(2) hypersurfaces (& their complete intersections) in toric manifolds.

Specific example : degree d hypersurface in $\mathbb{CP}^{N-1}$

$$M = \left\{ (z_1, \cdots, z_N) \neq 0 \;\middle|\; \underset{\substack{\uparrow \\ \text{degree } d \\ \text{polynomial.}}}{G(z_1, \cdots, z_N)} = 0 \right\} \Big/ \mathbb{C}^* \subset \mathbb{CP}^{N-1}$$

(M smooth : $G = \partial_i G = 0 \Rightarrow \forall z_i = 0$)

LSM: $P, \Phi_1, \cdots, \Phi_N$ chiral superfields

V real superfields

gauge transf : $\begin{cases} P \to e^{-idA} P, \; \Phi_j \to e^{iA} \Phi_j \\ V \to V - iA + i\bar{A}. \end{cases}$

$$\mathcal{L} = \int d^4\theta \left(\bar{P} e^{-dV} P + \sum_{i=1}^{N} \bar{\Phi}_i e^{V} \Phi_i - \frac{1}{2e^2} \bar{\Sigma}\Sigma \right)$$

$$+ \mathrm{Re} \int d^2\tilde{\theta} \, (-t\Sigma) + \mathrm{Re} \int d^2\theta \, \underbrace{P G(\Phi_1, \cdots, \Phi_N)}_{\text{Gauge invariant.}}.$$

Potential

$$U(p, \phi_i, \sigma) = d^2 |\sigma|^2 |p|^2 + \sum_{i=1}^{N} |\phi_i|^2 |\sigma|^2$$
$$+ \frac{e^2}{2} \left(-d|p|^2 + \sum_{i=1}^{N} |\phi_i|^2 - r \right)^2$$
$$+ |G(\phi)|^2 + \sum_{i=1}^{N} |p|^2 \left| \frac{\partial G}{\partial \phi_i} \right|^2$$

$\underline{r>0}$ $\quad \phi_i \neq 0 \Rightarrow \sigma = 0$

$$|G|^2 = |p||\partial_i G|^2 = 0 \overset{M:\text{smooth}}{\Longrightarrow} p=0, \ G(\phi)=0.$$

$$M_{vac} = \left\{ \begin{matrix} \phi \\ \sigma = p = 0 \end{matrix} \middle| \begin{matrix} G(\phi)=0 \\ \sum_{i=1}^{N} |\phi_i|^2 = r \end{matrix} \right\} \Big/ U(1) = \{\phi \neq 0 \mid G(\phi)=0\} / \mathbb{C}^* = M.$$

$\underline{r<0}$ $\quad p \neq 0 \Rightarrow \sigma = 0$

$$G = p\,\partial_i G = 0 \Rightarrow \phi_i = 0$$

$$M_{vac} = \left\{ \sigma = \phi = 0, p \,\middle|\, |p|^2 = -\frac{r}{d} \right\} \Big/ U(1) = 1 \text{ point}.$$

$$|p| = \sqrt{-\frac{r}{d}} \Rightarrow U(1) \xrightarrow{\text{broken}} \mathbb{Z}_d$$

$\underline{r \to -\infty}$: P mass $\sim e\sqrt{-r} \to \infty$ $\quad P$ frozen $\left(\text{at } \overset{\langle p \rangle}{\overset{\shortparallel}{\sqrt{\frac{-r}{d}}}}\right)$

massless : $\Phi_1, \cdots, \Phi_N$.

$$\left.\begin{matrix} W = \langle p \rangle G(\Phi_1, \cdots, \Phi_N) \\ \text{gauge group} = \mathbb{Z}_d \end{matrix}\right] \text{ LG orbifold}$$

$\exists\ \mathbb{Z}_d$ quantum symmetry.

$\underline{r=0}$ $\quad p = \phi_i = 0$, σ ... free ... $\underline{\text{non-compact}}$.

$\sim$ singularity of the theory.

r

... At $E \ll e\sqrt{r}, \sqrt{\frac{\partial G}{\partial \phi_i}}$

NLSM with target M

$\exists$ "continuation" to negative Kähler class but separated by a singularity

0 --- σ arbitrary (singular)

so, not really "continuation" (classically)

$-\infty$ --- LG orbifold $W = \langle p \rangle G(\Phi_1, \cdots, \Phi_N)$

$\Gamma = \mathbb{Z}_d$

($\exists$ $\mathbb{Z}_d$ quantum symmetry)

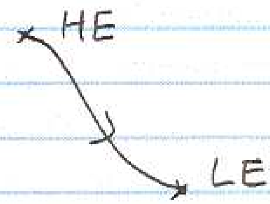
— This is a classical picture:

