Komaba Lectures on Mirror Symmetry

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Abstract

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1 Introduction

This is a part of the notes for the lectures at the Graduate School of Mathematical Sciences, the University of Tokyo on August 26-28, 2016. A full version will be available soon.

2 2d (2,2) Supersymmetric QFTs

Let us first describe the basics of quantum field theories (QFTs) in two dimensions with $(2,2)$ supersymmetry, with emphasis on the mathematical aspects.

2.1 2d (2,2) supersymmetry

When formulated on the Minkowski spacetime, a $2d$ $(2,2)$ supersymmetric QFT has the following symmetry operators: the time translation H (Hamiltonian), the space translation P (momentum), the Lorentz transformation M, the supercharges $Q_+, \overline{Q}_+, Q_-, \overline{Q}_-$, and possibly the vector R-charge F_V and/or the axial R-charge F_A. They act on the Hilbert space of states $\mathcal{H}$ which is $\mathbb{Z}_2$-graded. The supercharges $Q_\pm$ and $\overline{Q}_\pm$ are odd and are the adjoint of each other, while the other operators are even and self-adjoint. With respect to the Lorentz group, H and P form a vector, $i[M, H \pm P] = \mp 2(H \pm P)$, the supercharges are spinors, $i[M, Q_\pm] = \mp Q_\pm$, $i[M, \overline{Q}_\pm] = \mp \overline{Q}_\pm$, and the R-charges are scalars, $[M, F_V] = [M, F_A] = 0$. The supercharges obey[1]

$$\{Q_\pm, \overline{Q}_\pm\} = H \pm P, \tag{2.1}$$

$$\text{all other anticommutators} = 0. \tag{2.2}$$

R-charges are phase rotations of the supercharges

$$[F_V, Q_\pm] = -Q_\pm, \quad [F_V, \overline{Q}_\pm] = \overline{Q}_\pm, \tag{2.3}$$

$$[F_A, Q_\pm] = \mp Q_\pm, \quad [F_A, \overline{Q}_\pm] = \pm \overline{Q}_\pm. \tag{2.4}$$

2.2 A and B

Let us put

$$Q_A := \overline{Q}_+ + Q_-, \qquad Q_B := \overline{Q}_+ + \overline{Q}_-. \tag{2.5}$$

[1] There is a possible modification to (2.2) by central terms which we do not consider in these notes.

Then, $(Q, F) = (Q_A, F_A)$ or (Q_B, F_V) obey

$$Q^2 = 0, \qquad [F, Q] = Q. \tag{2.6}$$

This means that the space of states forms a complex with differential Q and grading F.[1] The same applies also for the space of local operators. In particular, cohomology classes of local operators form a ring called the *chiral ring*, which we denote by $\mathcal{R}_A$ for (Q_A, F_A) and $\mathcal{R}_B$ for (Q_B, F_V). It is a graded commutative algebra.

When formulated on a half of the Minkowski space, say, where the space coordinate is bounded as $x \leq 0$ and the time t is unbounded, a boundary condition on the fields must be specified at $x = 0$. There are essentially two types of boundary conditions that preserve maximal number of supercharges:

$$\begin{aligned} &\textit{A-type}: \qquad Q_A \text{ and } Q_A^\dagger \text{ conserved.} \\ &\textit{B-type}: \qquad Q_B \text{ and } Q_B^\dagger \text{ conserved.} \end{aligned}$$

Boundary conditions of such types are called *A-branes* and *B-branes* respectively. The pair $(Q, F) = (Q_A, F_A)$ (*resp.* (Q_B, F_V)) acts on local operators inserted on the boundary with an A-type (*resp.* B-type) boundary condition, obeying the same relation as (2.6). The cohomology classes form an algebra, which is non-commutative in general. Two differnt boundary conditions of the same type can be placed on the boundary with a local operator inserted inbetween. The pair (Q, F) acts also on such local operators obeying (2.6),[2] and we may consider the cohomology classes. Then, we have a *category*, which is denoted by $\mathcal{C}_A$ for A-branes and $\mathcal{C}_B$ for B-branes. Objects are boundary conditions and morphisms between boundary conditions are Q-cohomology classes of local operators inserted between them, with the composition represented by the product of operators.

2.3 RG flow

In a general QFT, the behaviour of observables depends very much on the energy scale, or inversely, the distance scale. For example, the correlation function of two operators inserted as two points of distance r is in general a complicated function of r. The behaviour at small r i.e. short distance, or equivalently, high energy or ultra-violet (UV), is in general very much different from the behaviour at large r i.e. long distance, or equivalently, low energy or infra-red (IR). If the length scale is increased (i.e. the energy scale is lowered),

[1]In this subsection, we make statements assuming that F_A or F_V is present, but that is not necessary. We can use the $\mathbb{Z}_2$-grading instead.

[2]There is a potential anomaly to the relations [1].

the behaviour of the theory changes — it may be identified as the behaviour of a different theory before the change of the scale. This change of the theory under the change of the scale is called the *renormalization group flow*, or *RG flow* in short. A theory is scale invariant if it is invariant under the RG flow. The two point correlation function of an operator $\mathcal{O}$ in such a theory depends on the distance as its power,

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{\mathrm{dist}(x,y)^{2\Delta_{\mathcal{O}}}}. \tag{2.7}$$

The number $\Delta_{\mathcal{O}}$ is called the *dimension* of the operator.

In two-dimensions, scale invariance of a QFT is proven to be equivalent to conformal invariance. For a general QFT, we have conformally invariant field theories (CFTs) in the UV and IR limits. An invariant of a CFT is its central charge c, and it is known that it descreases under the RG flow, $c_{\mathrm{UV}} \geq c_{\mathrm{IR}}$.

The same applies of course to QFTs with supersymmetry. In $2d$ $(2,2)$ supersymmetric QFTs, there are a class of observables which are invariant under the RG flow, even if the theory is not scale invariant. The chiral ring and the category of branes, which are introduced above, are examples of such observables which are "protected" from renormalization. In a $(2,2)$ superconformal field theory (SCFT), it is convenient to use $\widehat{c} = c/3$ for the central charge.

2.4 Deformations

A QFT can be deformed by adding a local operator $\mathcal{O}$ to its Lagrangian density. If the theory is scale invariant, the deformation is called *irrelevant*, *marginal* and *relevant* if the dimension of the operator minus the dimension of the spacetime, which is $\Delta_{\mathcal{O}} - 2$ in a 2d theory, is positive, zero and negative, respectively. Under the RG flow, an irrelevant deformation decays and a relevant deformation grows. A marginal deformation is called *exactly marginal* if it remains invariant under the RG flow, while it is *marginally irrelevant* (*resp. marginally relevant*) if it decays (*resp.* grows) under the RG. The moduli space of conformal field theories is coordinatized by exactly marginal deformation parameters. It has a natural metric G called the Zamolodchikov metric [2]: Let v_1 and v_2 be tangent vectors at one theory that correspond to exactly marginal operators $\mathcal{O}_1$ and $\mathcal{O}_2$. Then, their inner product $G(v_1, v_2)$ is provided (for 2d) by

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle \;=\; \frac{G(v_1,v_2)}{\mathrm{dist}(x,y)^4}. \tag{2.8}$$

In a 2d (2,2) supersymmetric QFT, deformation operators that preserve the supersymmetry are of the following three types,

$$\Delta_D \mathcal{L} = Q_+ Q_- \overline{Q}_- \overline{Q}_+ \mathcal{K}, \tag{2.9}$$

$$\Delta_A \mathcal{L} = Q_+ \overline{Q}_- \mathcal{O}_A \text{ and its adjoint}, \tag{2.10}$$

$$\Delta_B \mathcal{L} = Q_+ Q_- \mathcal{O}_B \text{ and its adjoint}, \tag{2.11}$$

for scalar operators $\mathcal{K}$, $\mathcal{O}_A$ and $\mathcal{O}_B$, where $\mathcal{K}$ is arbitrary, $\mathcal{O}_A$ is *A-chiral*, $\overline{Q}_+ \mathcal{O}_A = Q_- \mathcal{O}_A = 0$, and $\mathcal{O}_B$ is *B-chiral*, $\overline{Q}_\pm \mathcal{O}_B = 0$. We shall call them *D-term*, *A-term*, and *B-term*, respectively.[1] We see from (2.1)-(2.2) that a D-term is an A-term and a B-term at the same time, up to total derivatives. It turns out that A-term deformations modulo D-term deformations are in one to one correspondence with elements of the chiral ring $\mathcal{R}_A$. Similarly, B-term deformations modulo D-term deformations are in one to one correspondence with elements of the chiral ring $\mathcal{R}_B$.

It follows from the algebra (2.1)-(2.2) that D-terms and A-terms are Q_B-exact while D-terms and B-terms are Q_A-exact, up to total derivatives. In particular, the ring $\mathcal{R}_B$ and the category $\mathcal{C}_B$ are invariant under D-term and A-term deformations, while $\mathcal{R}_A$ and the category $\mathcal{C}_A$ are invariant under D-term and B-term deformations.

In a (2,2) SCFT, the D-term deformations are irrelevant, since each supercharge carry dimension $\frac{1}{2}$. Only a part of A-term and B-term deformations are marginal or relevant.

The spaces of parameters of the theory corresponding to A-term deformations and B-term deformations are denoted by $\mathfrak{M}_A$ and $\mathfrak{M}_B$. They have complex structures: tangent vectors of $\mathfrak{M}_A$ (*resp.* $\mathfrak{M}_B$) of type (1,0) correspond to operators of the form $Q_+ \overline{Q}_- \mathcal{O}_A$ (*resp.* $Q_+ Q_- \mathcal{O}_B$) and can naturally be identified as elements of the chiral ring $\mathcal{R}_A$ (*resp.* $\mathcal{R}_B$). For a (2,2) SCFT, the subspaces of exacly marginal parameters, $\mathfrak{M}_A^0 \subset \mathfrak{M}_A$ and $\mathfrak{M}_B^0 \subset \mathfrak{M}_B$, are complex submanifolds. They are also submanifolds of the moduli space of conformal field theories. The Zamolodchikov metric induced on $\mathfrak{M}_A^0$ and $\mathfrak{M}_B^0$ are known to be Kähler [3].

2.5 Topological Twists

The theory can be formulated not just on the (half of) Minkowski space. For example, we can consider the cylinder or the strip, again with the Minkowski metric, which yield closed or open string states. We may also formulate the theory on these manifolds with Euclidean metric, by Wick rotation of the time line. Furthermore, we may formulate the

[1]The standard terminology is: *twisted F-term* instead of A-term, and *F-term* intead of B-term.

system on a two-dimensional manifold with an arbitrary metric and spin structure, via the standard covariantization. Does the supersymmetry survive? We can certainly extend the definition of supersymmetry transformation of fields $\mathcal{O}$

$$\delta\mathcal{O} = i\epsilon_+ Q_- \mathcal{O} - i\epsilon_- Q_+ \mathcal{O} - i\bar{\epsilon}_+ \overline{Q}_- \mathcal{O} + i\bar{\epsilon}_- \overline{Q}_+ \mathcal{O}. \tag{2.12}$$

by covariantization of the expressions $Q_\pm\mathcal{O}$ and $\overline{Q}_\pm\mathcal{O}$, and by taking the variational parameters $\epsilon_\pm$ and $\bar{\epsilon}_\pm$ to be sections of the spin bundles $S_\pm$. However, invariance of the covariantized action requires the variational parameters to be covariantly constant, which is impossible on a curved manifold. There are several ways to restore a part of the supersymmetry. One is the *topological twisting* which we now describe.

Let us assume that $F = F_V$ or F_A is conserved and has *charge integrality*, that is, it generates a $U(1)$ symmetry group under which the non-spinorial and spinorial fields have even and odd charges respectively. The topological twisting is to replace a field of R-charge q with values in a vector bundle E by a field with values in $E \otimes T_\Sigma^{\otimes q/2}$, when we consider the theory on an oriented Riemannian manifold Σ. Here, T_Σ is the holomorphic tangent bundle equipped with the Levi-Civita connection. Note that, due to the constraint on the parity of the R-charge, $E \otimes T_\Sigma^{\otimes q/2}$ makes sense without choice of spin structure of Σ. The same change occurs also for the variational parameters $\epsilon_\pm$ and $\bar{\epsilon}_\pm$, and some of them become scalars. We can take such a parameter to be constant, and the corresponding supercharge is conserved. It is called the *A-twist* for $F = F_V$ and *B-twist* for $F = F_A$. In the A-twisted (*resp.* B-twisted) theory, the supercharges $\overline{Q}_+$ and Q_- and hence their sum $Q = Q_A$ (*resp.* $\overline{Q}_\pm$ and their sum $Q = Q_B$) are conserved. Forthermore, the correlation functions of Q-closed operators depend only on the Q-cohomology classes of the operators and are invariant under deformation of the Riemannian metric on Σ. In particular, they depend only on the topology of Σ — we obtain a topological field theory.

