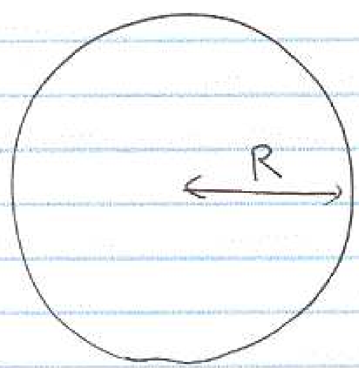
Mirror Symmetry

idea of derivation

$$\begin{tikzcd} \text{LSM} \arrow[r, "\text{T-duality}"] \arrow[d, "e^2\to\infty"'] & \text{Mirror LSM} \arrow[d, "e^2\to\infty"] \\ \text{NLSM}/\cdots/(\text{LG})\text{orbifold} & ?\ (\text{the mirror}) \end{tikzcd}$$

T-duality



Sigma model with target

S^1_R (circle of radius R)

State of a string is characterized by

- momentum $p = \frac{l}{R}$ $l \in \mathbb{Z}$
- winding # $m \in \mathbb{Z}$
- oscillation detail

Wave function e^{ipx}

single valued as $x \to x + 2\pi R$

(Spacetime) energy E of a string without oscillation

this part doesn't depend on R.

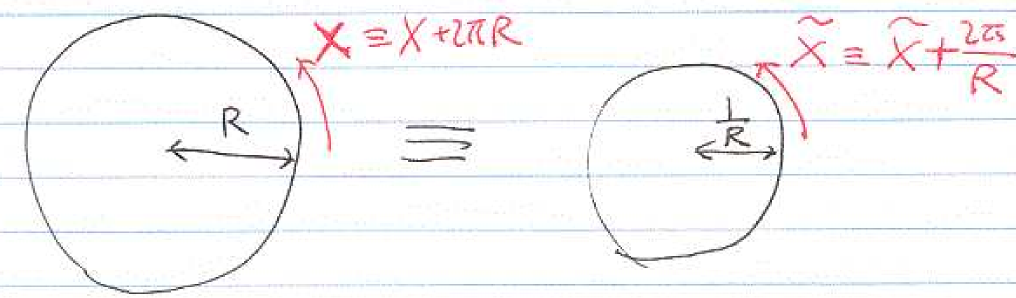
$$E^2 = p^2 + M^2 = \left(\frac{\ell}{R}\right)^2 + (Rm)^2$$

— invariant under

$$R \longleftrightarrow \frac{1}{R}$$

$$\ell \longleftrightarrow m$$

This implies equivalence of (NL) sigma models



momentum $\longleftrightarrow$ winding #

winding # $\longleftrightarrow$ momentum

$$\Rightarrow \quad \partial_t X = \partial_\sigma \tilde{X}$$

$$\partial_\sigma X = \partial_t \tilde{X}$$

This equivalence is called T-duality

Path-integral derivation

Consider a system of a Riemann surface Σ with metric h

$$\varphi \in \mathrm{Map}(\Sigma, \mathbb{R}/2\pi\mathbb{Z})$$

$$B \in \Omega^1(\Sigma, \mathbb{R})$$

$$S' = \frac{1}{2\pi}\int_\Sigma \frac{1}{2R^2}\|B\|^2 + \frac{i}{2\pi}\int_\Sigma B \wedge d\varphi$$

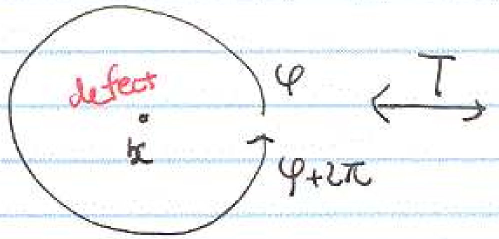
$\int \mathcal{D}B$, $B = iR^2 * d\varphi$

$\int \mathcal{D}\varphi$, $\boxed{B = d\tilde{\varphi} \quad \tilde{\varphi} \in \mathrm{Map}(\Sigma, \mathbb{R}/2\pi\mathbb{Z})}$ *

$$S = \frac{1}{2\pi}\int \frac{R^2}{2}\|d\varphi\|^2 \qquad\qquad \tilde{S} = \frac{1}{2\pi}\int \frac{1}{2R^2}\|d\tilde{\varphi}\|^2$$