* to be modified by quantum ~~correction~~ effects.
  * renormalization
  * gauge dynamics / θ-angle

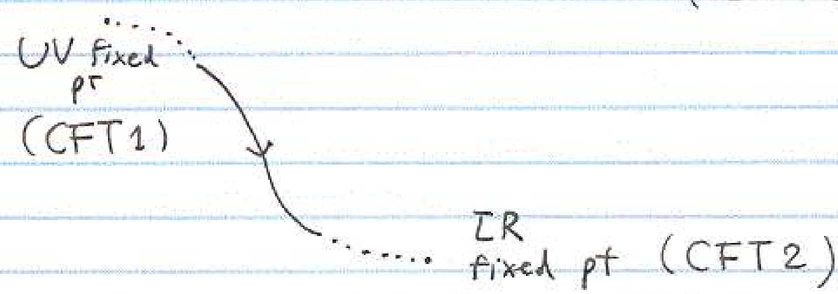
Basic way of thinking in QFT (K. Wilson)

— Start with high energy & end at low energy

- basic problem: Given a theory at high energy, where does it go to ? (at low energy)



- Extreme HE & LE limits: UV & IR fixed points (CFT's)



Consider a system of variable $\phi(x)$, coupling const g & action $S(X, g)$. Expand $\phi(x)$ in Fourier modes

$$\phi(x) = \int \frac{d^2k}{(2\pi)^2} e^{ikx} \hat{\phi}(k)$$

Remove the part with $|k| > \Lambda_{UV}$ $\left(\hat{\phi}(k) = 0 \text{ if } |k| > \Lambda_{UV}\right)$

Λ_{UV}: UV cut-off.

Take some lower energy $\mu < \Lambda_{UV}$ & separate $\phi(x)$ into two parts

$$\phi(x) = \underbrace{\int_{|k|<\mu} e^{ikx}\hat{\phi}(k)}_{\phi_L(x)} + \underbrace{\int_{\mu<|k|<\Lambda_{UV}} e^{ikx}\hat{\phi}(k)}_{\phi_H(x)}$$

Let's ~~just~~ integrate over only ϕ_H ; obtaining a functional for ϕ_L :

$$\int \mathcal{D}\phi_H \, e^{-S(\phi_L+\phi_H,\, g)} = e^{-\underbrace{S_{eff}(\phi_L,\, g)}_{\text{called effective action at energy } \mu}}$$

$$\begin{array}{ll} E=\Lambda_{UV} & S(X,g) \\ & \downarrow \\ E=\mu & S_{eff}(X,g) \end{array}$$

This is the <u>renormalization group flow</u>

In order to "standardize" the kinetic term & coupling term in $S_{eff}(\phi_L, g)$, we need to redefine ϕ_L & g (e.g. $\phi_L = Z_\phi \phi'$, $g = Z_g g'$)

This redefinition is <u>renormalization</u>. (Not unique! choice of ren.scheme)

μ : fixed

When $S_{eff}[\phi', g']$ <u>is</u> finite in the limit $\Lambda_{UV}\to\infty$

can be made

The theory is called <u>renormalizable</u>.

In a SUSY QFT, there are <u>two parts</u> that are "protected from renormalization".

$$S = \int d^2x\, d^4\theta\, K(\Phi, \overline{\Phi}, \widetilde{\Phi}, \overline{\widetilde{\Phi}}, F, g_D) + \mathrm{Re}\int d^2x\, d^2\theta\, W(\Phi, \lambda)$$
$$+ \mathrm{Re}\int d^2x\, d^2\tilde{\theta}\, \widetilde{W}(\widetilde{\Phi}, \tilde{\lambda})$$

SUSY $\Rightarrow$ W cannot include $\overline{\Phi}, \widetilde{\Phi}, \overline{\widetilde{\Phi}}, F$

$\widetilde{W}$ —— $\Phi, \overline{\Phi}, \overline{\widetilde{\Phi}}, F$

$\Rightarrow$ W_{eff} cannot include $\overline{\Phi}_L, \widetilde{\Phi}_L, \overline{\widetilde{\Phi}}_L, F_L, g_D, \bar{\lambda}, \tilde{\lambda}, \bar{\tilde{\lambda}}$.

$\widetilde{W}_{eff}$ —— $\Phi_L, \overline{\Phi}_L, \overline{\widetilde{\Phi}}_L, F_L, g_D, \lambda, \bar{\lambda}, \bar{\tilde{\lambda}}$.