We can further consider *topological string theory* by coupling the twisted theory to a certain theory of 2d gravity called *topological gravity*. A g-loop amplitude is obtained by integration over the moduli space of curves of genus g.

2.6 Examples

Non-linear sigma model

Let $X = (X, g)$ be a compact Kähler manifold. Then, there is a 2d (2,2) supersymmetric QFT called the *non-linear sigma model* with target X. As a part of the data, we also choose a class $[B] \in \mathrm{H}^2(X, \mathbb{R}/2\pi\mathbb{Z})$ called the *B-field*.

The field variables on a surface Σ consist of a map $\phi : \Sigma \to X$ and a Dirac fermion ψ with values in $\phi^* TX$. On the Minkowski spacetime ($\Sigma = \mathbb{R}^2$ with time and space coordinates x^0 and x^1), the Lagrangian is[1]

$$\begin{aligned}\mathcal{L} &= g_{i\bar{j}}\left(\partial_0\phi^i\partial_0\overline{\phi}^{\bar{j}} - \partial_1\phi^i\partial_1\overline{\phi}^{\bar{j}} + i\overline{\psi}^{\bar{j}}_-(D_0+D_1)\psi^i_- + i\overline{\psi}^{\bar{j}}_+(D_0-D_1)\psi^i_+\right)\\ &\quad + R_{i\bar{j}k\bar{l}}\psi^i_+\psi^k_-\overline{\psi}^{\bar{j}}_-\overline{\psi}^{\bar{l}}_+ + 2\pi(\phi^*B)_{10}. \end{aligned} \tag{2.14}$$

Here, we have chosen local holomorphic coordinates z^i of X, and ϕ^i etc are the components $\phi^i = z^i \circ \phi$ etc. $D_\mu\psi^i_\pm = \partial_\mu\psi^i_\pm + \partial_\mu\phi^j\Gamma^i_{jk}\psi^k_\pm$ is the covariant derivative with respect to the pull back of the Levi-Civita connection on TX, and $R_{i\bar{j}k\bar{l}} = -g_{\bar{l}m}(R_{i\bar{j}})^m{}_k$ is the Riemann curvature of TX. The Lagrangian is invariant under the supersymetry transformation $\delta = i\epsilon_+Q_- - i\epsilon_-Q_+ - i\overline{\epsilon}_+\overline{Q}_- + i\overline{\epsilon}_-\overline{Q}_+$:

$$\delta\phi^i = \epsilon_+\psi^i_- - \epsilon_-\psi^i_+, \tag{2.15}$$

$$\delta\psi^i_\pm = \pm i\overline{\epsilon}_\mp(\partial_0 \pm \partial_1)\phi^i + \epsilon_\pm\Gamma^i_{jk}\psi^j_+\psi^k_-. \tag{2.16}$$

The transformation of the complex conjugate variables are obtained from these by the rule $(AB)^\dagger = B^\dagger A^\dagger$. For example, $\delta\overline{\phi}^{\bar{\imath}} = \overline{\psi}^{\bar{\imath}}_-\overline{\epsilon}_+ - \overline{\psi}^{\bar{\imath}}_+\overline{\epsilon}_- = -\overline{\epsilon}_+\overline{\psi}^{\bar{\imath}} + \overline{\epsilon}_-\overline{\psi}^{\bar{\imath}}_+$.

The model classically has both vector and axial $U(1)$ R-symmetries with charge integrality, but the axial R-symmetry is anomalous if the first Chern class $c_1(X)$ is non-zero: the axial rotation shifts $[B]$ by $c_1(X)$. The model is classically scale invariant, but the target metric changes under the RG flow. The Kähler class[2] runs according to the Ricci flow: $[\omega] \to [\omega'] = [\omega] + c_1(X)\log(\mu'/\mu)$ for the change $\mu \to \mu'$ of the energy scale. When $c_1(X) = 0$, the theory flows in the infra-red limit to an SCFT with central $\widehat{c} = \dim_{\mathbb{C}} X$. The chiral ring etc of the model are

$$\begin{aligned}\mathcal{R}_A &= \mathrm{QH}^*(X) \quad \text{quantum cohomology ring,}\\ \mathcal{R}_B &= \mathrm{H}^*(X, \wedge^* T_X) \quad \text{cohomology ring of polyvector fields,}\\ \mathcal{C}_A &= \mathrm{Fuk}(X) \quad \text{Fukaya category,}\\ \mathcal{C}_B &= \mathrm{D}^b_{Coh}(X) \quad \text{derived category of sheaves with coherent cohomologies}\end{aligned}$$

[1]In the present notes, we take the convention that the action is the integral of Lagragian density *divided by* 2π:

$$S = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2x\,\mathcal{L}. \tag{2.13}$$

The B-field enters into the Euclidean action as $-i\int_\Sigma \phi^*B$, or equivalently, it enters into the path-integral weight as $\exp\left(i\int_\Sigma \phi^*B\right)$. This explains the periodicity of the B-field.

[2]We normalize the Kähler class as $\omega = \frac{i}{2\pi}g_{i\bar{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^{\bar{j}}$, so that the path-integral weight for a holomorphic map $\phi : \Sigma \to X$ is $\exp\left(-\int_\Sigma \phi^*(\omega - iB)\right)$.

$$
\begin{aligned}
\mathfrak{M}_A^{0,c} &= \text{the space of complexified Kähler class } [\omega - iB] \in \mathrm{H}^2(X, \mathbb{R}/2\pi i\mathbb{Z}),\\
\mathfrak{M}_B^{0,c} &= \text{the moduli space of complex structures of } X.
\end{aligned}
$$

$\mathfrak{M}_{A/B}^{0,c}$ is a submanifold of $\mathfrak{M}_{A/B}$ that corresponds to the marginal deformations of the classical system. When $c_1(X) = 0$, the space $\mathfrak{M}_A^{0,c} \times \mathfrak{M}_B^{0,c}$ is identified as an open subset of the moduli space of the IR SCFTs. When $c_1(X) \neq 0$, there is an RG low on $\mathfrak{M}_A^{0,c}$ in the direction of $c_1(X)$, and the shift in the direction of $ic_1(X)$ is absorbed by the axial rotaion.

As the model has vector $U(1)$ R-symmetry with charge integrality, A-twist is always possible. The corresponding topological string theory is known as the *Gromov-Witten theory* in mathematics. B-twist is possible if and only if X is a Calabi-Yau manifold, that is, the canonical bundle K_X is trivial as a holomorphic line bundle.

Landau-Ginzburg model

Let $W(x)$ be a polynomial of N variables $x = (x^1, \ldots, x^N)$ with complex coefficients, having only isolated critical points. Then, there is a 2d (2,2) supersymmetric QFT called the *Landau-Ginzburg model* with superpotential $W(x)$.

The field variables consist of complex scalar fields $x^1, \ldots, x^N$ and Dirac fermions $\psi^1, \ldots, \psi^N$. On the Minkowski spacetime, the Lagrangian is

$$
\begin{aligned}
\mathcal{L} &= \sum_{i=1}^N \Big(|\partial_0 x^i|^2 - |\partial_1 x^i|^2 + i\overline{\psi_-^i}(\partial_0 + \partial_1)\psi_-^i + i\overline{\psi_+^i}(\partial_0 - \partial_1)\psi_+^i \Big)\\
&\quad - \frac{1}{4}\sum_{i=1}^N |\partial_i W(x)|^2 - \frac{1}{2}\sum_{i,j=1}^N \left(\psi_+^i \psi_-^j \partial_i \partial_j W(x) + \text{c.c.} \right), \qquad (2.17)
\end{aligned}
$$

and the supersymmetry transformation is

$$
\delta x^i = \epsilon_+ \psi_-^i - \epsilon_- \psi_+^i, \qquad (2.18)
$$

$$
\delta \psi_\pm^i = \pm i\overline{\epsilon}_\mp (\partial_0 \pm \partial_1) x^i - \frac{1}{2}\epsilon_\pm \overline{\partial_i W(x)}. \qquad (2.19)
$$

The model always have an axial $U(1)$ R-symmetry with charge integrality. A vector R-symmetry exists if and only if $W(x)$ is quasi-homogeneous, that is, with a change of coordinates if necessary, there are some numbers $R = (R_1, \ldots, R_N)$ such that $W(\lambda^R x) = \lambda^2 W(x)$, where $\lambda^R x = (\lambda^{R_1} x_1, \ldots, \lambda^{R_N} x_N)$. In that case, the theory flows in the infra-red limit to an SCFT with $\widehat{c} = \mathrm{tr}(1 - R)$. The chiral ring etc of the model are

$$
\mathcal{R}_A \;=\; ?,
$$

$$
\begin{aligned}
\mathcal{R}_B &= \mathrm{Jac}(W) \quad \text{Jocobi ring } \mathbb{C}[x_1, \ldots, x_N]/(\partial_1 W, \ldots, \partial_N W), \\
\mathcal{C}_A &= \mathrm{Fuk}(W) \quad \text{Fukaya category controled by } W, \\
\mathcal{C}_B &= \mathrm{MF}(W) \quad \text{category of matrix factorizations of } W \\
\mathfrak{M}_A &= ?, \\
\mathfrak{M}_B &= \text{the moduli space of versal deformations of } W,
\end{aligned}
$$

The lecturer does not know what $\mathcal{R}_A$ and $\mathfrak{M}_A$ are at this moment.

As the model has axial $U(1)$ R-symmetry with charge integrality, B-twist is always possible. The corresponding topological string theory at the tree level (genus zero) is closely related to K. Saito's theory of primitive forms. When W is a Morse function, A. Givental proposed a recipe to construct the higher genus amplitudes, and C. Teleman proved that they satisfy a mathematical axiom of topological string theory.

When W is quasi-homogeneous, there is also a vector R-symmetry. However, it does not possess the charge integrality and the A-twist is not possible. That problem may be cured by orbifolding. Gauge the system by a finite group $\Gamma \subset GL(N, \mathbb{C})$ of symmetries of $W(x)$ that include $\mathrm{e}^{\pi i R}$ as its element. Then, the charge integrality holds for gauge invariant fields, and the A-twist becomes possible. The corresponding topological string theory is called the *FJRW theory*.

2.7 Mirror Symmetry

The 2d (2,2) supersymmetry algebra has an automorphism: $Q_- \leftrightarrow \overline{Q}_-$, $F_V \leftrightarrow F_A$, and the other generators kept intact. A pair of 2d (2,2) supersymmetric QFTs are said to be *mirror* to each other when there is an isomorphism between them under which the supersymmetry generators undergo the above automorphism. There are immediate consequences of the mirror symmetry: the ring $\mathcal{R}_A$ of one theory is isomorphic to the ring $\mathcal{R}_B$ of the mirror, the category $\mathcal{C}_A$ of one theory is equivalent to the category $\mathcal{C}_B$ of the mirror, the parameter space $\mathfrak{M}_A$ of one theory is isomorphic to the parameter space $\mathfrak{M}_B$ of the mirror, and the topological A-model (the A-twisted model or the corresponding topological string theory) of one theory is isomorphic to the topological B-model of the other.

The most famous example of mirror symmetry is the one for the sigma models with Calabi-Yau targets, say X and Y. As a part of the above consequences, we have the relation between the Hodge numbers, $h^{p,q}(X) = h^{n-p,q}(Y)$, where $n = \dim X = \dim Y$. Other well known examples are mirror symmetry between the sigma model with a non-Calabi-

Yau target and the Landau-Ginzburg model, and the one between Landau-Ginzburg orbifolds. Some of the consequences in these examples are mathematically proven.

