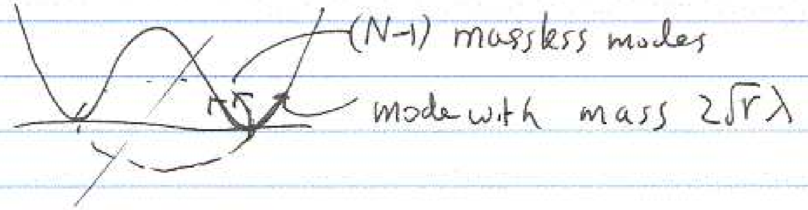
Polar coordinate (ρ, Ω_i) : $\sum_{i=1}^{N} (\Omega_i)^2 = 1$; $\phi_i = \rho \Omega_i$

$$S = \int d^2x \left\{ \frac{1}{2}(\partial_0 \rho)^2 - \frac{1}{2}(\partial_1 \rho)^2 - \frac{\lambda^2}{2}(\rho^2 - r)^2 + \frac{\rho^2}{2} \sum_{i=1}^{N} \left((\partial_0 \Omega_i)^2 - (\partial_1 \Omega_i)^2 \right) \right\}$$

$r>0$: expand around $\rho = \sqrt{r}$ (vacuum):

- Ω_i - direction (S_r^{N-1}) : massless
- ρ : $\frac{\lambda^2}{2}(\rho^2 - r)^2 = \frac{\lambda^2}{2}\left((\sqrt{r} + \delta\rho)^2 - r\right)^2 = \frac{\lambda^2}{2}\left(2\sqrt{r}\,\delta\rho + (\delta\rho)^2\right)^2$

$$= \frac{1}{2} \underbrace{(2\sqrt{r}\lambda)^2}_{\text{mass of } \delta\rho} (\delta\rho)^2 + 2\lambda^2 \sqrt{r}\, (\delta\rho)^3 + \frac{\lambda^2}{2} (\delta\rho)^4$$



At energies $E \ll 2\sqrt{r}\lambda$, the massive mode is hard to excite. $\Rightarrow \rho$ can be regarded as a fixed background.

You are left with the theory of massless modes.

— This is NLSM on S_r^{N-1} !

This system is called the Linear Sigma Model corresponding to S^{N-1}.

* The description is not patch-by-patch
* This allows to treat global & non-perturbative aspects.
* The theory makes sense also for $r \le 0$ (although this distinction is actually vacuous by renormalization effect)

How about $\mathbb{C}P^{N-1} = \{ |z_1|^2 + \cdots + |z_N|^2 = r \} / U(1)$?

$$(z_1, \cdots, z_N) \equiv (e^{i\alpha} z_1, \cdots, e^{i\alpha} z_N)$$

Try $(\phi_1, \cdots, \phi_N)$: $\mathbb{C}^N$-valued

$$S = \int d^2x \left\{ \sum_{i=1}^{N} \left(|\partial_0 \phi_i|^2 - |\partial_1 \phi_i|^2 \right) - \frac{\lambda^2}{2} \left(\sum_{i=1}^{N} |\phi_i|^2 - r \right)^2 \right\}$$

and declare

$$(\phi_1(x), \cdots, \phi_N(x)) \equiv (e^{i\alpha(x)} \phi_1(x), \cdots, e^{i\alpha(x)} \phi_N(x)).$$

Does it make sense?

- potential term is invariance ✓
- Kinetic term : $|\partial_\mu \phi_i|^2$

$$\partial_\mu \phi_i(x) = \partial_\mu (e^{i\alpha(x)} \phi_i(x)) = e^{i\alpha(x)} (\partial_\mu \phi_i + i \partial_\mu \alpha \phi_i)$$

$|\partial_\mu \phi_i|^2$ Not invariant !

$\rightsquigarrow$ introduce a new field v_μ that transforms as $v_\mu(x) \to v_\mu(x) - \partial_\mu \alpha(x)$, so that

$$D_\mu \phi_j := \partial_\mu \phi_j + i v_\mu \phi_j \to e^{i\alpha} (\partial_\mu \phi_j + \cancel{i \partial_\mu \alpha \phi_j} + i(v_\mu - \cancel{\partial_\mu \alpha}) \phi_j)$$
$$= e^{i\alpha} D_\mu \phi_j$$

$|D_\mu \phi_j|$ invariant

covariant derivative.