— <u>decoupling thm</u> ($\Leftrightarrow$ decoupling the th top. cor funs)

In simple case $W_{eff}(\Phi, \lambda) \equiv W(\Phi, \lambda)$

not changing at all!

— <u>non-renormalization thm</u>.

~~Basic~~ elementary quantum effects in LSM

① Renormalization of r.

Consider $U(1)$ gaug theory with $\Phi_1, \cdots, \Phi_N$, charge $Q_1, \cdots, Q_N$.

r (FI parameter) is renormalized:

$$r(\mu) = r(\Lambda_{UV}) + (Q_1 + \cdots + Q_N)\log\left(\frac{\mu}{\Lambda_{UV}}\right)$$

explanation $\quad \mathcal{L} \supset -r_{(\Lambda_{UV})} D + \sum_{i=1}^{N} Q_i D|\phi_i|^2 + \frac{1}{2e^2}D^2$

$\qquad + (\text{D-independent terms})$

$$\int \mathcal{D}\Phi_H \mathcal{D}V_H \Rightarrow \sum_{i=1}^{N} Q_i D|\phi_i|^2 \longrightarrow \sum_{i=1}^{N} Q_i D_L\left(|\phi_{L,i}|^2 + \langle|\phi_i|^2\rangle_L^H\right)$$

$$\langle|\phi_i|^2\rangle_L^H = \int_{\mu \le |k| \le \Lambda_{UV}} \frac{d^2k}{(2\pi)^2}\frac{2\pi}{k^2} = \log\left(\frac{\Lambda_{UV}}{\mu}\right)$$

$$\mathcal{L} \rightarrow \underbrace{-rD_L + \sum_{i=1}^{N} Q_i D_L \langle|\phi_i|^2\rangle_L^H}_{} + \sum_{i=1}^{N} Q_i D_L|\phi_{L,i}|^2 + \frac{1}{2e_L^2}D_L^2$$

$$-\underbrace{\left(r_{(\Lambda_{UV})} - \sum_{i=1}^{N} Q_i \log\left(\frac{\Lambda_{UV}}{\mu}\right)\right)}_{=r(\mu)} D_L$$

When $\sum_{i=1}^{N} Q_i \neq 0$, r "runs" i.e. changes under change in scale μ .... it is <u>NOT</u> a good parameter of the theory by itself.

- One needs to say "$r = r(\mu)$ at $E = \mu$".
- Or one may take <u>the energy scale Λ at which $r = 0$</u> as the parameter of the theory, so that $r(\mu) = \sum_{i=1}^{N} Q_i \log\left(\frac{\mu}{\Lambda}\right)$ ... <u>dimensional transmutation</u>

When $\sum_{i=1}^{N} Q_i = 0$, r is a good & free parameter of theory.

e.g. LSM for degree d hypersurface in $\mathbb{CP}^{N-1}$: $\sum_i Q_i = N - d$.

$E \uparrow$	$d < N$	$d = N$	$d > N$
	$r > 0$ NLSM on M (degree d $\subset \mathbb{CP}^{N-1}$)	$r = r(\mu)$ doesn't run	$r < 0$ LG orbifold $W = G(\Phi)$, $\Gamma = \mathbb{Z}_d$ (with some perturbation)
	Λ - - - - - -	$r \gg 0$: NLSM $r \ll 0$: LG orb.	Λ - - - - - -
	$r < 0$ LG orbifold $W = G(\Phi_i)$ $\Gamma = \mathbb{Z}_d$		$r > 0$ NLSM on degree d hypersurface.

Cf: also similar for theory with $W = 0$.

② $U(1)_A$ anomaly

$$t = r - i\theta$$

$$\mathcal{L} = \int d^4\theta \left(\sum_{i=1}^{N} \bar{\Phi}_i e^{Q_i V} \Phi_i - \frac{1}{2e^2}|\Sigma|^2 \right) + \mathrm{Re}\int d^2\tilde{\theta}\,(-t\Sigma) + \mathrm{Re}\int d^2\theta\, P G(\Phi)$$

… invariant under $U(1)_A$ $\quad V(x, \theta^\pm, \bar{\theta}^\pm) \to V(x, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm)$

$$\Phi_i(x, \theta^\pm, \bar{\theta}^\pm) \to \Phi_i(x, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm)$$

$\underline{\Sigma = \bar{D}_+ D_- V \text{ charge } 2}$

$$\psi_{i,\pm} \to e^{\mp i\beta}\psi_{i,\pm}, \quad \lambda_\pm \to e^{\mp i\beta}\lambda_\pm, \quad \sigma \to e^{2i\beta}\sigma$$