3 T-duality

[4, 5]

3.1 T-duality

Let us consider the two-dimensional non-linear sigma model whose target space is the circle S^1_R of circumference $2\pi R$. The field variable is the map $X : \Sigma \to \mathbb{R}/2\pi R\mathbb{Z}$ and the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_0 X)^2 - \frac{1}{2}(\partial_1 X)^2. \tag{3.1}$$

When formulated on the cylinder $\Sigma = \mathbb{R} \times S^1$ where the spatial coordinate is periodic, $x^1 \equiv x^1 + 2\pi$, there are two conserved quantities. One is the target momentum associated with the translational symmetry $X \to X + \Delta X$, and the other is the winding number of the map X from the spatial circle to the target circle S^1_R. They are measured respectively by

$$p = \frac{1}{2\pi}\int_{S^1} \partial_0 X \, \mathrm{d}x^1, \qquad w = \frac{1}{2\pi}\int_{S^1} \partial_1 X \, \mathrm{d}x^1. \tag{3.2}$$

Note that w has quantized value Rm for some integer m. In the quantum theory, the momentum is also quantized as $p = l/R$ for some integer l, as one can see from the fact that the translation operator $\exp(ip\Delta X)$ must be trivial for $\Delta X = 2\pi R$. The energy E and (worldsheet) momentum P of the states in the sector with a given l and m are

$$E = \frac{1}{2}\left(\left(\frac{l}{R}\right)^2 + (Rm)^2\right) + E_{\rm osc}, \qquad P = lm + P_{\rm osc}, \tag{3.3}$$

where $E_{\rm osc}$ and $P_{\rm osc}$ are the contributions from the oscillation and are independent of l, m and R.

We see that the energy-momentum spectrum is invariant under

$$R \longrightarrow \frac{1}{R}, \tag{3.4}$$

provided l and m are exchanged. In other words, the model with target S^1_R has the same spectrum as the model with target $S^1_{1/R}$. It turns out that the two theories are equivalent. This is the *T-duality*. Let $\widetilde{X}$ be the field for the “dual model” with target $S^1_{1/R}$, and let us define $\widetilde{p}$ and $\widetilde{w}$ in the same way as (3.2). Then, the momentum of the original model corresponds to the winding number of the dual, $p \leftrightarrow \widetilde{w}$, and the winding number of the original model corresponds to the momentum of the dual, $w \leftrightarrow \widetilde{p}$. In other words,

$$\partial_0 X \;\longleftrightarrow\; \partial_1 \widetilde{X}, \qquad \partial_1 X \;\longleftrightarrow\; \partial_0 \widetilde{X}. \tag{3.5}$$

T-duality can be derived from the following path integral argument [6]. Let us consider a theory with a map $X : \Sigma \to S^1_R$ and a one-form $J \in \Omega^1(\Sigma; \mathbb{R})$ as the variable and the Lagrangian

$$\mathcal{L}' \;=\; \frac{1}{2}\left(J_0^2 - J_1^2\right) + J_1 \partial_0 X - J_0 \partial_1 X. \tag{3.6}$$

If we integrate out the J-field first, which is done by completing the square with respect to J and performing the Gaussian integral; or equivalently, by solving the Euler-Lagrange equation for J, which gives $J_0 = \partial_1 X$ and $J_1 = \partial_0 X$, and inserting the answer back to the Lagrangian, we obtain the Lagrangian $\mathcal{L}$ in (3.1) of the original system. On the other hand, if we integrate out X first, we obtain the constraint $J_\mu = \partial_\mu \widetilde{X}$ for a map $\widetilde{X} : \Sigma \to S^1_{1/R}$.[1] Inserting the result to the remaining terms in $\mathcal{L}'$, we obtain

$$\widetilde{\mathcal{L}} = \frac{1}{2}(\partial_0 \widetilde{X})^2 - \frac{1}{2}(\partial_1 \widetilde{X})^2. \tag{3.7}$$

This the the Lagrangian for the dual theory. Comparing the two expressions for the J-field, we obatin the relation (3.5). This derivation has the advantage that it can be used to derive T-duality rule in more complicated situations, as we will encounter repeatedly.

<u>T-duality on D-branes</u>

It is instructive to see how the boundary conditions, or D-branes, transform under T-duality. Let us formulate the sigma model with target S^1_R on the left-half Minkowski spacetime $\Sigma = \mathbb{R} \times (-\infty, 0]$. As the typical boundary conditions, we may consider Neumann boundary condition (N) and the Dirichlet boundary condition (D):

$$(\mathrm{N}): \;\partial_1 X|_{\partial\Sigma} = 0, \qquad (\mathrm{D}): \;\partial_0 X|_{\partial\Sigma} = 0. \tag{3.8}$$

In view of the relation (3.5), we see that the Neumann boundary condition for X corresponds to the Dirichlet boundary condition for $\widetilde{X}$, and vice versa. To be precise, there is

[1] The logic for this, in particular, that this $\widetilde{X}$ has peridicity $2\pi/R$ is non-trivial and is interesting. It roughly goes as follows. Integral over the continuous part of X yields the constraint that J is a closed one-form. The path-integral over X also includes the sum over the winding numbers of X along closed one-cycles of Σ, and this yields the constraint that the period of J along closed one-cycles are $2\pi/R$ times integers. This means that $J = \mathrm{d}\widetilde{X}$ for a map $\widetilde{X} : \Sigma \to S^1_{1/R}$.

a one parameter family of boundary conditions, for both (N) and (D). (D) means that the boundary value of X is fixed, and hence is parametrized by that fixed position. That is, (D) is parametrized by $[x] \in S^1_R$. We shall call this boundary condition "the D0-brane at $[x]$". On the other hand, under (N), we may add $S_{\mathrm{bdry}} = \int_{\partial\Sigma} a\, \mathrm{d}X = \int_{\partial\Sigma} a\partial_0 X \mathrm{d}x^0$ to the action. It means that the "boundary particle", that is, the particle in S^1_R with trajectory $X|_{\partial\Sigma}$, is charged under the $U(1)$ gauge potential $A = a\, \mathrm{d}x$. Since the invariant of a $U(1)$ connection on a circle is its holonomy, which is $\mathrm{e}^{ia\cdot 2\pi R}$, we see that a is equivalent to $a + 1/R$. That is, (N) is parametrized by $[2\pi a] \in S^1_{1/R}$. We shall call this boundary condition "the D1-brane wrapped on S^1_R and supporting the $U(1)$ gauge potential with holonomy $\mathrm{e}^{ia\cdot 2\pi R}$". Then, the precise statement of T-duality is: *the D1-brane wrapped on the cricle S^1_R corresponds to the D0-brane in $S^1_{1/R}$, and the holonomy of the former corresponds to the position of the latter.* This can be proved by looking at the spectrum, or, via the path-integral argument as above.

One important lesson is that the dual manifold $S^1_{1/R}$ can be realized as the moduli space of D1-branes wrapped on S^1_R in the original model. From this emerges Strominger-Yau-Zaslow's picture of mirror symmetry.

Abelian Duality

T-duality is an example of the so called the "Abelian duality" that generalizes the electric-magnetic duality in the free Maxwell theory in four dimension. In each dimension D, there is a free field theory $\mathcal{F}^k_D(e)$ of a k-form gauge potential A whose action on a D-manifold M is $S = \frac{1}{4\pi e^2}\int_M \mathrm{d}A \wedge *\mathrm{d}A$. To be precise, A is a collection of k-forms defined locally in such a way that $\mathrm{d}A$ is a globally defined closed $(k+1)$-form whose cohomology class takes values in $\mathrm{H}^{k+1}(M, 2\pi\mathbb{Z})$. Abelian duality states that $\mathcal{F}^k_D(e)$ is equivalent to $\mathcal{F}^{D-k-2}_D(1/e)$. This can be shown by the path-integral argument as in T-duality, where J is a $(k+1)$-form variable. T-duality is $\mathcal{F}^0_2(1/R) \cong \mathcal{F}^0_2(R)$ and the electric-magnetic duality is $\mathcal{F}^1_4(e) \cong \mathcal{F}^1_4(1/e)$.

Generalizations

We may consider the product of circles, say, $S^1_{R_1} \times S^1_{R_2}$, as the target space of the sigma model and apply T-duality to various circle factors. For example, applying it on the first, second and both circles, we obtain the equaivalence of the original model and the sigma models with targets $S^1_{1/R_1} \times S^1_{R_2}$, $S^1_{R_1} \times S^1_{1/R_2}$ and $S^1_{1/R_1} \times S^1_{1/R_2}$. We may also consider the product of the circle sigma model and arbitrary QFT, and apply T-duality on the circle.

For example, for any Riemannian manifold $\mathcal{B}$, the sigma models with targets $\mathcal{B}\times S^1_R$ and $\mathcal{B}\times S^1_{1/R}$ are equivalent. More generally, let us consider the sigma model whose target space is topologically the product $\mathcal{B}\times T^n$ of a space $\mathcal{B}$ and the n-torus $T^n=(S^1)^n$, but with the metric and the B-field having mixed components. The path-integral argument can be applied also in such a situation, as along as the metric and the B-field is invariant under translations in the torus directions. Let x^i be the coordinates of T^n with periodicity 2π and let x^a be arbitrary local coordinates of $\mathcal{B}$. Let us put $E=g+2\pi B$, and let E^{ij} be the inverse matrix to $E_{ij}=g_{ij}+2\pi B_{ij}$. Let $\widetilde{x}_i$ be the coordinates of the dual torus $\widetilde{T}^n$ with periodicity 2π. Then, the T-dual theory is the sigma model whose target space is $\mathcal{B}\times\widetilde{T}^n$ with $\widetilde{E}=\widetilde{g}+2\pi\widetilde{B}$ given by

$$\widetilde{E}^{ij}=E^{ij},\qquad \widetilde{E}^{i}{}_{b}=E^{ij}E_{jb},\qquad \widetilde{E}_{a}{}^{j}=-E_{ai}E^{ij},\qquad \widetilde{E}_{ab}=E_{ab}-E_{ai}E^{ij}E_{jb}. \tag{3.9}$$

3.2 T-duality as Mirror Symmetry

Let us consider the two-dimensional $(2,2)$ supersymmetric non-linear sigma model whose target space is the cylinder $\mathbb{R}\times S^1_R$, which is a flat Kähler manifold with metric $\mathrm{d}s^2=|\mathrm{d}z|^2$ where z is a complex coordinate with periodicity $z\equiv z+2\pi Ri$. The field variables are the map $\phi:\Sigma\to\mathbb{C}/2\pi Ri\mathbb{Z}$ and a Dirac fermion ψ, and the Lagrangian is

$$\mathcal{L}=\frac{1}{2}|\partial_0\phi|^2-\frac{1}{2}|\partial_1\phi|^2+\frac{i}{2}\overline{\psi}_-(\partial_0+\partial_1)\psi_-+\frac{i}{2}\overline{\psi}_+(\partial_0-\partial_1)\psi_+. \tag{3.10}$$

This system is the product of the sigma model with target S^1_R and another system consisting of the real part of ϕ (an ordinary scalar field) and the Dirac fermion ψ. Applying T-duality to S^1_R, we obtain the product of the sigma model with target $S^1_{1/R}$ and the same system of $\mathrm{Re}(\phi)$ and ψ. In other words, the product of the sigma model with target $\mathbb{R}\times S^1_{1/R}$ and a Dirac fermion. The dual system, being equivalent to the original, must also have $(2,2)$ supersymmetry. What is the supersymmetry transformation of the field variables?

We recall the supersymmetry tranformation in the original system from (2.15) and (2.16):

$$\delta\phi=\epsilon_+\psi_--\epsilon_-\psi_+, \tag{3.11}$$

$$\delta\psi_\pm=\pm i\overline{\epsilon}_\mp(\partial_0\pm\partial_1)\phi. \tag{3.12}$$

Note that ϕ is B-chiral,

$$\overline{Q}_+\phi=\overline{Q}_-\phi=0. \tag{3.13}$$

Let us write $\phi = \mathrm{Re}(\phi) + iX$, where X is a map to $S^1_R = \mathbb{R}/2\pi R\mathbb{Z}$. Denoting the T-dual variable of X as $\widetilde{X}$ as before, we may write the dual complex variable as $\widetilde{\phi} = \mathrm{Re}(\phi) + i\widetilde{X}$. Recalling the relation (3.5), we see that ϕ and $\widetilde{\phi}$ are related by

$$(\partial_0 + \partial_1)\phi = (\partial_0 + \partial_1)\widetilde{\phi}, \qquad (\partial_0 - \partial_1)\overline{\phi} = (\partial_0 - \partial_1)\widetilde{\phi}. \tag{3.14}$$

Applying (3.11) and employing the equation of motion $(\partial_0 \pm \partial_1)\psi_\mp = 0$, we find $\delta(\partial_0 \pm \partial_1)\widetilde{\phi} = (\partial_0 \pm \partial_1)(-\overline{\epsilon}_+\overline{\psi}_- - \epsilon_-\psi_+)$. Integrating, we find the supersymmetry transformation in the dual system,

$$\delta\widetilde{\phi} = -\overline{\epsilon}_+\overline{\psi}_- - \epsilon_-\psi_+, \tag{3.15}$$

$$\delta\psi_+ = i\overline{\epsilon}_-(\partial_0 + \partial_1)\widetilde{\phi}, \quad \delta\overline{\psi}_- = i\epsilon_+(\partial_0 - \partial_1)\widetilde{\phi}. \tag{3.16}$$

We see that it does not take the standard form as (2.15) and (2.16) (or (3.11) and (3.12)). In particular, $\widetilde{\phi}$ is not B-chiral but A-chiral,

$$\overline{Q}_+\widetilde{\phi} = Q_-\widetilde{\phi} = 0. \tag{3.17}$$

However, if we rename the variational parameters as $-\overline{\epsilon}_+ \to \epsilon'_+$ and $\epsilon_- \to \epsilon'_-$ (and hence $-\epsilon_+ \to \overline{\epsilon}'_+$ and $\overline{\epsilon}_- \to \overline{\epsilon}'_-$), and the supercharges as

$$Q_+ \to Q'_+, \quad Q_- \to \overline{Q}'_-, \quad \overline{Q}_+ \to \overline{Q}'_+, \quad \overline{Q}_- \to Q'_-, \tag{3.18}$$

then the transformation by $\delta = i\epsilon'_+Q'_- - i\epsilon'_-Q'_+ - i\overline{\epsilon}'_+\overline{Q}'_- + i\overline{\epsilon}'_-\overline{Q}'_+$ takes the standard form. In other words, there is an equivalence between the $(2,2)$ supersymmetric sigma models with targets $\mathbb{R} \times S^1_R$ and $\mathbb{R} \times S^1_{1/R}$ under which the supercharges are mapped as (3.18). That is, they are mirror to each other.