Another invariant

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu \quad \text{curvature}$$

In the theory with action

$$S = \int d^2x \left\{ \frac{1}{2e^2}(F_{01})^2 + \sum_{i=1}^{N}\left(|D_0\phi_i|^2 - |D_1\phi_i|^2\right) - \frac{\lambda^2}{2}\left(\sum_{i=1}^{N}|\phi_i|^2 - r\right)^2 \right\}$$

it makes sense to identify

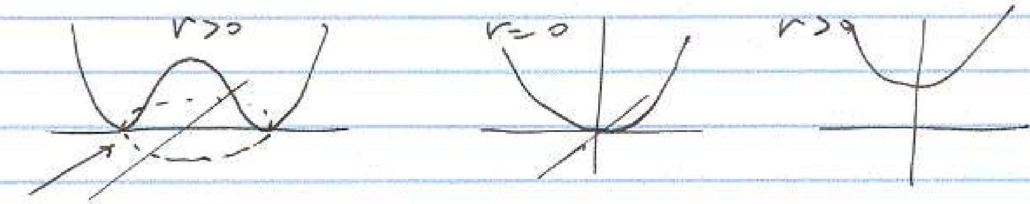
$$(\phi_1(x), \cdots, \phi_N(x), v_\mu(x)) \equiv (e^{i\alpha(x)}\phi_1(x), \cdots, e^{i\alpha(x)}\phi_N(x), v_\mu(x) - \partial_\mu\alpha(x))$$

—— this is called <u>gauging</u> $U(1)$ symmetry

v_μ -- <u>gauge field</u>

Theory obtained is <u>$U(1)$ gauge theory</u>.

<u>$N=1$</u>



$$\phi = \rho e^{i\varphi}, \quad D_\mu\phi = e^{i\varphi}\left(\partial_\mu\rho + i\rho(v_\mu + \partial_\mu\varphi)\right)$$

$$v'_\mu := v_\mu + \partial_\mu\varphi \quad \underline{\text{gauge invariant}}$$

$$F_{\mu\nu} = \partial_\mu v'_\nu - \partial_\nu v'_\mu = F'_{\mu\nu}$$

$$\mathcal{L} = \frac{1}{2e^2}(F'_{01})^2 - \rho^2(v'_\mu)^2 - (\partial_\mu\rho)^2 - \frac{\lambda^2}{2}(\rho^2 - r)^2$$

$$\overset{\rho=\sqrt{r}+\delta\rho}{=} \frac{1}{2e^2}(F'_{01})^2 - r(v'_\mu)^2 - (\partial_\mu\delta\rho)^2 + (\sqrt{2r}\lambda)^2(\delta\rho)^2$$

$$+ (\text{cubic or higher in } \delta\rho,\ v'_\mu).$$

$\underline{v'_\mu \cdots \text{ mass } \sqrt{2r}\,e}$ $\qquad \delta\rho \cdots \text{ mass } \sqrt{2r}\lambda$

The gauge field acquires a mass by eating the angular mode φ $\left(\begin{smallmatrix} v_\mu \to \\ v'_\mu = v_\mu + \partial_\mu\varphi \end{smallmatrix}\right)$.

— Higgs Mechanism.

$\underline{N>1}$ * mode transverse to $\sum_{i=1}^{N}|\phi_i|^2 = r$ ... mass $\sqrt{2r}\lambda$

* mode tangent to $U(1)$ gauge orbit
— eaten by gauge field which acquire mass $\sqrt{2r}\,e$.

* The rest .... NLSM on $\mathbb{CP}^{N-1}_r$ (with Fubini-Study metric).

At $E \ll \lambda\sqrt{2r},\ e\sqrt{2r}$, the system can be regarded as NLSM on $\mathbb{CP}^{N-1}$ — LSM for $\mathbb{CP}^{N-1}$.