For a gauge configuration with $-\frac{1}{2\pi}\int F_v = k$

$$\mathcal{D}(\psi_i, \lambda)\mathcal{D}(\bar{\psi}_i, \bar{\lambda}) \to e^{-2ki\sum_{i=1}^{N} Q_i\beta}\, \mathcal{D}(\psi_i\lambda)\mathcal{D}(\bar{\psi}_i\bar{\lambda})$$

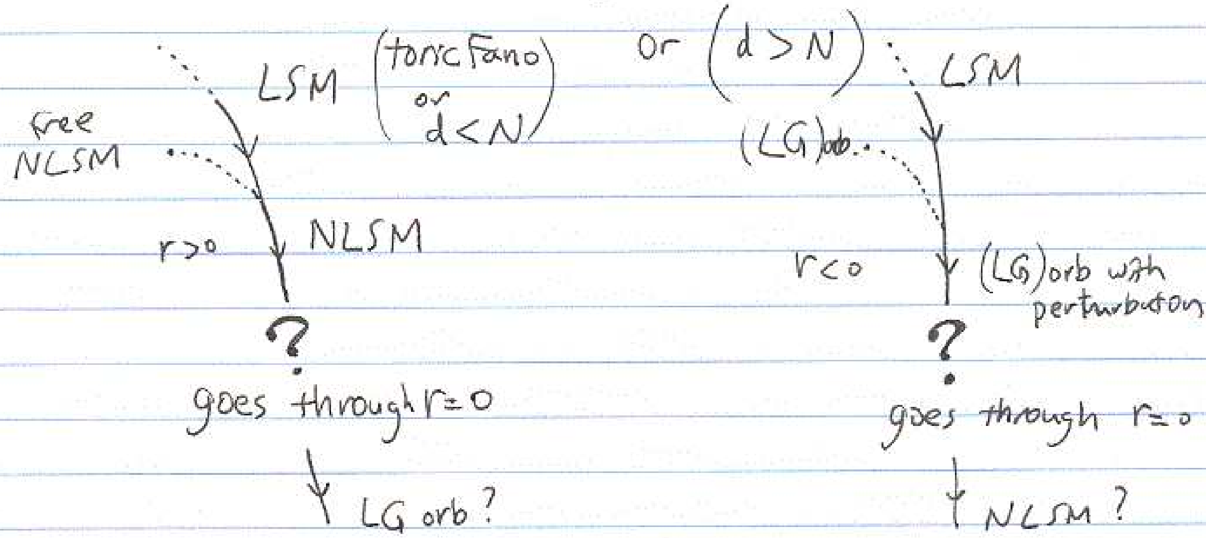
— $U(1)_A$ anomaly if $\sum_{i=1}^{N} Q_i \neq 0$

Same effect as $\quad \theta \to \theta - 2\beta \sum_{i=1}^{N} Q_i$

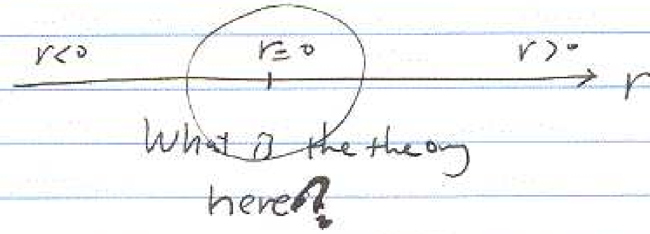
When $\sum_{i=1}^{N} Q_i \neq 0$, θ angle can be absorbed by field redefinition (phase rotation). θ is $\underline{\text{NOT}}$ a genuine parameter of the theory.

When $\sum_{i=1}^{N} Q_i = 0$, $U(1)_A$ anomaly free, θ genuine free parameter.

Low energy theory



d=N or CY sigma model



Near $r \sim 0$, classical potential for σ is nearly flat.

$\Longrightarrow$ Motivates us to probe the region with large σ.

For large σ, charged fields are heavy (by $U \supset Q_i^2 |\sigma|^2 |\phi_i|^2$)

— appropriate to integrate out.

$$e^{-S_{eff}(\Sigma)} = \int_{\mathcal{D}\Phi} e^{-S(V,\Phi)}$$

By supersymmetry $S_{eff}(\Sigma) = \int d^2x\, d^4\theta\, K_{eff}(\Sigma, \bar{\Sigma}) + \mathrm{Re}\int d^2x\, d^2\tilde{\theta}\, \widetilde{W}_{eff}(\Sigma)$.

By decoupling thm, $W = W(\Phi)$ cannot affect $\widetilde{W}_{eff}(\Sigma)$.