Let us see how D-branes transform under this T-duality. First, let us consider the D1-brane wrapped on the circle S^1_R at a fixed position in $\mathrm{Re}(\phi)$, supporting a flat $U(1)$ gauge potential. This is a Lagrangian submanifold of the cylinder and hence is an A-brane. T-duality maps this to a D-brane at the same positon in $\mathrm{Re}(\phi)$ and at the position in the dual circle $S^1_{1/R}$ corresponding to the holonomy of the gauge potential. That is, it is a D0-brane at a point of the dual cylinder. This is a complex submanifold and hence is a B-brane. Next, let us consider the D1-brane, again an A-brane, at a fixed poition in the circle S^1_R but is extending in $\mathrm{Re}(\phi)$. T-duality maps this to a D2-brane extending both in $S^1_{1/R}$ and $\mathrm{Re}(\phi)$, where the holonomy in the $S^1_{1/R}$ corresponds to the position of the original D1-brane in S^1_R. This is a B-brane. We see that A-branes in the original model are mapped to B-branes in the dual. This is indeed a property of mirror symmetry.

The same holds also when $\mathrm{Re}(\phi)$ is a periodic variable as well — the $(2,2)$ supersymmetric sigma models with targets $S^1_{R'} \times S^1_R$ and $S^1_{R'} \times S^1_{1/R}$ are mirror to each other. We may also consider the sigma model on the two-torus $S^1 \times S^1$ with a general (flat) metric and a (flat) B-field. Let x^1 and x^2 be the coordinates of the first and the second circles, both with periodicity 2π. The general metric and the B-field can be written as

$$\mathrm{d}s^2 = R^2 \left|\mathrm{d}x^1 + \tau \mathrm{d}x^2\right|^2 \qquad B = B_{12}\mathrm{d}x^1 \wedge \mathrm{d}x^2, \tag{3.19}$$

where $\tau = \tau_1 + i\tau_2$ is in the upper half-plane, $\tau_2 > 0$. This τ determines the complex structure of the torus, while the complexified Kähler class may be parametrized as

$$\omega - iB = \frac{1}{2\pi i}\rho\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \quad \text{with} \quad \rho = 2\pi B_{12} + iR^2\tau_2. \tag{3.20}$$

Let us now apply T-duality on the x^1 circle, with x^2 kept intact. The metric and the B-field of the T-dual theory can be read from (3.9). Denoting the T-dual coordinate by $\widetilde{x}_1$, again with periodicity 2π, they are

$$\mathrm{d}\widetilde{s}^2 = \frac{1}{R^2}\left|\mathrm{d}\widetilde{x}_1 + \left(2\pi B_{12} + iR^2\tau_2\right)\mathrm{d}x^2\right|^2, \qquad \widetilde{B} = \frac{\tau_1}{2\pi}\mathrm{d}\widetilde{x}_1 \wedge \mathrm{d}x^2. \tag{3.21}$$

In particular, the τ and ρ parameters of the dual theory are

$$\widetilde{\tau} = 2\pi B_{12} + iR^2\tau_2 = \rho, \qquad \widetilde{\rho} = 2\pi\frac{\tau_1}{2\pi} + i\frac{1}{R^2}R^2\tau_2 = \tau. \tag{3.22}$$

The complex structure and the complexfied Kähler parameters are indeed exchanged between the dual models. This is another property of mirror symmetry.

This generalizes to higher dimensions. Let us consider the sigma model whose target space is $X = \mathcal{B} \times T^n$ with a metric g and a B-field B, possibly with mixed components. Adding fermionic variables, we can make it into a supersymmetric sigma model, and it has (2,2) supersymmetry provided that X admits a complex structure such that g is Kähler and that the B-field is flat, $\mathrm{d}B = 0$. The supersymmetric model has T-duality to $\widetilde{X} = \mathcal{B} \times \widetilde{T}^n$ with the metric $\widetilde{g}$ and the B-field $\widetilde{B}$ given by the same formula as in the bosonic case (3.9). Then, one can show that this T-duality is a mirror symmetry under the condition that T^n at each point of $\mathcal{B}$ is a Lagrangian submanifold of X on which the B-field vanishes, $B|_{T^n} = 0$. (In this case, we shall say that $X \to \mathcal{B}$ is a *Lagrangian torus fibration.*) Note that this is nothing but the condition for the Dn-brane wrapped on T^n at a point of $\mathcal{B}$ supporting a flat $U(1)$ gauge potential to be an A-brane. This is perfectly consistent with the fact that mirror symmetry exchanges A-branes and B-branes: As we have seen, such a Dn-brane is mapped under T-duality to the D0-brane at a point of $\widetilde{X}$. Since the latter is a B-brane, if the T-duality is a mirror symmetry, the original Dn-brane must be an A-brane. Note also that the dual space $\widetilde{X}$ can be reconstructed as the moduli space of Dn-brane wrapped on T^n in the original model.

Strominger-Yau-Zaslow [9] empolyed D-branes to argue that mirror symmetry between Calabi-Yau manifolds is a T-duality in the same sense. That is, each of the mirror pair has a structure of Lagragian torus fibration and the mirror symmetry is T-duality along fibers, so that the mirror manifold can be obtained as the moduli space of the D-brane wrapped on the torus fibers of the original manifold.[1] In general, a Calabi-Yau manifold does not admit a smooth torus fibration, and there will be quantum corrections to the rule like (3.9) which are significant near the singular fibers. The main problem is how to correctly evaluate the quantum corrections.

3.3 A Puzzle and a Solution

As the simplest example with a Lagrangian torus fibration with singular fibers, let us consider the case where the target space X is the complex projective line $\mathbb{CP}^1$ which has the topology of a two sphere. It has a structure of a circle fibration over the segment, as

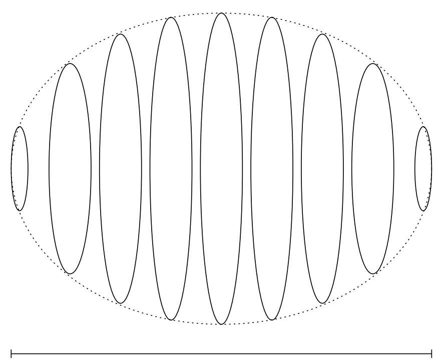

Figure 1: $\mathbb{CP}^1$ as circle fibration over a segment.

shown in Fig. 1. Note that $\mathbb{CP}^1$ is not a Calabi-Yau manifold and hence the set-up is not precisely the same as SYZ, but we can still ask the same question: what happens when we apply T-duality to circle fibers and what is the quantum correction from the singular fibers? The size of the circle is the largest over the mid point of the segment, decreases as you move away from the middle, and vanishes at the two ends. Since T-duality inverts the radius of the circle (3.4), in the dual side, the size of the circle is the smallest over the mid point of the segment, increases as you move away from the middle, and blows up at the two ends. Therefore, the dual space $\widetilde{X}$ should look like Fig. 2, having the topology of a cylinder. If this is taken seriously, then, the non-linear sigma model whose target space is $X = \mathbb{CP}^1$ must be dual to the non-linear sigma model with target space $\widetilde{X}$ as in Fig. 2.

[1] In [9], another important element — stability of D-branes — is also imposed, and the Calabi-Yau manifold is argued to have the structure of a *special* Lagrangian torus fibration.

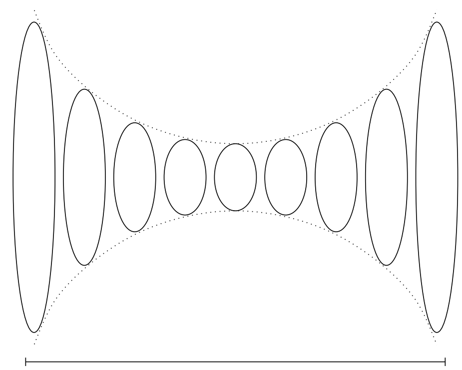

Figure 2: The naïve T-dual of the fibration.

Is this correct? As a test, let us compare the conserved quantities of the two theories. The original system has $U(1)$ symmetry that rotates the circle fibers, and hence there is an associated Noether charge — the momentum in the circle direction. In the dual side, the space $\widetilde{X}$ has two holes at the ends of the segment, and hence the winding number in the circle direction is conserved. This is consistent with the fact that T-duality maps the momentum of one theory to the winding number of the dual. On the other hand, there is no conserved winding number in the original system since S^2 is simply connected — a loop winding around the circle fiber can be shrunk by going to the either ends on the segment. In the dual side, there seems to be a $U(1)$ symmmetry that rotates the circle fibers, and hence the momentum seems to be conserved. Thus, the symmetries of the two theories do *not* match! There is another significant difference between the two theories: $\mathbb{CP}^1$ is compact whereas $\widetilde{X}$ is non-compact. When formulated on the cylinder $\Sigma = \mathbb{R} \times S^1$, this means that the energy spectrum of the $\mathbb{CP}^1$ sigma model is certainly discrete whereas the spectrum for the $\widetilde{X}$ sigma model is continuous. This cannot be the case when the two theories are equivalent. So, the naïve duality does not seem to hold.

In fact, there is a major quantum correction to this naïve T-duality that comes from the degeneration of the circle fiber. It is the generation of a superpotential. Let y be the complex coordinate of $\widetilde{X}$ with periodicty $y \equiv y + 2\pi i$, so that $\mathrm{Im}(y)$ parametrizes the circle fiber while $\mathrm{Re}(y)$ parametrizes the horizontal segment. It turns out that $\mathrm{Re}(y)$ can take values in the entire real line $\mathbb{R}$, where the left and the right ends of the segment corresponds to $-\infty$ and $+\infty$. Then, the dual theory has the superpotential [10]

$$W = \mathrm{e}^{-y} + \mathrm{e}^{y}. \tag{3.23}$$

The effect is indeed large near the dual of the singular fibers — the first and the second terms blow up at the left and the right ends respectively. The presence of this superpotential solves both of the two problems of the naïver T-duality. First, this superpotential is not invariant under the shift of $\mathrm{Im}(y)$, and hence the fiber momentum is not conserved in the dual theory. This corresponds to the non-conservation of the winding number in the $\mathbb{CP}^1$ sigma model. Second, the associated potential $U(y) = g^{y\overline{y}}|\partial_y W|^2$ is proper, that is, the subset $\{U(y) < E\}$ is compact for each E. This means that the spectrum of the theory formulated on $\Sigma = \mathbb{R} \times S^1$ is discrete, just as in the $\mathbb{CP}^1$ sigma model.

This leads us to claim that the mirror of the $\mathbb{CP}^1$ sigma model is the Landau-Ginzburg model with a cylinder target and superpotential (3.23). The superpotential generation will be derived in Section 5. As we will see there, this mirror construction generalizes to wider classes of target spaces including toric varieties and hyperplanes or complete intersections thereof in toric varieties. The main tool in the derivation is a class of supersymmetric gauge theories called the *gauged linear sigma models* to which we turn next.

4 Gauged Linear Sigma Models

In this section, we provide an introduction to a class of 2d (2,2) supersymmetric gauge theories called gauged linear sigma models (GLSMs) which was introduced in [11].