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the ~~radi~~ angular mode φ is absorbed into the gauge field v'_μ and acquire a mass

-- Higgs mechanism

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SUPERSYMMETRIZE.

phase rotation ~~recall~~
$$\phi(x) \to e^{i\alpha(x)} \phi(x)$$

$$\partial_\mu \phi(x) \to e^{i\alpha(x)} (\partial_\mu \phi + i \partial_\mu \alpha \phi)$$ $|\partial_\mu \phi|^2$ NOT inv.

$$\rightsquigarrow \quad v_\mu \to v_\mu - \partial_\mu \alpha$$

$$(\partial_\mu + i v_\mu)\phi \to e^{i\alpha}(\partial_\mu + i v_\mu)\phi$$

so $|D_\mu \phi|^2 = |\partial_\mu \phi + i v_\mu \phi|^2$ is inv.

Supersymmetric theory

Φ chiral superfield

Kinetic term $\int d^4\theta \, \bar{\Phi}\Phi$

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Consider $\Phi \to e^{iA}\Phi$ A chiral $\Rightarrow e^{iA}\Phi$ chiral.

but $\bar{\Phi}\Phi \to \bar{\Phi}e^{-i\bar{A}}e^{iA}\Phi \neq \bar{\Phi}\Phi$

$A-\bar{A}=0$ impossible (A chiral, $\bar{A}$ antichiral)

introduce (real $V^\dagger = V$)

$\rightsquigarrow \quad V \to V + i\bar{A} - iA$

Then $\bar{\Phi}e^{V}\Phi \to \bar{\Phi}e^{-i\bar{A}}e^{V+i\bar{A}-iA}e^{iA}\Phi = \bar{\Phi}e^{V}\Phi$

invariant

This V is an analogue of gauge field. in supersymmetric theory.

~~Using gauge symmetry, one can eliminat~~

$V = f_0 + \theta^+ f_+ + \theta^- f_- + \bar{\theta}^+ \bar{f}_+ + \bar{\theta}^- \bar{f}_- + \cdots$

Using gauge symmetry, one can eliminate the lower component of the θ-expansion. and may assume

$$V = \theta^-\bar{\theta}^-(v_0 - v_1) + \theta^+\bar{\theta}^+(v_0+v_1) - \theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma}$$
$$+ i\,\theta^-\theta^+(\bar{\theta}^-\bar{\lambda}_- + \bar{\theta}^+\bar{\lambda}_+) + i\,\bar{\theta}^+\bar{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+)$$
$$+ \theta^-\theta^+\bar{\theta}^+\bar{\theta}^- D$$

$$\begin{cases} v_\mu & \text{one-form} \\ \sigma & \text{complex scalar} \\ \lambda_\pm, \bar\lambda_\pm & \text{Dirac fermion} \\ D & \text{real scalar} \end{cases}$$

This is called the Wess-Zumino gauge.

<u>Residual gauge transformation</u> (gauge transf. that keeps this form)

$$A = \alpha(x)$$

$$\begin{cases} v_\mu \to v_\mu - \partial_\mu \alpha \\ \sigma \to \sigma \\ \lambda_\pm, \bar\lambda_\pm \to \lambda_\pm, \bar\lambda_\pm \\ D \to D \end{cases}$$

"Ordinary" gauge transformation.

What's the curvature?

$\Sigma = \bar{D}_+ D_- V$ invariant under $V \to V + i\bar{A} - iA$

$$\overset{\text{WZ gauge}}{=} \sigma(\tilde y) + i\theta^+ \lambda_+(\tilde y) - i\bar\theta^- \lambda_-(\tilde y) + \theta^+\bar\theta^-(D(\tilde y) - iF_{01}(\tilde y))$$

$\uparrow$ Ordinary curvature.

$$\bar{D}_+ \Sigma = D_- \Sigma = 0$$

twisted chiral superfield.

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Supersymmetric Lagrangian

$$\mathcal{L}=\int d^4\theta\left(\bar{\Phi}e^V\Phi-\frac{1}{2e^2}\bar{\Sigma}\Sigma\right)+\frac{1}{2}\left(\int d^2\tilde{\theta}\left(-t\Sigma\right)+\text{c.c.}\right)$$

$$t=r-i\theta$$

WZ gauge

$$\overset{\downarrow}{=}\ -|D_\mu\phi|^2+i\bar{\psi}_-(D_0+D_1)\psi_-+i\bar{\psi}_+(D_0-D_1)\psi_++D|\phi|^2+|F|^2$$