For the purpose of computing $\widetilde{W}_{eff}(\Sigma)$, one can set $W=0$. Then $S(V,\Phi)$ is quadratic in Φ & the path-integral can be performed exactly. $\Rightarrow$

$$\widetilde{W}_{eff} = -\sum_{i=1}^{N} Q_i \Sigma \left(\log\left(\frac{Q_i \Sigma}{\mu}\right) - 1 \right) - \underset{\substack{\uparrow \\ \text{any scale}}}{t(\mu)} \Sigma$$

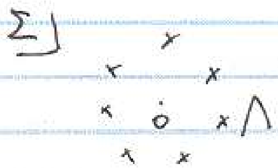
<u>Example</u> $\mathbb{C}P^{N-1}$. $Q_1 = \cdots = Q_N = 1$, $W=0$.

$$\widetilde{W}_{eff}(\Sigma) = -N\Sigma\left(\log\left(\frac{\Sigma}{\mu}\right) - 1\right) - t(\mu)\Sigma$$

Critical pts: $0 = \widetilde{W}_{eff}'(\Sigma) = -N \log\left(\frac{\Sigma}{\mu}\right) - t(\mu)$

$$\left(\frac{\Sigma}{\mu}\right)^N = e^{-t(\mu)} \quad \therefore \Sigma^N = \Lambda^N \quad \left(t(\mu) = N \log\left(\frac{\mu}{\Lambda}\right) \right)$$

$$\Sigma = \Lambda e^{\frac{2\pi i}{N} l} \quad l = 0, 1, \cdots, N-1$$



<u>N-vacua</u> $\Longleftrightarrow$ <u>$\dim H^*(\mathbb{C}P^{N-1}) = N$</u>

$\mathbb{Z}_{2N}$ axial R-symmetry spontaneously broken.

Example degree $d = N$ hypersurface in $\mathbb{C}P^{N-1}$ (CY mfd)

$$\widetilde{W}_{eff} = -N\Sigma\left(\log\left(\frac{\Sigma}{\mu}\right)-1\right) + \underset{\overset{\|}{N}}{d}\,\Sigma\left(\log\left(\frac{-\overset{\overset{N}{\|}}{d}\Sigma}{\mu}\right)-1\right) - t(\mu)\Sigma$$

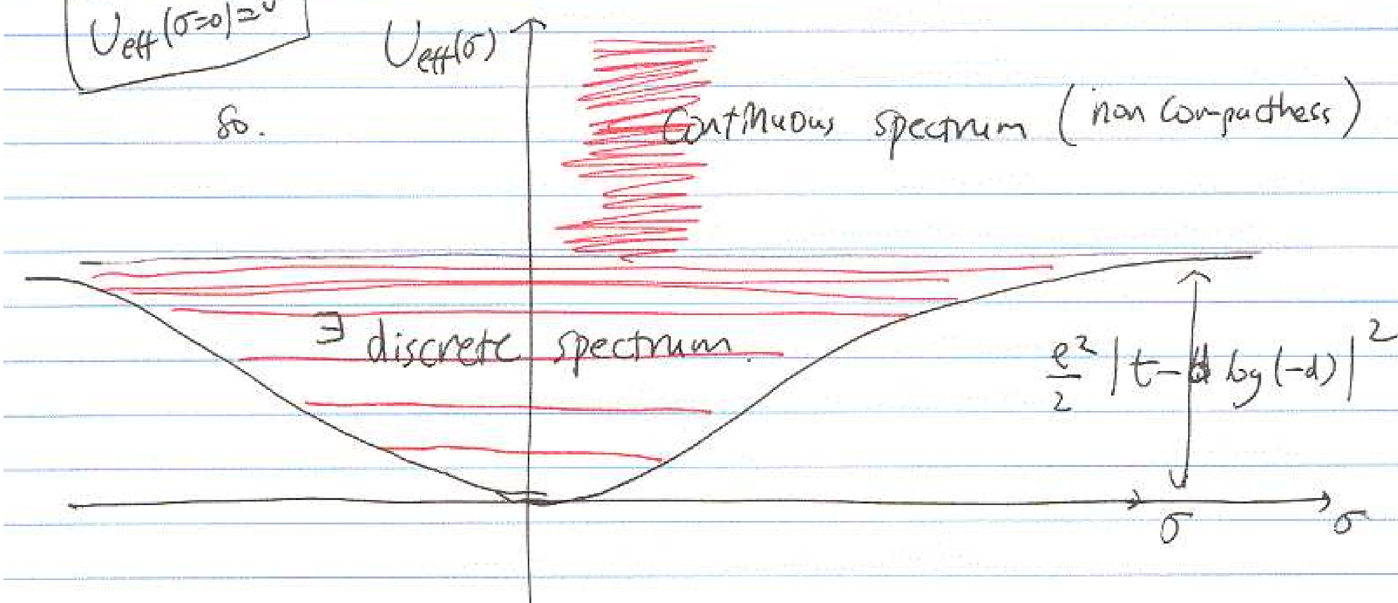
$$= -\left(t - d\log(-d)\right)\Sigma$$

U(1) gauge dynamics $\Rightarrow$ $U_{eff}(\sigma) = \frac{e_{eff}^2}{2}\left|t - d\log(-d)\right|^2$ at large σ.