For a compact Lie group G, we write T and Z_G for a maximal torus and the center, respectively. We write the Weyl group of G by W. We write $\mathfrak{g} \supset \mathfrak{t} \supset \mathfrak{z}$ for the Lie algebras of $G \supset T \supset Z_G$ and regard them "pure imaginary". "Reals" are $i\mathfrak{g} \supset i\mathfrak{t} \supset i\mathfrak{z}$ in the complexfied Lie algebras $\mathfrak{g}_{\mathbb{C}} \supset \mathfrak{t}_{\mathbb{C}} \supset \mathfrak{z}_{\mathbb{C}}$. Thus, for a $\mathfrak{g}_{\mathbb{C}}$ valued quantity X, we write $X = \mathrm{Re}(X) + i\mathrm{Im}(X)$ where both $\mathrm{Re}(X)$ and $\mathrm{Im}(X)$ are $i\mathfrak{g}$ valued, and write $\overline{X} = \mathrm{Re}(X) - i\mathrm{Im}(X)$. The weight lattice of T is denoted by $\mathrm{P} \subset i\mathfrak{t}^*$.

4.1 The Data

A 2d (2,2) gauge theory is specified by a choice of

- *gauge group* G: a compact Lie group,
- *matter representation* V: a finite dimensional complex representation of G,
- *superpotential* $W(\phi)$: a G invariant polynomial function of $\phi \in V$, and
- *twisted superpotential* $\widetilde{W}(\sigma)$: a G invariant polynomial function of $\sigma \in \mathfrak{g}_{\mathbb{C}}$.

As a minor part of the data, we also choose a G-invariant inner product on $i\mathfrak{g}$ and a G-invariant hermitian inner product on V, both positive definite.

A vector $U(1)$ R-symmetry exists when there is a linear map $R : V \to V$ commuting with the G-action such that

$$W(\lambda^R \phi) = \lambda^2 W(\phi). \tag{4.1}$$

The charge integrality holds when $e^{\pi i R} : V \to V$ is the same as the action of a gauge group element, say $J \in G$. An axial $U(1)$ R-symmetry with charge integrality exists at the classical level when $\widetilde{W}(\sigma)$ is linear, and it remains to be a symmetry of the quantum system under *Calabi-Yau condition*: $G \subset SL(V)$. In the present notes, we assume all of the above but the Calabi-Yau condition. We write the linear twisted superpotential as

$$\widetilde{W}(\sigma) = -t(\sigma), \tag{4.2}$$

for an adjoint invariant linear form

$$t \ = \ \zeta - i\theta \in \mathfrak{g}_{\mathbb{C}}^{*G}, \tag{4.3}$$

where ζ and θ, both in $i\mathfrak{g}^{*G}$, are called the Fayet-Illiopoulos (FI) parameter and the theta parameter respectively. Note that ζ and θ can also be regarded as elements of $i\mathfrak{t}^{*\mathrm{W}}$ or $i\mathfrak{z}^*$ thanks to the natural isomorphisms $\mathfrak{g}^{*G} \cong \mathfrak{t}^{*\mathrm{W}} \cong \mathfrak{z}^*$. To be precise, the theta parameter is subject to a discrete identification,

$$\theta \ \equiv \ \theta + 2\pi n, \tag{4.4}$$

for an image n of a character $G \to U(1)$ under the differential map $\mathrm{Hom}(G, U(1)) \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{u}(1)) = i\mathfrak{g}^*$. Therefore, the space of theta parameter (or theta *angle*) is the compact torus $i\mathfrak{g}^{*G}/2\pi\Lambda_G$, where $\Lambda_G := \mathrm{Image}(\mathrm{Hom}(G, U(1)) \to i\mathfrak{g}^{*G})$. When G is connected, Λ_G is isomorphic to $\mathrm{Hom}(G, U(1))$ and is equal to the lattice P^{W} of Weyl invariant weights of T embedded in $i\mathfrak{g}^{*G}$ via $i\mathfrak{t}^{*\mathrm{W}} \cong i\mathfrak{g}^{*G}$.

4.2 Lagrangian

The theory consists of two sets of fields called a *matter multiplet* and a *gauge multiplet*. The gauge multiplet consists of a G connection v_μ, as well as a scalar σ, a Dirac fermion λ and a scalar D with values in $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}$ and $i\mathfrak{g}$. The matter multplet consists of a scalar ϕ, a Dirac fermion ψ and a scalar F, all with values in V. The gauge connection v_μ is "real-valued" so that the curvature is $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + i[v_\mu, v_\nu]$ and the gauge covariant derivative is $D_\mu \phi = \partial_\mu \phi + i v_\mu \phi$.

We describe the supersymmetry and the Lagrangian on the Minkowski space, with time and space coordinates x^0 and x^1 and the metric $\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2$. We often use the light-cone coordinates $x^\pm = x^0 \pm x^1$.

The supersymmetry transformation $\delta = i\epsilon_+ Q_- - i\epsilon_- Q_+ - i\overline{\epsilon}_+ \overline{Q}_- + i\overline{\epsilon}_- \overline{Q}_+$ is

$$\begin{aligned}
\delta v_\pm &= \frac{i}{2}\overline{\epsilon}_\pm \lambda_\pm + \frac{i}{2}\epsilon_\pm \overline{\lambda}_\pm, \\
\delta\sigma &= -i\overline{\epsilon}_+ \lambda_- - i\epsilon_- \overline{\lambda}_+, \\
\delta\lambda_\pm &= i\epsilon_\pm \left((D \pm iv_{01}) \pm \frac{1}{2}[\sigma, \overline{\sigma}] \right) + \epsilon_\mp (D_0 \pm D_1)\sigma_\mp, \\
\delta D &= \frac{1}{2}\epsilon_+ \big((D_0 - D_1)\overline{\lambda}_+ + i[\sigma, \overline{\lambda}_-] \big) + \frac{1}{2}\epsilon_- \big((D_0 + D_1)\overline{\lambda}_- + i[\overline{\sigma}, \overline{\lambda}_+] \big) + c.c.,
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\delta\phi &= \epsilon_+ \psi_- - \epsilon_- \psi_+, \\
\delta\psi_\pm &= \pm i\overline{\epsilon}_\mp (D_0 \pm D_1)\phi \mp \overline{\epsilon}_\pm \sigma_\mp \phi + \epsilon_\pm F, \\
\delta F &= -i\overline{\epsilon}_+ (D_0 - D_1)\psi_+ - i\overline{\epsilon}_- (D_0 + D_1)\psi_- + \overline{\epsilon}_+ \overline{\sigma}\psi_- + \overline{\epsilon}_- \sigma\psi_+ + i(\overline{\epsilon}_- \overline{\lambda}_+ - \overline{\epsilon}_+ \overline{\lambda}_-)\phi.
\end{aligned} \tag{4.6}$$

where we use the notation $\sigma_+ = \sigma$ and $\sigma_- = \overline{\sigma}$ just in here. Note that σ is A-chiral and ϕ is B-chiral

$$\overline{Q}_+ \sigma = Q_- \sigma = 0, \qquad \overline{Q}_+ \phi = \overline{Q}_- \phi = 0. \tag{4.7}$$

Before writing down the Lagrangian, let us prepare some notations. We denote the G-invariant innder product on $i\mathfrak{g}$ and the G-invariant hermitian inner product on V by

$$(X, Y) \in i\mathfrak{g} \times i\mathfrak{g} \longmapsto \frac{1}{e^2} XY \in \mathbb{R}, \qquad (\phi_1, \phi_2) \in V \times V \longmapsto \overline{\phi}_1 \phi_2 \in \mathbb{C}. \tag{4.8}$$

We shall also write $\frac{1}{e^2}XX = \frac{1}{e^2}X^2$, $\overline{\phi}\phi = |\phi|^2$, etc. Note that there are as many parameters as the number of the irreducible components of $i\mathfrak{g}$ and V in these inner products. For example if $i\mathfrak{g} = \oplus_{k \in K} i\mathfrak{g}_k$ is the irreducible decomposition, we can write $\frac{1}{e^2}X^2 = \sum_{k \in K} \frac{1}{e_k^2}(X_k)^2$ where X_k is the $i\mathfrak{g}_k$ component of $X \in i\mathfrak{g}$. The constant e_k is called the *gauge coupling constant* for the k-th factor. The collection $e = (e_k)_{k \in K}$ may simply be called the gauge coupling constant as well. The inner produce on V defines a G-invariant symplectic structure on V, and we denote by $\mu : V \to i\mathfrak{g}^*$ the moment map that vanishes at the origin.

The supersymmetric Lagrangian is

$$\begin{aligned}
\mathcal{L} &= Q_+ Q_- \overline{Q}_+ \overline{Q}_- \left(-\frac{1}{2e^2}|\sigma|^2 + |\phi|^2 \right) + \mathrm{Re}\, Q_+ Q_- W(\phi) + \mathrm{Re}\, Q_+ \overline{Q}_- \Big(-t(\sigma) \Big) \\
&\qquad\qquad + \text{total derivative} \\
&= \mathcal{L}_{\mathrm{g}} + \mathcal{L}_{\mathrm{m}} + \mathcal{L}_W + \mathcal{L}_t,
\end{aligned} \tag{4.9}$$

where

$$\mathcal{L}_{\mathrm{g}} = \frac{1}{2e^2}\Big[\, |D_0 \sigma|^2 - |D_1 \sigma|^2 + i\overline{\lambda}_- (D_0 + D_1)\lambda_- + i\overline{\lambda}_+ (D_0 - D_1)\lambda_+$$

$$+ (v_{01})^2 + D^2 - \frac{1}{4}[\sigma, \overline{\sigma}]^2 - \lambda_+[\sigma, \overline{\lambda}_-] + [\overline{\sigma}, \lambda_-]\overline{\lambda}_+ \Big], \tag{4.10}$$

$$\begin{aligned}
\mathcal{L}_{\mathrm{m}} &= |D_0\phi|^2 - |D_1\phi|^2 + i\overline{\psi}_-(D_0 + D_1)\psi_- + i\overline{\psi}_+(D_0 - D_1)\psi_+ \\
&\quad + |F|^2 + \overline{\phi}D\phi - \frac{1}{2}|\sigma\phi|^2 - \frac{1}{2}|\overline{\sigma}\phi|^2 - \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_- \\
&\quad - i\overline{\phi}\lambda_-\psi_+ + i\overline{\phi}\lambda_+\psi_- + i\overline{\psi}_+\overline{\lambda}_-\phi - i\overline{\psi}_-\overline{\lambda}_+\phi,
\end{aligned} \tag{4.11}$$

$$\mathcal{L}_W = \mathrm{Re}\left[F^i\frac{\partial W}{\partial\phi^i} - \psi_+^i\psi_-^j\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\right] \tag{4.12}$$

$$\mathcal{L}_t = -\zeta(D) + \theta(v_{01}) \tag{4.13}$$

The Lagrangian (4.9) is manifestly supersymmetric since σ is A-chiral and ϕ is B-chiral (4.7).

The fields D and F are "auxiliary fields" — they do not have the kinetic terms. They can be eliminated by the equations of motion. After doing that, we obtain the scalar potential (4.14).

In view of the convention (2.13), the theta term enters into the action as $\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}^2x\, \theta(v_{01}) = \int_{\mathbb{R}^2} \theta(\frac{i}{2\pi}F_v)$ where $F_v = iv_{01}\mathrm{d}x^0 \wedge \mathrm{d}x^1$ is the $\mathfrak{g}$-valued ("imaginary") curvature two-form. This explains the periodicity of θ.