S'_R $\qquad B = \cdots \Rightarrow R\, d\varphi = i\frac{1}{R} * d\tilde{\varphi}$ $\qquad S'_{1/R}$

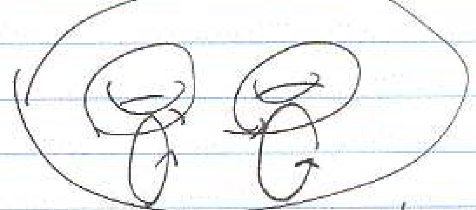
Winding # in φ = momentum in $\tilde{\varphi}$



$\xleftrightarrow{T}$ $\quad \cdot\, e^{i\tilde{\varphi}(x)}$

proof of *

$$\varphi : \Sigma \to \mathbb{R}/2\pi\mathbb{Z}$$



φ may ~~jump~~ change by $2\pi\mathbb{Z}$ along nontrivial loops

$$d\varphi = d\varphi_0 + 2\pi \sum_i m_i \omega_i$$

$$\varphi_0 : \Sigma \to \mathbb{R}, \ \{\omega_i\} \in H^1(\Sigma, \mathbb{Z}) \text{ basis}, \ m_i \in \mathbb{Z}.$$

$$\int \mathcal{D}\varphi_0 \Rightarrow dB = 0 \Rightarrow B = d\tilde{\varphi}_0 + \sum_j a_j \omega_j$$

$$\tilde{\varphi}_0 : \Sigma \to \mathbb{R}, \quad a_j \in \mathbb{R}.$$

$$\sum_{\{m_1 \cdots m_{2g}\}} e^{\frac{i}{2\pi}\int \left(d\tilde{\varphi}_0 + \sum_j a_j \omega_j\right) \wedge \left(2\pi \sum_i m_i \omega_i\right)}$$

$$= \sum_{\{m_1, \cdots, m_{2g}\}} e^{i \sum_{i,j} a_j m_i \int_\Sigma \omega_j \wedge \omega_i}$$

integral invertible matrix ($\in Sp(2g, \mathbb{Z})$)

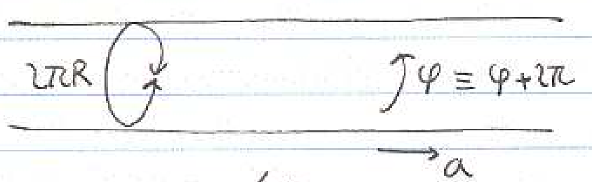
$$= \sum_{\{n_1 \cdots n_{2g}\}} \prod_j \delta_{a_j, 2\pi n_j}$$

($\tilde{\varphi}_0 : \Sigma \to \mathbb{R}$, $n_j \in \mathbb{Z}$)

$$\therefore \ B = d\tilde{\varphi}_0 + \sum_j 2\pi n_j \omega_j = d\tilde{\varphi}$$

$$\tilde{\varphi} : \Sigma \to \mathbb{R}/2\pi\mathbb{Z}.$$

(2,2) supersymmetric case



$\phi = a + i\varphi$

$\Phi = \phi + \theta^+\psi_+ + \theta^-\psi_- + \theta^+\theta^- F$

σ-model ~~action~~ Lagrangian

chiral, $\Phi \equiv \Phi + 2\pi i$

$$L = \int d^4\theta \, \frac{R^2}{2} \bar{\Phi}\Phi$$

(Roček-Verlinde)

Let us perform T-duality on φ ... Do it on superspace

$$L' = \int d^4\theta \left(-\frac{1}{4R^2} B^2 + \frac{1}{2} B(\Phi + \bar{\Phi}) \right)$$

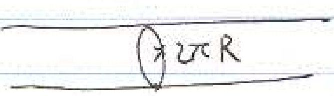
B : real superfield

Φ : chiral superfield

$\int \mathcal{D}B$

$B = R^2(\Phi + \bar{\Phi})$

$$L = \int d^4\theta \, \frac{R^2}{2} \bar{\Phi}\Phi$$



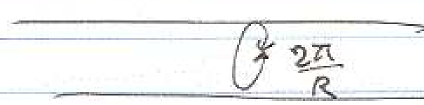
$\int \mathcal{D}\Phi$

$B = \tilde{\Phi} + \bar{\tilde{\Phi}}$

- $\tilde{\Phi}$ twisted chiral **

- $\tilde{\Phi} \equiv \tilde{\Phi} + 2\pi i$

$$\tilde{L} = \int d^4\theta \left(-\frac{1}{2R^2} \bar{\tilde{\Phi}}\tilde{\Phi} \right)$$