$$-|\sigma|^2|\phi|^2-\bar{\psi}_-\sigma\psi_+-\bar{\psi}_+\bar{\sigma}\psi_--i\bar{\phi}\lambda_-\psi_+-i\bar{\phi}\lambda_+\psi_-+i\bar{\psi}_+\bar{\lambda}_-\phi$$
$$-i\bar{\psi}_-\bar{\lambda}\phi$$

$$+\frac{1}{2e^2}\left[-|\partial_\mu\sigma|^2+i\bar{\lambda}_-(\partial_0+\partial_1)\lambda_-+i\bar{\lambda}_+(\partial_0-\partial_1)\lambda_++F_{01}^2+D^2\right]$$

$$-rD+\theta F_{01}$$

Notice : D has not kinetic term (auxilrary field)

$\longrightarrow$ integrate out (complete the square)

$$\mathcal{L}=-\frac{1}{2e^2}\left(+|\partial_\mu\sigma|^2+F_{01}^2\right)-|D_\mu\phi|^2-U(\sigma,\phi)+\theta F_{01}$$
+ fermions

here $U(\sigma,\phi)=|\sigma|^2|\phi|^2+\frac{e^2}{2}\left(|\phi|^2-r\right)^2$

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N of Φ's:

$$\mathcal{L} = \int d^4\theta \left(\sum_{i=1}^{N} \bar{\Phi}_i e^V \Phi_i - \frac{1}{2e^2}|\Sigma|^2 \right) + \mathrm{Re} \int d^2\tilde{\theta} \left(-t\Sigma \right)$$

$$= -\frac{1}{2e^2}\left(|\partial_\mu \sigma|^2 + F_{01}^2 \right) - \sum_{i=1}^{N} |D_\mu \phi_i|^2 - U(\sigma, \phi_i) + \theta F_{01}$$

$$U(\sigma,\phi_i) = \sum_i |\sigma|^2 |\phi_i|^2 + \frac{e^2}{2}\left(\sum_{i=1}^{N} |\phi_i|^2 - r \right)^2$$

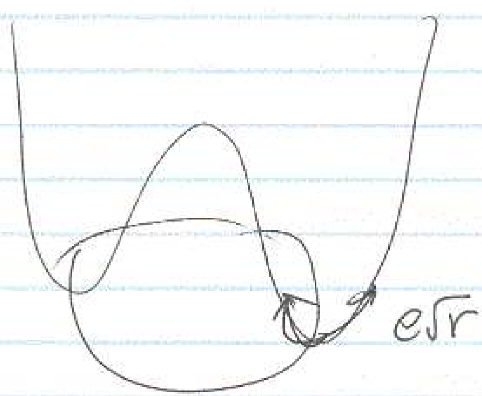
"Low energy" theory

Suppose $r > 0$

Look at U minimum, $U = 0 \Rightarrow \sum_{i=1}^{N} |\phi_i|^2 = r \Rightarrow \sigma = 0$

$$\therefore \mathcal{M}_{\text{vacuum}} = \left\{ \sigma = 0, \sum_{i=1}^{N} |\phi_i|^2 = r \right\} / U(1) \cong \mathbb{CP}^{N-1}$$

~~at energies~~



at energies $\mu \ll e\sqrt{r}$: NLσM on $\mathbb{CP}^{N-1}$

$e \to \infty$ limit (classical)

× $\sum_{i=1}^{N} |\phi_i|^2 = r$ is enforced.

× No gauge kinetic term ... $v_\mu, \sigma, \lambda_\pm$ algebraic in S

$\Rightarrow$ constraints:

· $$v_\mu = \frac{i}{2\sum_{i=1}^{N}|\phi_i|^2} \sum_{j=1}^{N} \left(\bar{\phi}_j \partial_\mu \phi_j - \partial_\mu \bar{\phi}_j \phi_j \right) + \text{fermion}$$

$$\quad = (\phi^* A)_\mu$$

A : a connection of $\mathcal{O}(1)$ over $\mathbb{CP}^{N-1}$

· $$\sigma = \frac{1}{\sum_{i=1}^{N}|\phi_i|^2} \sum_{j=1}^{N} \psi_{j-} \bar{\psi}_{j+}$$