On the other hand $\xrightarrow{\sigma\to\infty} \frac{e^2}{2}\left|t - d\log(-d)\right|^2 > 0$ if $\underline{t \neq d\log(-d)}$

We know $U_{eff}(\sigma=0) = 0$

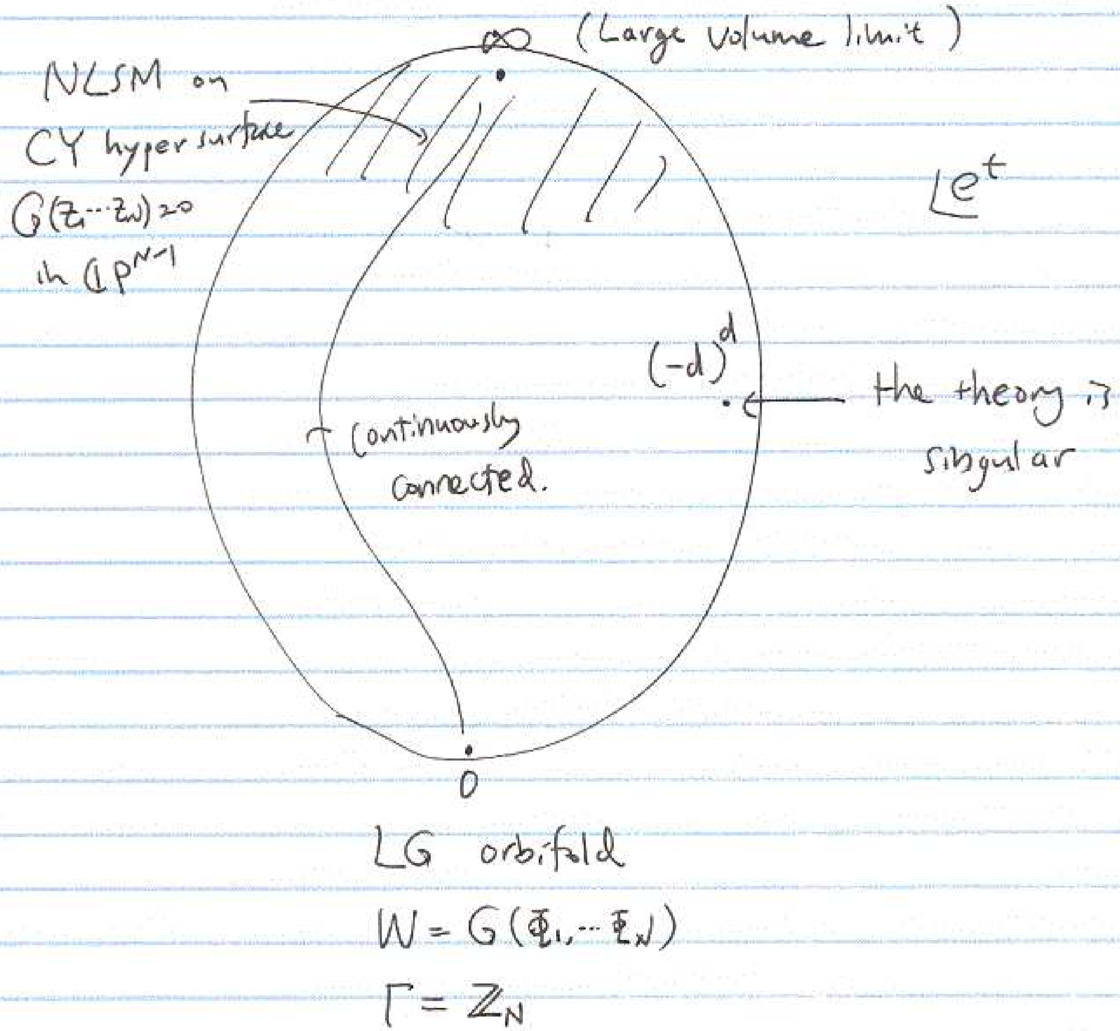
So.



At low energies, one can only see the discrete spectrum. $\Rightarrow$ "sound" theory.