Φ chiral $\xleftrightarrow{T}$ $\tilde{\Phi}$ twisted chiral

T-duality is mirror symmetry!

proof of **

$$\int d^4\theta \, B \Phi \sim \int d^2\theta \, (\bar{D}_+ \bar{D}_- B) \Phi$$

$$\int d^4\theta \, B \bar{\Phi} \sim \int d^2\bar{\theta} \, (D_+ D_- B) \bar{\Phi}$$

$$\int \mathcal{D}\Phi \mathcal{D}\bar{\Phi} \Rightarrow \text{constraints} \quad \bar{D}_+ \bar{D}_- B = D_+ D_- B = 0$$

general soln: $B = \tilde{\Phi} + \bar{\tilde{\Phi}}$

$\uparrow$ twisted chiral //

Comparison for $B \Rightarrow$

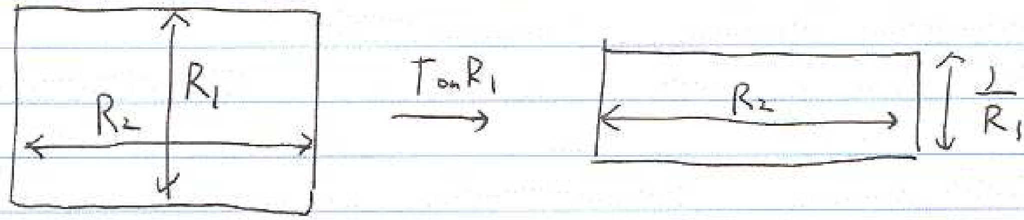
$$R(\Phi + \bar{\Phi}) = \frac{1}{R}(\tilde{\Phi} + \bar{\tilde{\Phi}})$$

$\phi = a + i\varphi$ $\qquad \cup \qquad$ $\tilde{\phi} = \tilde{a} + i\tilde{\varphi}$

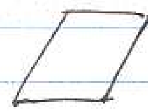
$$R a = \frac{1}{R} \tilde{a}$$

$$\left.\begin{aligned} R\,\partial_t \varphi &= \frac{1}{R} \partial_\sigma \tilde{\varphi} \\ R\,\partial_\sigma \varphi &= \frac{1}{R} \partial_t \tilde{\varphi} \end{aligned}\right\} \text{indeed T-duality along } \varphi.$$

Compact torus



Complex structure	Complex structure
$T = \dfrac{R_2}{R_1}$	$\tilde{T} = \dfrac{R_2}{\frac{1}{R_1}} = R_1 R_2$
Kähler class	Kähler class
$\omega = R_1 R_2$	$\omega = \dfrac{1}{R_1} \cdot R_2 = \dfrac{R_2}{R_1}$

* true also for slanted torus 
with B-field.

—— indeed, it's a Mirror Symmetry.

The Mirror dual of Linear Sigma Model

LSM .... V_a, Φ_i

We consider T-duality along the phases of Φ_i

along torus fibres.

Two points

- Φ_i is charged.
- At $\Phi_i = 0$, the circle shrink to zero size.

T-dual on a phase of charged field

$$|D_\mu \phi|^2 = |(\partial_\mu + i v_\mu)\phi|^2 \qquad \phi = \rho e^{i\varphi}$$

$$= (\partial_\mu \rho)^2 + \rho^2 (\partial_\mu \varphi + v_\mu)^2$$

T-dualize this φ. — Path-integral

For now consider $\rho \neq 0$ fixed.

$$S' = \frac{1}{2\pi}\int \frac{1}{4\rho^2}\|B\|^2 + \frac{i}{2\pi}\int B \wedge (d\varphi + \upsilon)$$

$$\int \mathcal{D}B \swarrow \qquad \searrow \int \mathcal{D}\varphi, \quad B = d\tilde{\varphi}$$

$$S = \frac{1}{2\pi}\int \rho^2 \| d\varphi + \upsilon \|^2 \qquad \tilde{S} = \frac{1}{2\pi}\int \frac{1}{4\rho^2}\|d\tilde{\varphi}\|^2 + \underbrace{\frac{i}{2\pi}\int d\tilde{\varphi}\wedge\upsilon}_{= -\frac{i}{2\pi}\int \tilde{\varphi} F_\upsilon}$$

$\tilde{\varphi}$: dynamical theta angle.