· $$\sum_{j=1}^{N} \bar{\phi}_j \psi_{j\pm} = \sum_{j=1}^{N} \bar{\psi}_{j\pm} \phi_j = 0 \Rightarrow \psi_\pm, \bar{\psi}_\pm \text{ tangent to } \mathbb{CP}^{N-1}$$

metric: $$ds^2 = \sum_{i=1}^{N} \frac{1}{2\pi} |D\phi_i|^2 \qquad \left(\begin{array}{c}\text{action is divided} \\ \text{by } 2\pi\end{array} \right)$$

v_μ above $\Rightarrow$ $$\sum_{j=1}^{N} \left(\bar{\phi}_j D_\mu \phi_j - D_\mu \bar{\phi}_j \phi_j \right) = 0$$

$$\Leftrightarrow \quad D_\mu \phi_j \perp U(1) \text{ gauge orbit.}$$

$$ds^2 = \frac{r}{2\pi} (\text{Fubini–Study metric})$$

$$\omega = \frac{r}{2\pi} \omega_{FS}$$

Cf. This process is so called <u>Kähler quotient</u>.

$G \circlearrowright (X, \omega)$ symmetry of a symplectic mfd.

$\mu : X \to \mathfrak{g}^*$ moment map $\begin{cases} d\mu(a) = i_{X_a}\omega \\ g^*\mu(a) = \mu(g^{-1}ag) \end{cases} \forall a \in \mathfrak{g}$

$\omega|_{\mu^{-1}(0)}$ decends to a symplectic form $\bar{\omega}$ on $\mu^{-1}(0)/G$

$(\mu^{-1}(0)/G, \bar{\omega})$ — symplectic quotient

J : compatible cplx str $\begin{cases} \omega(Jv, Jw) = \omega(v, w) \\ g(v, w) = \omega(v, Jw) > 0 \end{cases}$

$G \circlearrowright (X, \omega, J)$ symmetry ← Kähler metric

$\Rightarrow \exists$ compatible cplx str $\bar{J}$ in $(\mu^{-1}(0)/G, \bar{\omega})$

$\bar{g}(\bar{v}, \bar{w}) = \bar{\omega}(\bar{v}, \bar{J}\bar{w}) = g(v, w)$:

v, w lifts to $\mu^{-1}(0)$ orthogonal to G-orbit.

— Kähler quotient.

<u>Fact</u>

$$\mu^{-1}(0)/G \cong (X \setminus \text{bad orbits})/G_{\mathbb{C}} = X /\!\!/ G_{\mathbb{C}}$$

"G.I.T. quotient"

e.g. $X = \mathbb{C}^N$, $G = U(1)$ $(z_i \mapsto e^{i\alpha} z_i)$

$\mu = \sum_{i=1}^{N} |z_i|^2 - r$

$\mu^{-1}(0)/U(1) = (\mathbb{C}^N - 0)/\mathbb{C}^\times = \mathbb{CP}^{N-1}$ $\quad \bar{\omega}$ = Fubini-Study form, $\bar{g}$ = FS metric.

B-field : $\upsilon_\mu = (\phi^* A)_\mu$

A ... a connection of $\mathcal{O}(1) \to \mathbb{C}P^{N-1}$

$iF_A = \omega_{FS}$, $\frac{i}{2\pi} F_A = \frac{\omega_{FS}}{2\pi}$ generates $H^2(\mathbb{C}P^{N-1}; \mathbb{Z})$.

$\rightsquigarrow \quad F_{01} = (\phi^* \omega_{FS})_{01}$

$$\frac{1}{2\pi} \int d^2x \, \theta F_{01} = \frac{\theta}{2\pi} \int \phi^* \omega_{FS}$$

$$\therefore \quad \underline{B = \frac{\theta}{2\pi} \omega_{FS}}$$

Complexified Kähler class :

$$[\omega] - i[B] = r\left[\frac{\omega_{FS}}{2\pi}\right] - i\theta \left[\frac{\omega_{FS}}{2\pi}\right]$$

$$= \underbrace{(r - i\theta)}_{\overset{\parallel}{t}} \underbrace{\left[\frac{\omega_{FS}}{2\pi}\right]}_{H^2(\mathbb{C}P^{N-1}; \mathbb{Z}) \text{ generator.}}$$

Generalization

(1) toric manifolds

(2) hypersurfaces (& their complete intersections) in toric manifolds.