4.3 Phases

The classical potential for the scalar fields is

$$U(\sigma, \phi) = \frac{1}{8e^2}[\sigma, \overline{\sigma}]^2 + \frac{1}{2}|\sigma\phi|^2 + \frac{1}{2}|\overline{\sigma}\phi|^2 + \frac{e^2}{2}\left(\mu(\phi) - \zeta\right)^2 + |\mathrm{d}W(\phi)|^2. \tag{4.14}$$

Note that each term is non-negative. The space of zero points of U, called *classical vacua*, provides us with a first hint to understand the low energy behaviour of the theory. The vacuum equation $U = 0$ reads

$$[\sigma, \overline{\sigma}] = 0, \quad \sigma\phi = \overline{\sigma}\phi = 0, \quad \mu(\phi) = \zeta, \quad \mathrm{d}W(\phi) = 0. \tag{4.15}$$

The last two equations require ϕ to be in $\mathrm{Crit}(W) \cap \mu^{-1}(\zeta)$ and the first two equations require σ to be in the Cartan subalgebra of the stabilizer subgroup at ϕ. The space of the FI parameter ζ is separated into chambers, called *phases*, according to the topology of the G-space $\mathrm{Crit}(W) \cap \mu^{-1}(\zeta)$. Inside a phase, typically, the stabilizer subgroup is finite at each point of $\mathrm{Crit}(W) \cap \mu^{-1}(\zeta)$, so that σ is forced to vanish — the space of classical vacua is the quotient $X_\zeta = (\mathrm{Crit}(W) \cap \mu^{-1}(\zeta))/G$, called the *Higgs branch*. If that is the case, the theory reduces at low energies to the Landau-Ginzburg model $(\mu^{-1}(\zeta)/G, W_\zeta)$

where W_ζ is the function on $\mu^{-1}(\zeta)/G$ induced from W. If, in addition, W_ζ is Bott-Morse, the theory reduces further to the non-linear sigma model on $\mathrm{Crit}(W_\zeta)$, which is nothing but the Higgs branch X_ζ. Such a phase is called a *geometric phase*. On a wall between chambers (*phase boundary*), there are continuous stabilizer subgroups at some loci of $\mathrm{Crit}(W) \cap \mu^{-1}(\zeta)$. There develops a component of the space of classical vacua, called the *Coulomb branch*, in which σ can take any value in the Cartan subalgebra of the stabilizer subgroup. Emergence of this non-compact space may be regarded as a singularity.

Quantum effects will yield significant modification of this picture. In particular, classical Coulomb branch may be lifted, or quantum Coulomb vacua may emerge even in the absence of classical one. To see this, we explore the region in the field space where σ takes large generic values in a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. Then, the second and the third terms of (4.14) provide masses to many of the components of ϕ, typically all (which we assume for now). Integrating out the massive modes, we obtain the effective theory consisting of the gauge multiplet of the maximal torus T only, with the effective twisted superpotential

$$\widetilde{W}_{\mathrm{eff}}(\sigma) = -t(\sigma) + 2\pi i \rho(\sigma) - \sum_i Q_i(\sigma) \left(\log(Q_i(\sigma)/\mu) - 1\right). \tag{4.16}$$

Here, $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$ is half the sum of positive roots of $\mathfrak{g}$, Q_i's are the weights of V, and t is the FI-theta parameter at the scale μ. The gauge coupling constant of the effective theory is a complicated function $e_{\mathrm{eff}}(\sigma)$ but it approaches the given value e at large values of $|\sigma|$. The effective potential is

$$U_{\mathrm{eff}}(\sigma) = \min_{n \in \mathrm{P}} \frac{e_{\mathrm{eff}}^2(\sigma)}{2} \left| \mathrm{d}\widetilde{W}_{\mathrm{eff}}(\sigma) + 2\pi i n \right|^2 \tag{4.17}$$

Note that the choice of the branch of the logarithms in (4.16) has no physical effect — a different choice would shift $\widetilde{W}_{\mathrm{eff}}(\sigma)$ by an element of $2\pi i \mathrm{P}(\sigma)$, but that does not affect (4.17). A point σ is a true Coulomb vacuum in the quantum theory when $U_{\mathrm{eff}}(\sigma) = 0$, that is, $\mathrm{d}\widetilde{W}_{\mathrm{eff}}(\sigma) \in 2\pi i \mathrm{P}$. In particular, a classical Coulomb branch may not survive in the quantum theory, or true Coulomb vacua might appear even in the absence of classical Coulomb branch. For completeness, one should also explore the region in the field space where σ takes large generic values in a Cartan subalgebra for a subgroup $H \subset G$ and large values of H-neutral components of ϕ, and examine whether there are true mixed Coulomb-Higgs vacua.

The character of the theory depends very much on whether the infinitesimal version of the Calabi-Yau condition, $\mathfrak{g} \subset \mathfrak{sl}(V)$, is satisfied or not. That is, whether b_1 defined by

$$b_1(X) := \mathrm{tr}_V(X) \qquad X \in \mathfrak{g} \tag{4.18}$$

is zero or not. Note that b_1 may be regarded as an element of $\mathrm{P}^W \cong \mathrm{P}_{Z_G}$. As an element of P^W, it can also be written as $b_1 = \sum_i Q_i$.

Calabi-Yau case

Suppose it is satisfied, $b_1 = 0$. In this case, the FI parameter is invariant under the renormalization group and the axial $U(1)$ R-symmetry exists in the quantum theory. Accordingly, $\widetilde{W}_{\rm eff}$ in (4.16) is independent of the parameter μ. In particular, the vacuum equation $\mathrm{d}\widetilde{W}_{\rm eff}(\sigma) \in 2\pi i\mathrm{P}$ is invariant under the scaling, $\sigma \to \lambda\sigma$ for $\lambda \in \mathbb{C}^\times$. This means that if σ is a Coulomb vacuum, then, any of its scaling is also. In particular, the space of such vacua, the Coulomb branch, must be non-compact. Also, presence of Coulomb branch imposes a non-trivial constraint on the FI-theta parameter t. In fact, the equation $\mathrm{d}\widetilde{W}_{\rm eff}(\sigma) \in 2\pi i\mathrm{P}$ produces a parametric representation of t in terms of ratio of σ coordinates, defining a complex hypersurface in the space of t. Let $\Delta \subset \mathfrak{g}_{\mathbb{C}}^{*G}/2\pi i\Lambda_G$ be the discriminant locus on which there is a Coulomb branch and/or mixed branches. It is a union of hyeprsurfaces. When projected to the the ζ space, the discriminant locus Δ projects to an amoeba which asymptotes to the phase boundary. The space of regular values of t is thus

$$\mathfrak{M}_t = \mathfrak{g}_{\mathbb{C}}^{*G}/2\pi i\Lambda_G - \Delta. \tag{4.19}$$

Since $\Delta \subset \mathfrak{g}_{\mathbb{C}}^{*G}/2\pi i\Lambda_G$ has complex codimension one, one can go from one phase to another without meeting it. In particular, there is no sharp transition between different phases.

The theory flows in the infra-red limit to an SCFT with $\widehat{c} = \mathrm{tr}_V(1-R) - \dim G$, and the FI-theta parameter (*resp.* parameters of W) are exactly marginal A-term (*resp.* B-term) deformation parameters of the SCFT. That is, they parameterize submanifolds of the moduli space of SCFTs

$$\mathfrak{M}_t \subset \mathfrak{M}_A^0, \qquad \mathfrak{M}_W \subset \mathfrak{M}_B^0. \tag{4.20}$$

Quite often, the inclusion $\subset$ is equality $=$.

In a geometric phase, $\mathfrak{M}_t$ and $\mathfrak{M}_W$ are respectively (parts of) the moduli space of complexified Kähler class and the moduli space of complex structures of the Higgs branch, respectively.

Non Calabi-Yau case

Suppose the condition is violated, $b_1 \neq 0$. In this case, the FI parameter runs under the renormalization group — for a change of energy scale $\mu \to \mu'$ it changes as

$$\zeta \to \zeta' = \zeta + \log(\mu'/\mu) b_1, \tag{4.21}$$

and the classical axial $U(1)$ R-symmetry is anomalous — the axial rotation $\mathrm{e}^{i\beta} \in U(1)_A$ shifts the theta angle as

$$\theta \to \theta + 2\beta b_1. \tag{4.22}$$

Accordingly, $\widetilde{W}_{\text{eff}}$ in (4.16) depends non-trivially on μ. Since the vacuum equation $\mathrm{d}\widetilde{W}_{\text{eff}}(\sigma) \in 2\pi i \mathrm{P}$ has no scaling invariance, the space of Coulomb vacua does not have to be non-compact. Quite often, there are isolated Coulomb vacua. Such a vacuum cannot be found by the classical analysis of $U(\sigma, \phi)$ but should be taken into account as a sound vacuum of the quantum theory. Of course, there can be vacua at special values of σ, such as $\sigma = 0$, which can be found by the classical analysis.

The theory flows in the infra-red limit to one of the isolated Coulomb vacua, which is typically a massive vacuum, or to the Higgs branch theory $(\mu^{-1}(\zeta_{\text{IR}})/G, W_{\zeta_{\text{IR}}})$ at $\sigma = 0$ where ζ_{IR} is the IR value of the FI parameter, or to a mixture of these two types. Some of the Higgs branch theory can be a non-trivial SCFT.

One should be careful for the use of the term "phase" for two reasons; one is that the FI parameter runs under the renormalization group and another is that there are other vacua at different regions of the field space, such as Coulomb vacua. When we say "phase", we mean the theory at certain range of energy scales where ζ is in a certain chamber *and* in the region of the field space where the stabilizer subgroup is finite and the classical analysis is valid. When we want to make it clear, we shall sometimes use the term "regime" instead. In a geometric regime, the flow of ζ corresponds to the flow of the Kähler class and the axial shift of θ corresponds to that of the B-field.

4.4 Examples

The $\mathbb{C}\mathbb{P}^{N-1}$ Model

Let us consider the model with

$$\begin{aligned} G &= U(1), \\ V &= \mathbb{C}(1)^{\oplus N} \ni \phi = (\phi_1, \ldots, \phi_N), \end{aligned}$$

$$
\begin{aligned}
W &= 0, \\
\widetilde{W} &= -t\sigma.
\end{aligned}
$$

where $\mathbb{C}(i)$ is the representation of $U(1)$ of weight i ($g \in U(1)$ acts by multiplication by g^i) and $t = \zeta - i\theta \in \mathbb{C}/2\pi i\mathbb{Z}$. No superpotential is allowed since there is no gauge invariant polynomial of ϕ_i's. There is a vector $U(1)$ R-symmetry with charge integrality, say, with R-charge zero for all ϕ_i's.

The scalar potential reads

$$
U = |\sigma\phi|^2 + \frac{e^2}{2}\Big(|\phi|^2 - \zeta\Big)^2. \tag{4.23}
$$

When $\zeta > 0$, the vacuum equation $U = 0$ requires that $|\phi|^2 = \zeta$. In particular ϕ has a non-zero value, and the stabilizer subgroup of the gauge group is trivial. In such a situation, we say "the vacuum breaks the gauge group completely". The other equation $|\sigma\phi|^2 = 0$ then requires that σ must vanish. In fact, this is linked to the fact that gauge group is broken to a finite subgroup. Therefore, the space of classical supersymmetric vacua is the space of ϕ's solving $|\phi|^2 = \zeta$ modulo the gauge group action, that is, the complex projective space $\mathbb{CP}^{N-1}$. All modes transverse to this vacuum manifold have a mass of the order of $e\sqrt{\zeta}$, and hence the theory reduces at energies below $e\sqrt{\zeta}$ to the non-linear sigma model with target $\mathbb{CP}^{N-1}$. We read from the Lagrangian that the Kähler and the B-field classes are given by

$$
[\omega] \simeq \zeta H \in \mathrm{H}^2(\mathbb{CP}^{N-1}, \mathbb{R}), \qquad [B] \simeq [\theta H] \in \mathrm{H}^2(\mathbb{CP}^{N-1}, \mathbb{R}/2\pi\mathbb{Z}), \tag{4.24}
$$

where H is the hyperplane class of $\mathbb{CP}^{N-1}$.
When $\zeta < 0$, the equation $U = 0$ has no solution. That is, there is no classical supersymmetric vacuum, and we say "supersymmetry is (classically) broken." The potential is minimized by $\phi = 0$ and σ arbitrary, and the vacuum energy is $e^2\zeta^2/2$.

There is a significant quantum correction to this classical picture. First, the FI parameter runs according to the renormalization group as $\zeta' = \zeta + N\log(\mu'/\mu)$. It is large positive at high energies and large negative at low energies. This means that the interpretation of the theory as the $\mathbb{CP}^{N-1}$ sigma model is valid only at high enough energies (but below $e\sqrt{\zeta}$) where ζ is large positive. It is invalid at lower energies where ζ approaches zero. At even lower energies ζ becomes negative. There, as we have seen above, the classical vacua, $\phi = 0$ and σ arbitrary, have positive energy $e^2\zeta^2/2$ and supersymmetry is broken. However, the scalar potential receives a quantum correction. In the region where σ is large and the ϕ_i multiplets are heavy, we should integrate out the massive

ϕ_i multiplets, rather than just set them to zero, and study the effective Lagrangian for the gauge multiplet. The gauge coupling constant becomes a function $e_{\rm eff}(\sigma)$ of σ which approaches the classical value e at large σ. The effective twisted superpotential can be computed exactly and is given by

$$\widetilde{W}_{\rm eff}(\sigma) = -t\sigma - N\sigma\log(\sigma/\mu) = -N\sigma\log(\sigma/\Lambda), \tag{4.25}$$

where

$$\Lambda^N := \mathrm{e}^{-t}\mu^N \tag{4.26}$$

is a renormalization group invariant scale parameter. The effective potential is

$$U_{\rm eff}(\sigma) = \min_{n\in\mathbb{Z}} \frac{e_{\rm eff}^2(\sigma)}{2}\left|\partial_\sigma \widetilde{W}_{\rm eff}(\sigma) + 2\pi i n\right|^2, \tag{4.27}$$

and the vacuum equation is $\partial_\sigma \widetilde{W}_{\rm eff}(\sigma) = -N\log(\sigma/\Lambda) \equiv 0 \mod 2\pi i\mathbb{Z}$, that is,

$$\sigma^N = \Lambda^N. \tag{4.28}$$

We see that there are N solutions. That is, there are N supersymmetric vacua in the quantum theory, contrary to the conclusion from the classical analysis. Expansion of the potential around each solution has a quadratic term, and hence each vacuum has massive excitations only — we say "each vacuum has a mass gap."