But exactly at $t = d\log(-d)$, the gap vanishes $\Longrightarrow$ Continuous spectrum from $E=0$
(absence of discrete spectrum)
... singularity of the theory.

Moduli space of theories



Example degree $d \neq N$ hypersurface in $\mathbb{CP}^{N-1}$ $\begin{pmatrix} d<N \text{ Fano} \\ d>N \text{ general type} \end{pmatrix}$

$$\widetilde{W}'_{\text{eff}} = -(N-d)\log\left(\frac{\Sigma}{\mu}\right) - t(\mu) + d\log(-d) = 0$$

$$\Rightarrow \left(\frac{\Sigma}{\mu}\right)^{N-d} = e^{-t(\mu)}(-d)^d \quad \text{--- } \underline{|N-d| \text{ solutions}}$$

But this is <u>not</u> all. ... we haven't looked at small σ.

At small σ, the classical description applies (analysis)

E	$d<N$	$d>N$
	NLSM on degree d hypersurface	LG orbifold $W = G(\Phi)/\mathbb{Z}_d$ + perturbation
Λ		
	$(N-d)$ massive vacua + LG orbifold $W = G(\Phi)/\mathbb{Z}_d$	$(d-N)$ massive vacua + NLSM on degree d hypersurface.

<u>Check</u> * Witten index — must be the same

* central charge of UV/IR fixed pts — must be $c_{UV} > c_{IR}$

<u>Witten index</u>

$$\mathrm{Tr}_{NLSM}(-1)^F = \chi(M) = \frac{(1-d)^N - 1}{d} + N$$

$$\mathrm{Tr}_{LGO}(-1)^F = \frac{(1-d)^N - 1}{d} + d$$

<u>Central charge</u>

$$\hat{c}_{NLSM} = \dim M = (N-2)$$

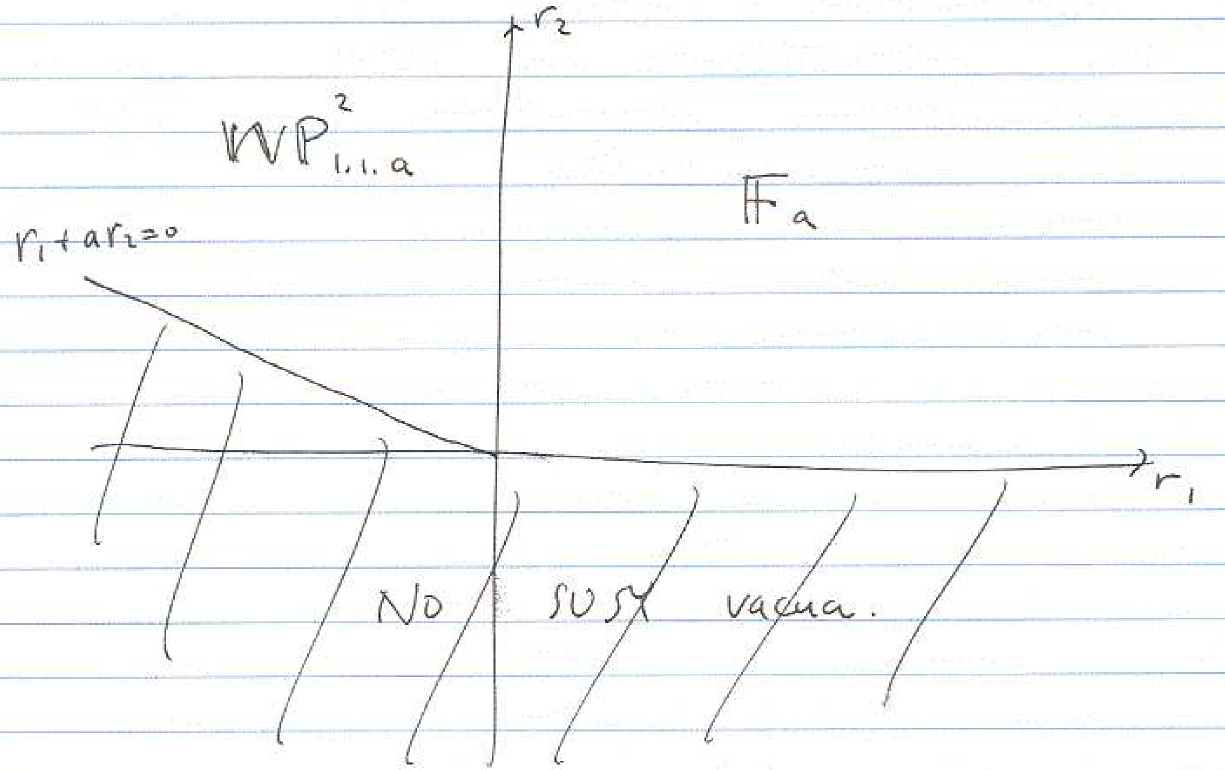
$$\hat{c}_{LGO} = N - \frac{2N}{d}$$

	$d < N$		$d > N$	
	$\mathrm{Tr}(-1)^F$	$\hat{c}$	$\mathrm{Tr}(-1)^F$	$\hat{c}$