<u>SUSY case</u>

$$\mathcal{L} = \int d^4\theta \left(\bar{\Phi} e^V \Phi - \frac{1}{2e^2}|\Sigma|^2 \right) + \mathrm{Re}\int d^2\tilde{\theta}\,(-t\Sigma) \quad \overset{T}{\longleftrightarrow} \quad \tilde{\mathcal{L}} = \int d^4\theta \left(-K(Y,\bar{Y},\Sigma,\bar{\Sigma}) - \frac{1}{2e^2}|\Sigma|^2 \right) + \mathrm{Re}\int d^2\tilde{\theta}\,(Y - t)\Sigma$$

Φ chiral $\longleftrightarrow$ Y : Twisted chiral

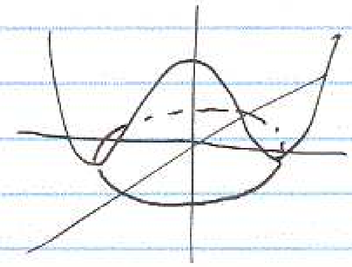
$$\arg\Phi \equiv \arg\Phi + 2\pi i \quad \overset{T}{\longleftrightarrow} \quad \mathrm{Im}\,Y \equiv \mathrm{Im}\,Y + 2\pi i$$

$$\bar{\Phi} e^V \Phi \longleftrightarrow Y + \bar{Y}$$

$$Y \sim \frac{1}{2}\rho^2 - i\tilde{\varphi}$$

effect of $\Phi = 0$

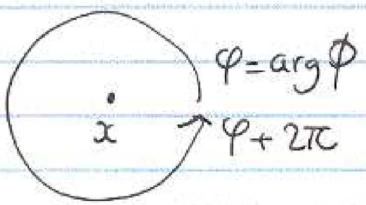
(Φ, V) system has potential $U = \frac{e^2}{2}\left(|\phi|^2 - r\right)^2$



$|\phi|^2 \longrightarrow r$ at ∞. (it appears there is no problem)

But there are Vortex configurations

e.g. $\phi(z,\bar{z}) = f(|z|)\frac{z}{|z|} \sim \begin{cases} \text{const}\cdot z & \text{near } z=0 \\ \sqrt{r}\,\frac{z}{|z|} & \text{near } |z|=\infty \end{cases}$



$\xleftrightarrow{T}$ $\cdot$ $e^{i\tilde{\varphi}(x)}$-insertion.

SUSY-vortex $\longleftrightarrow$ e^{-Y} $\cdot$ twisted superpotential term.

$$\widetilde{W} = (Y - t)\Sigma + e^{-Y}$$

This is the exact dual superpotential

proof: The only thing that respects

$\cdot$ symmetry, $\cdot$ asymptotic condition, $\cdot$ holomorphy

& the effect of vortex instanton.

$U(1)^k$ gauge theory with $\Phi_1, \cdots, \Phi_N$

$$Q_1^a, \cdots, Q_N^a$$

$$L = \int d^4\theta \left(\sum_{i=1}^{N} \bar{\Phi}_i e^{Q_i^a V_a} \Phi_i - \frac{1}{2e_a^2} \bar{\Sigma}_a \Sigma_a \right)$$

$$+ \mathrm{Re} \int d^2\tilde{\theta} \left(- t^a \Sigma_a \right) \xrightarrow{e_a^2 \to \infty} \text{NLSM on toric mfd.}$$

$\arg \Phi_i \xrightarrow{\text{T-dual}} Y_i$, twisted chiral, $Y_i \equiv Y_i + 2\pi i$

$$\tilde{W} = \sum_{a=1}^{N} \Big(\underbrace{\sum_{i=1}^{N} Q_i^a Y_i}_{\Phi_i \text{ charged}} - \underbrace{t^a}_{\text{classical}} \Big) \Sigma_a + \underbrace{e^{-Y_1} + \cdots + e^{-Y_N}}_{\text{by vortex instanton}}$$

This is the mirror LSM.
twisted superpotential of

In the limit $e_a^2 \to \infty$, ~~where~~ LSM reduces to NLSM on a toric mfd $\mathbb{C}^N /\!/ (\mathbb{C}^*)^k = \left\{ \phi \,\middle|\, \sum_{i=1}^{N} Q_i^a |\phi_i|^2 = r^a \right\} / U(1)^k$

In this limit, Σ_a become heavy, & it is appropriate to integrate them out.