This is the conclusion for the gauge linear sigma model. However, since the model can be interpreted as the $\mathbb{CP}^{N-1}$ sigma model at certain window of energy scales, provided $e \gg \Lambda$, this can also be regarded as the conclusion for the $\mathbb{CP}^{N-1}$ sigma model. Indeed, supersymmetric vacua of the sigma model are known to be in one to one correspondence with cohomology classes of the target space,[1] and hence the conclusion that there are N supersymmetric vacua is consistent with the fact that the cohomology group of $\mathbb{CP}^{N-1}$ is of rank N.

The Model $\mathbf{T}^{U(1)}_{N,d}$

Let us consider the model $\mathrm{T}^{U(1)}_{N,d}$ labelled by two positive integers N and d:

$$\begin{aligned} G &= U(1), \\ V &= \mathbb{C}(-d) \oplus \mathbb{C}(1)^{\oplus N} \ni (p, x_1, \ldots, x_N), \\ W &= pf(x_1, \ldots, x_N), \\ \widetilde{W} &= -t\sigma. \end{aligned}$$

[1] This follows from the fact that the sigma model is A-twistable and hence there is a one to one correspondence between supersymmetric vacua and elements of $\mathcal{R}_A$, and that $\mathcal{R}_A$ is isomorphic as a vector space to the cohomology group of the target space.

$f(x_1,\ldots,x_N)$ is a degree d polynomial which is generic in the sense that $\partial f/\partial x_i = 0$ for all i implies $x_1 = \cdots = x_N = 0$. The R-charge is unique up to gauge, $R = (2-d\epsilon, \epsilon, \ldots, \epsilon)$, and satisfies the charge integrality condition with $J = \mathrm{e}^{\pi i \epsilon}$.

The scalar potential reads

$$U = |-d\sigma p|^2 + |\sigma x|^2 + \frac{e^2}{2}\Big(-d|p|^2 + |x|^2 - \zeta\Big)^2 + |f(x)|^2 + |p\,\mathrm{d}f(x)|^2. \tag{4.29}$$

The space of classical vacua and the pattern of gauge symmetry breaking depend on whether the FI parameter ζ is positive, negative, or zero.
For $\zeta > 0$, vanishing of the middle term requires that x has a non-zero value which breaks the gauge group completely. Accordingly, σ must vanish. Also, since $f(x)$ is generic, vanishing of $|p\,\mathrm{d}f(x)|^2$ and $x \neq 0$ requires $p = 0$. Therefore, the space of classical vacua is the space of x's solving $|x|^2 = \zeta$ and $f(x) = 0$ modulo the gauge $U(1)$ action. This is nothing but the hypersurface X_f of $\mathbb{CP}^{N-1}$ defined by $f = 0$. All modes transverse to this vacuum manifold is massive and hence the theory reduces at low energies to the non-linear sigma model with target X_f. The Kähler and the B-field classes are given by

$$[\omega] \;\simeq\; \zeta H \in \mathrm{H}^2(X_f, \mathbb{R}), \qquad [B] \;\simeq\; [(\theta + \pi d)H] \in \mathrm{H}^2(X_f, \mathbb{R}/2\pi\mathbb{Z}). \tag{4.30}$$

The shift $\theta \to \theta + \pi d$ comes from integrating out the massive p multiplet [12]. It turns out that this relation may have further quantum corrections which are exponentially small in the $\zeta \to +\infty$ limit, $[\omega - iB] = [(t - d\pi i)H] + O(\mathrm{e}^{-t})$.
For $\zeta < 0$, vanishing of the middle term forces p to have a non-zero value which breaks the gauge group to the subgroup $\mathbb{Z}_d \subset U(1)$ consisting of d-th roots of unity. Accordingly, σ must vanish. Also, by the genericity of $f(x)$, vanishing of $|p\,\mathrm{d}f(x)|^2$ requires x to vanish. Thus, the vacuum manifold is one point, $p = \sqrt{-\zeta/d} =: \langle p \rangle$, $x = 0$ and $\sigma = 0$, with stabilizer $\mathbb{Z}_d$. This time, the x modes are massless (for $d \geq 3$), and the theory reduces at low energies to the model of x-fields with the superpotential $W = \langle p \rangle f(x)$, that is, the Landau-Ginzburg orbifold $(\mathbb{C}^N/\mathbb{Z}_d, W = \langle p \rangle f)$ at low energies.
For $\zeta = 0$, vanishing of the middle term and the fourth term requires $p = 0$ and $x = 0$. Then, the rest of the vacuum equation imposes no condition on σ. That is, σ can take any value. The vacuum manifold is the non-compact space $\mathbb{C}$. This is the Coulomb branch.
To summarize, the space of FI-parameter $i\mathfrak{z}^* \cong \mathbb{R}$ has two phases — the geometric phase $\zeta > 0$ and the Landau-Ginzburg phase $\zeta < 0$ — separated by the phase boundary $\zeta = 0$ where there is a non-compact Coulomb branch.

The FI parameter runs as $\zeta' = \zeta + (N - d)\log(\mu'/\mu)$ and the axial $U(1)$ R-symmetry is anomalous, except when the Calabi-Yau condition $d = N$ is satisfied. The effective twisted superpotential at large values of σ is

$$\widetilde{W}_{\rm eff}(\sigma) = -t\sigma + d\sigma(\log(-d\sigma/\mu) - 1) - N\sigma(\log(\sigma/\mu) - 1), \tag{4.31}$$

and the equation for the Coulomb vacuum is $\partial_\sigma \widetilde{W}_{\rm eff} \equiv 0 \bmod 2\pi i\mathbb{Z}$, or

$$\sigma^{N-d} = (-d)^d \, {\rm e}^{-t} \mu^{N-d}. \tag{4.32}$$

When $d = N$, we have a family of superconformal field theories with $\widehat{c} = N - 2$ parametrized by t as well as the parameters for f. Since the equation (4.32) has solutions (i.e. arbitrary $\sigma \neq 0$) only for ${\rm e}^t = (-N)^N$, the discriminant locus Δ is one point at $t \equiv N \log N + N\pi i$,

$$\mathfrak{M}_t = \mathbb{C}/2\pi i\mathbb{Z} - \{[N \log N + N\pi i]\}. \tag{4.33}$$

We see that the non-linear sigma model on the Calabi-Yau manifold X_f is continuously connected to the Landau-Ginzburg orbifold $(\mathbb{C}^N/\mathbb{Z}_N, f)$. In particular, the topological B-models of the two theories are equivalent, while the topological A-models are related by analytic continuation. This is the *CY/LG correspondence*. In the present model the inclusions in (4.20) are both equalities, $\mathfrak{M}_t = \mathfrak{M}_A^0$ and $\mathfrak{M}_W = \mathfrak{M}_B^0$.

When $d < N$, the FI parameter ζ runs from positive to negative in such a way that $\Lambda^{N-d} := {\rm e}^{-t}\mu^{N-d}$ is RG invariant. The theory at the energy scale μ with $e \gg \mu \gg |\Lambda|$ is the non-linear sigma model on the Fano manifold X_f. At lower energies below $|\Lambda|$ the sigma model description is no longer valid. There are $(N - d)$ Coulomb vacua with mass gap at $\sigma^{N-d} = (-d)^d \Lambda^{N-d}$, as well as one Higgs branch theory at $\sigma = 0$ (for $d > 1$) which is the Landau-Ginzburg orbifold $(\mathbb{C}^N/\mathbb{Z}_d, f)$. When $3 \leq d < N$, the Higgs branch theory further flows in the infra-red limit to a superconformal field theory with $\widehat{c} = N(1 - 2/d)$. When $d = 2$, the Higgs branch theory has two (*resp.* one) supersymmetric ground states with a mass gap for even (*resp.* odd) N.

When $d > N$, the FI parameter ζ runs from negative to positive in such a way that $\Lambda^{d-N} := {\rm e}^t \mu^{d-N}$ is RG invariant. The theory can be regarded as the superconformal field theory with $\widehat{c} = N(1 - 2/d)$ corrsponding to the Landau-Ginzburg orbifold $(\mathbb{C}^N/\mathbb{Z}_d, f)$ which is perturbed by a relevant operator of dimension $2N/d$. There are $(d-N)$ Coulomb vacua with mass gap at $\sigma^{d-N} = (-d)^{-d}\Lambda^{d-N}$ as well as one Higgs branch theory at $\sigma = 0$ which is the non-linear sigma model on the hypersurface X_f of general type.

The Model $\mathbf{T}^{U(1)}_{N,d}(0)$

We may also consider the model $\mathrm{T}^{U(1)}_{N,d}(0)$ with the same $(G, V, \widetilde{W})$ as in the previous example but with vanishing superpotential $W = 0$. The constraint on the vector R-charge (that W must have R-charge 2) is gone, and it turns out that the most natural choise is to assign $R = 0$ to all fields $p, x_1, \dots, x_N$.

The scalar potential lacks the last two terms in (4.29), and accordingly, the description of the classical phases changes.
For $\zeta > 0$, x is again non-zero and breaks the gauge group completely, resulting in $\sigma = 0$. However, this time, p does not have to vanish. The vacuum manifold is now the space of (p, x) obeying $-d|p|^2 + |x|^2 = \zeta$ modulo the gauge $U(1)$ action. It is the total space of the line bundle $\mathcal{O}_{\mathbb{CP}^{N-1}}(-d)$ over $\mathbb{CP}^{N-1}$, where p and x parametrize the fiber and the base. All the transverse modes are massive and the theory reduces at low energies to the non-linear sigma model with the non-compact target space $\mathrm{tot}[\mathcal{O}_{\mathbb{CP}^{N-1}}(-d)]$.
For $\zeta < 0$, p is non-zero and breaks the gauge group to $\mathbb{Z}_d$, resulting in $\sigma = 0$. Only the x fields are massless and we are left with the theory with $G = \mathbb{Z}_d$, $V = \mathbb{C}(1)^{\oplus N}$, $W = 0$ (and $\widetilde{W} = 0$ for a trivial reason). This is the free orbifold theory $\mathbb{C}^N/\mathbb{Z}_d$. Note that $\mathrm{tot}[\mathcal{O}_{\mathbb{CP}^{N-1}}(-d)]$ can be regarded as a resolution of the orbifold singularity at the origin.
For $\zeta = 0$, the vacuum manifold is the union of two branches — the Higgs branch where $\sigma = 0$ and the Coulomb branch where $x = p = 0$. They touch each other at the origin, $x = p = \sigma = 0$.

In the quantum theory, the FI running, the axial anomaly, and the effective twisted superpotential at large $\sigma 0$ are the same as in the model with superpotential. In particular, the equation for the Coulomb vacuum remains the same as (4.32). The detail depends on the relation between N and d.

When $d = N$, we have a family of non-compact superconformal field theories parametrized by e^t. The discriminant locus is $\mathrm{e}^t = (-N)^N$. We see that the sigma model with target $\mathrm{tot}[\mathcal{O}_{\mathbb{CP}^{N-1}}(-N)]$ is continuously connected to the free orbifold $\mathbb{C}^N/\mathbb{Z}_N$. In particular, topological B-models of the two are equivalent while the topologival A-models are related by analytic continuation. The case $N = 2$ is an example of *McKay correspondence* and the others are generalizations thereof.

When $d < N$, the FI parameter ζ runs from positive to negative in such a way that $\Lambda^{N-d} := \mathrm{e}^{-t}\mu^{N-d}$ is RG invariant. The theory at high energies is the non-linear sigma model with target $\mathrm{tot}[\mathcal{O}_{\mathbb{CP}^{N-1}}(-d)]$. There are $(N - d)$ Coulomb vacua with mass gap at $\sigma^{N-d} = (-d)^d\Lambda^{N-d}$, as well as one Higgs branch theory at $\sigma = 0$ (for $d > 1$) which is the free orbifold $\mathbb{C}^N/\mathbb{Z}_d$.