HE	$\frac{(1-d)^N-1}{d}+N$	$N-2$	$\frac{(1-d)^N}{d}+d$	$N-2\frac{N}{d}$
	$\parallel$	$\vee$	$\parallel$	$\vee$
LE	$(N-d)+\frac{(1-d)^N}{d}+d$	$N-2\frac{N}{d}$	$(d-N)+\frac{(1-d)^N}{d}+N$	$N-2$

Example Hirzebruch surface F_a

	Φ_1	Φ_2	Φ_3	Φ_4	
$U(1)_1$	1	1	$-a$	0	$\leftrightarrow r_1$
$\times$					
$U(1)_2$	0	0	1	1	$\leftrightarrow r_2$

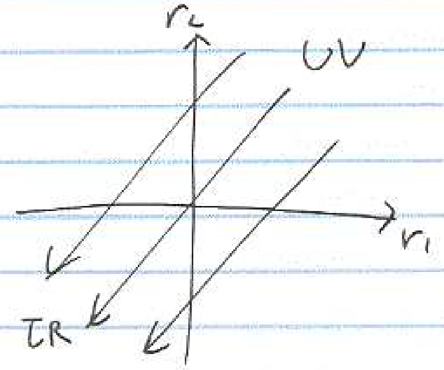
Classical parameter space



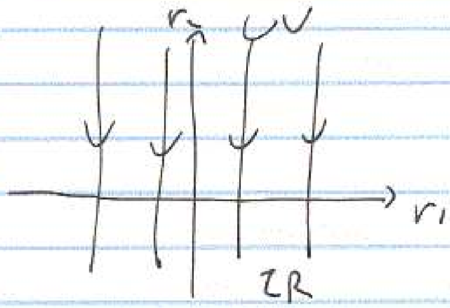
renormalization : $r_1(\mu) = (2-a)\log\mu$

$r_2(\mu) = 2\log\mu$

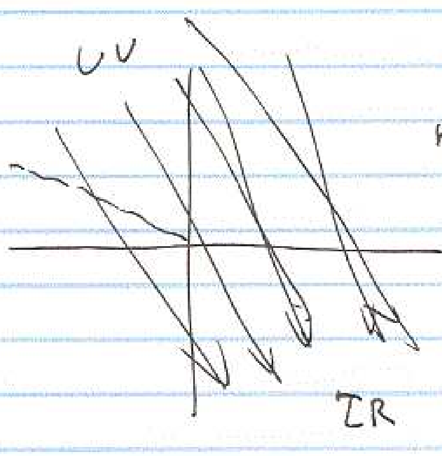
$a=0,1$ $F_a \cdots$ Fano ($\mathbb{CP}^1\times\mathbb{CP}^1$ or $\mathbb{CP}^2$ with 1pt blowup)



$a=2$ $F_a \cdots$ "nef" ($c_1 \geqslant 0$ $\exists =0$ direction)



$a \geqslant 3$ F_a has $c_1<0$ direction



starts with $\mathbb{WP}^2_{1,1,a}$

may pass through "F_a"

in intermediate energies.

$$\frac{\partial \widetilde{W}_{eff}}{\partial \Sigma_1} = \frac{\partial \widetilde{W}_{eff}}{\partial \Sigma_2} = 0$$

$$\Rightarrow \qquad \Sigma_1^2 (\Sigma_2 - a\Sigma_1)^{-a} = e^{-t_1}$$

$$(\Sigma_2 - a\Sigma_1)\Sigma_2 = e^{-t_2}$$

Σ_1 uniquely determine (by 2nd eqn) by $\Sigma_2' = \Sigma_2 - a\Sigma_1$

$$((\Sigma_2')^2 - e^{-t_2})^2 = a^2 e^{-t_1} (\Sigma_2')^{a+2}$$

solutions : 4 for $a = 0, 1, 2$

$a+2$ for $a \geq 3$

$\longleftrightarrow$ $\dim H^*(F_a) = 4$

$\dim \underset{\substack{\uparrow \\ \text{orb}}}{H^*} (WP^2_{1,1,a}) = a+2$