$\Longrightarrow$ Constraints $\sum_{i=1}^{N} Q_i^a Y_i = t^a \qquad a=1,\cdots,k$

left with $\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_N}$

Namely, we obtain a LG model with target

$$(\mathbb{C}^\times)^{N-k} = \left\{ (Y_1,\cdots,Y_N) \,\middle|\, \sum_{i=1}^{N} Q_i^a Y_i = t^a \right\}$$

& superpotential

$$\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_N}$$

— This is the mirror of toric NLSM.

<u>example</u> $U(1)$, $\underset{1}{\Phi_1}, \cdots, \underset{1}{\Phi_N} \overset{T}{\longleftrightarrow} Y_1,\cdots,Y_N$, $U(1)$

$\widetilde{W} = (Y_1+\cdots+Y_N - t) + e^{-Y_1} + \cdots + e^{-Y_N}$

$e^2 \to \infty \downarrow \qquad\qquad \downarrow e^2 \to \infty$

NLSM on $\mathbb{CP}^{N-1}$

$\omega = tH$

$H \in H^2(\mathbb{CP}^{N-1}, \mathbb{Z})$ generator

$Y_1 + \cdots + Y_N = t$

$\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_N}$

$= e^{-Y_1} + \cdots + e^{-Y_{N-1}} + e^{-t+Y_1+\cdots+Y_{N-1}}$

An affine toda superpotential.

example

$U(1)$ $\Phi_1, \cdots, \Phi_N, P$ $\longleftrightarrow$ $U(1)$ $Y_1, \cdots, Y_N, Y_p$

$1 \cdots 1, -d$

$$\widetilde{W} = (Y_1 + \cdots + Y_N - dY_p - t)\Sigma + e^{-Y_1} + \cdots + e^{-Y_N} + e^{-Y_p}$$

$e^2 \to \infty$ $\downarrow$ $\qquad$ $\downarrow e^2 \to \infty$

$t \gg 0$ NLSM on $\mathcal{O}(-d) \to \mathbb{CP}^{N-1}$

$t \to -\infty$ orbifold $\mathbb{C}^N/\mathbb{Z}_d$

$$\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_N} + e^{-Y_p}$$

$$Y_1 + \cdots + Y_N = dY_p + t$$

Solution : $Y_i = dZ_i \quad i = 1 \cdots N$

$Y_p = -\frac{t}{d} + Z_1 + \cdots + Z_N$

$Y \leftarrow Z \quad 1 : (\mathbb{Z}_d)^{N-1}$

$Z_i \to Z_i + 2\pi i \frac{m_i}{d} \quad m_i \in \mathbb{Z}_d$

$\sum_i m_i \equiv 0$.

The mirror is LG orbifold

$$\widetilde{W} = (e^{-Z_1})^d + \cdots + (e^{-Z_N})^d + e^{\frac{t}{d}} e^{-Z_1} \cdots e^{-Z_N}$$

$$\Gamma = (\mathbb{Z}_d)^{N-1} \quad : \quad e^{-Z_i} \to \omega_i e^{-Z_i} \quad \begin{pmatrix} \omega_i^d = 1 \\ \omega_1 \cdots \omega_N = 1 \end{pmatrix}$$

When LSM has $W = PG(\Phi_1, \cdots, \Phi_N)$

~~phase rotation sym of P, Φ~~ $\longleftrightarrow$ ~~winding # conservation of $Y_1 \cdots Y_N, Y_p$~~

$X_i = e^{-Z_i}$ will be allowed to take $X_i = 0$

X_i ... natural variable.

Claim The mirror of system with $W = PG(\Phi)$

is $\widetilde{W} = X_1^d + \cdots + X_N^d + e^{\frac{t}{d}} X_1 \cdots X_N$

$\widetilde{\Gamma} = (\mathbb{Z}_d)^{N-1} \quad : X_i \to \omega_i X_i \quad \begin{pmatrix} \omega_i^d = 1 \\ \omega_1 \cdots \omega_N = 1 \end{pmatrix}$

LG orbifold