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(1) $\Phi_1, \cdots, \Phi_N$ chiral superfields

$V_1, \cdots, V_k$ real superfields

gauge transf: $\Phi_i \to e^{iQ_i^a A_a} \Phi_i$

$$V_a \to V_a - iA_a + i\overline{A_a}$$

$$S = \int d^2x\, d^4\theta \left(\sum_{i=1}^{N} \bar{\Phi}_i e^{\sum_{a=1}^{k} Q_i^a V_a} \Phi_i - \sum_{a=1}^{k} \frac{1}{2e_a^2} \bar{\Sigma}_a \Sigma_a \right)$$

$$+ \mathrm{Re} \int d^2\tilde{\theta} \left(- \sum_{a=1}^{k} t^a \Sigma_a \right) \qquad t^a = r^a - i\theta^a$$

Potential $$U = \sum_{a=1}^{k} \frac{e_a^2}{2} \left(\sum_{i=1}^{N} Q_i^a |\phi_i|^2 - r^a \right)^2$$

$$+ \sum_{i=1}^{N} \left| \sum_{a=1}^{k} Q_i^a \sigma_a \right|^2 |\phi_i|^2$$

(r^a) in a "suitable" region

$\Rightarrow$ enough # of $|\phi_i| \neq 0 \Rightarrow \sigma_a = 0$

zero of U

$$= \left\{ (\phi_1, \cdots, \phi_N) \in \mathbb{C}^N \,\middle|\, \sum_{i=1}^N Q_i^a |\phi_i|^2 - r^a \right\} \Big/ U(1)^k$$

symplectic quotient of $(\mathbb{C}^N, \sum_i \frac{i}{2} dz_i \wedge d\bar{z}_i)$

by $U(1)^k : z_i \to e^{iQ_i^a \alpha_a} z_i$

$$\mu^a = \sum_{i=1}^N Q_i^a |z_i|^2 - r^a$$

$$\cong \left(\mathbb{C}^N \setminus \underbrace{\text{bad orbits}}_{\text{depends on } (r^a)} \right) \Big/ (\mathbb{C}^\times)^k = (\mathbb{C}^N) /\!/\!/ (\mathbb{C}^*)^k$$

$\cdots$ toric manifold of dimension $(N-k)$.

<u>e.g.</u> single $U(1)$

$P, \Phi_1, \cdots, \Phi_N$ $\mathbb{C}^{N+1}$ valued

Q_i: $-d, 1, \cdots, 1$

Vac ~~mfd = {~~ potential $= \frac{e^2}{2} \left(-d|p|^2 + \sum_{i=1}^N |\phi_i|^2 - r \right)^2$

$$+ d^2 |\sigma|^2 |p|^2 + \sum_{i=1}^N |\sigma|^2 |\phi_i|^2$$

$r > 0$: $\phi_i \neq 0 \Rightarrow \sigma = 0$

$$\{(p, \phi_1, \cdots, \phi_N) \mid -d|p|^2 + \underbrace{\sum_{i=1}^{N} |\phi_i|^2}_{V_0} = r\} / U(1)$$

$$= (\mathbb{C}^{N+1} \setminus \{(*, 0, \cdots, 0)\}) / \mathbb{C}^*$$

$= \text{total space of } \mathcal{O}(-d) \to \mathbb{CP}^{N-1}$

$r < 0$: $p \neq 0 \Rightarrow \sigma = 0$

$$\{(p, \phi_1, \cdots, \phi_N) \mid d|p|^2 = \sum_{i=1}^{N} |\phi_i|^2 + |r|\} / U(1)$$

$$= (\mathbb{C}^{N+1} \setminus \{(0, *, * \cdots, *)\}) / \mathbb{C}^*$$

$$= \{(p = 1, \phi_1, \cdots, \phi_N)\} / \mathbb{Z}_d$$

$\mathbb{Z}_d \subset \mathbb{C}^*$ subgroup leaving p inv.

$$= \mathbb{C}^N / \mathbb{Z}_d$$

$r = 0$: (p, ϕ_i) can be $0 \Rightarrow \sigma \neq 0$

<u>Vac mfd</u>