When $d > N$, the FI parameter runs from negative to positive in such a way that $\Lambda^{d-N} := \mathrm{e}^t\mu^{d-N}$ is RG invariant. The theory can be regarded as the relevant deformation of the free orbifold $\mathbb{C}^N/\mathbb{Z}_d$. There are $(d-N)$ Coulomb vacua with mass gap at $\sigma^{d-N} = (-d)^{-d}\Lambda^{d-N}$ as well as one Higgs branch theory at $\sigma = 0$ which is the non-linear sigma model with target $\mathrm{tot}[\mathcal{O}_{\mathbb{CP}^{N-1}}(-d)]$.

The Model $\mathbf{H}_n$

Finally, let us consider the model H_n labelled by a non-negtaive integer n:

$$\begin{aligned}
G &= U(1)\times U(1),\\
V &= \mathbb{C}(1,0)\oplus\mathbb{C}(1,0)\oplus\mathbb{C}(0,1)\oplus\mathbb{C}(-n,1)\ni(\phi_1,\phi_2,\phi_3,\phi_4),\\
W &= 0,\\
\widetilde{W} &= -t^1\sigma_1 - t^2\sigma_2.
\end{aligned}$$

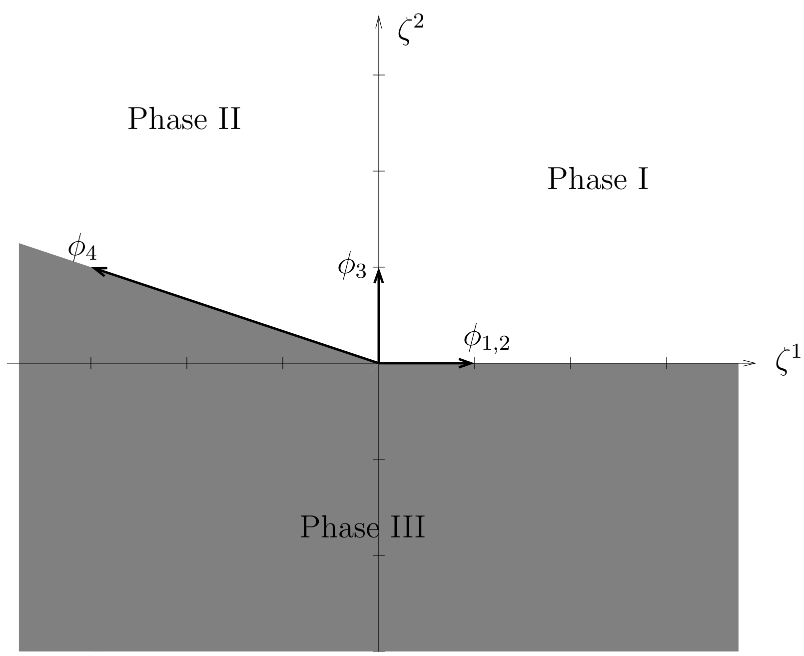

Figure 3: The phases of H_n (for $n = 3$)

Writing down the scalar potential is left as an exercise for the reader. The classical supersymmetric vacuum equation consists of $\sigma_1\phi_1 = \sigma_1\phi_2 = \sigma_2\phi_3 = (-n\sigma_1+\sigma_2)\phi_4 = 0$ and

$$|\phi_1|^2 + |\phi_2|^2 - n|\phi_4|^2 = \zeta^1, \qquad |\phi_3|^2 + |\phi_4|^2 = \zeta^2. \tag{4.34}$$

The space of FI parameter $\mathbb{R}^2 = \{(\zeta^1, \zeta^2)\}$ is separated into three phases, as depicted in Fig. 3. Phase I is the first quadrant, $\zeta^{1,2} > 0$, Phase II is the chamber $\zeta^1 < 0$ and $\zeta^1 + n\zeta^2 > 0$, and Phase III is the remaining region which is shaded in the figure.

<u>In Phase I</u>, both (ϕ_1, ϕ_2) and (ϕ_3, ϕ_4) must be non-zero, and the gauge group is completely broken. The theory reduces to the sigma model whose target space is the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n))$. This surface is Fano for $n = 0, 1$, nef for $n = 2$ and non-Fano for $n \geq 3$. Non-zero Hodge numbers are $h^{0,0} = 1$, $h^{1,1} = 2$, $h^{2,2} = 1$.

<u>In Phase II</u>, which is present for $n \geq 1$, both (ϕ_1, ϕ_2, ϕ_3) and ϕ_4 must be non-zero. The non-zero values of ϕ_4 break the gauge group to the subgroup $\{(g, g^n)\} \cong U(1)$ under which ϕ_1, ϕ_2, ϕ_3 have charges $1, 1, n$ respectively. This residual group is completely broken by the non-zero values of (ϕ_1, ϕ_2, ϕ_3) except at the points with $\phi_1 = \phi_2 = 0$ where it is broken to the $\mathbb{Z}_n$ subgroup. The theory reduces to the sigma model whose target space is the weighted projective space $\mathrm{W}\mathbb{P}^2_{1,1,n}$. When $n \geq 2$, the space $\mathrm{W}\mathbb{P}^2_{1,1,n}$ has one orbifold point of the type $\mathbb{C}^2/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts on $\mathbb{C}^2$ by $(z_1, z_2) \mapsto (\,\mathrm{e}^{2\pi i/n} z_1,\, \mathrm{e}^{2\pi i/n} z_2)$. Note that $\mathbb{F}_n$ in Phase I can be regarded as a resolution of this orbifold singularity. In fact, $\mathbb{F}_n \to \mathrm{W}\mathbb{P}^2_{1,1,n}$ is a compact version of the resolution $\mathrm{tot}[\mathcal{O}_{\mathbb{P}^1}(-n)] \to \mathbb{C}^2/\mathbb{Z}_n$ which appeared in the model $\mathrm{T}^{U(1)}_{N,d}$ with $N = 2$ and $d = n$.

<u>In Phase III</u>, the equation (4.34) has no solution. That is, supersymmetry is broken at the classical level.

In the quantum theory, the FI parameter runs as $\zeta^{1\prime} = \zeta^1 + (2 - n)\log(\mu'/\mu)$, $\zeta^{2\prime} = \zeta^2 + 2\log(\mu'/\mu)$. ζ^2 runs from large positive to large negative for any n, while the direction of ζ^1 running depends on whether $n \leq 1$ (positive to negative), $n = 2$ (no runing), $n \geq 3$ (negative to positive). These three cases will be discussed separately below. Computing the effective twisted superpotential at large values of σ's is left as an exercise for the reader. Here, we just write down the vacuum equations there:

$$\sigma_1^2 = q_1 \mu^{2-n}(-n\sigma_1 + \sigma_2)^n, \qquad \sigma_2(-n\sigma_1 + \sigma_2) = q_2 \mu^2, \tag{4.35}$$

where $q_a := \mathrm{e}^{-t^a}$. There are four solutions for $n = 0, 1$, generically four solutions for $n = 2$, and $n + 2$ solutions for $n \geq 3$ (see blow).

<u>$n = 0, 1$</u> The flow of the FI parameter is depicted in Fig. 4. In both cases, the UV limit of any flow is in Phase I that corresponds to $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ $(n = 0)$ or $\mathbb{F}_1 =$ one pont blow up of $\mathbb{CP}^2$ $(n = 1)$. Thus, the GLSM can be identified as the sigma model with these target spaces. Note that both target spaces are Fano manifolds and hence the sigma models are UV complete. In the model H_1, some of the flows go through Phase II in the intermediate energy scale. Recall that Phase II corresponds in this case to $\mathrm{W}\mathbb{CP}^2_{1,1,1} = \mathbb{CP}^2$. Let us discuss whether we can identify the $\mathbb{CP}^2$ sigma model as a part

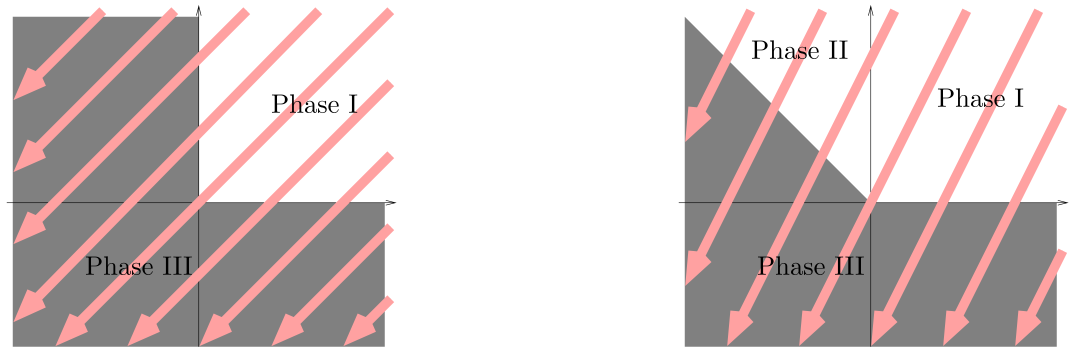

Figure 4: The RG flow for H_0 (left) and H_1 (right).

of the model H_1. We write the FI-theta parameters as $q_1 = \Lambda/\mu$ and $q_2 = q(\Lambda/\mu)^2$ in terms of RG invariant parameters q and Λ of the theory. Note that the theory is in Phase II when $q_1 \gg 1$ and $q_1 q_2 \ll 1$, i.e. for the window of energy scales $q^{\frac{1}{3}}\Lambda \ll \mu \ll \Lambda$, which is possible only if $q \ll 1$. If we take the limit $q \to 0$ holding $\Lambda_{\mathbb{P}^2}^3 := q\Lambda^3$ fixed, then the window becomes $\Lambda_{\mathbb{P}^2} \ll \mu < \infty$, just as in the $\mathbb{CP}^2$ sigma model. The equations (4.35) can be written as

$$\sigma_2 = \sigma_1 + \sigma_1^2/\Lambda, \qquad \sigma_1^3(\sigma_1 + \Lambda) = q\Lambda^4. \tag{4.36}$$

When $q \ll 1$, there are three solutions with $\sigma_1^3 \sim q\Lambda^3$, $\sigma_1 \sim \sigma_2$, and a single solution with $\sigma_1 \sim -\Lambda$, $\sigma_2 \sim q\Lambda$. In the above scaling limit, the three solutions stay finite, $\sigma_1^3 \sim \Lambda_{\mathbb{P}^2}^3$, while the single solution goes away to infinity. These suggest that the scaling limit takes out the $\mathbb{CP}^2$ sigma model out of H_1.

$\underline{n=2}$ The flow of the FI parameter is depicted in Fig. 5-Left. q_1 is an RG-invariant parameter but q_2 runs as $q_2 = (\Lambda/\mu)^2$. The theory describes the $\mathbb{F}_2$ sigma model when $q_1 \ll 1$ and $W\mathbb{P}^2_{1,1,2}$ sigma model when $q_1 \gg 1$. For $q_1 \neq 1/4$, the vacuum equation has four solutions

$$\sigma_1 = \frac{\epsilon_2\epsilon_1 q_1^{\frac{1}{2}}\Lambda}{(1+2\epsilon_1 q_1^{\frac{1}{2}})^{\frac{1}{2}}}, \qquad \sigma_2 = \epsilon_2\Lambda(1+2\epsilon_1 q_1^{\frac{1}{2}})^{\frac{1}{2}}, \qquad \epsilon_1, \epsilon_2 \in \{\pm 1\}. \tag{4.37}$$

As $q_1 \to 1/4$, two of them goes away to infinity; $q_1 = 1/4$ may be regarded as a discriminant locus. The $W\mathbb{P}^2_{1,1,2}$ and $\mathbb{F}_2$ models are connected along a path in the q_1 space that goes from $q_1 \gg 1$ to $q_1 \ll 1$ avoiding $q_1 = 1/4$. The $\mathbb{C}^2/\mathbb{Z}_2$ singularity (A_1 singularity) of $W\mathbb{P}^2_{1,1,2}$ is resolved by inserting a $\mathbb{CP}^1$ in this process. We see that McKay correspondence is realized as a part of this compact model.

$\underline{n \geq 3}$ The flow of the FI parameter is as in Fig. 5-Right. The UV limit of any flow is in Phase II. Thus, the GLSM can be identified as the $W\mathbb{P}^2_{1,1,n}$ sigma model. Some of

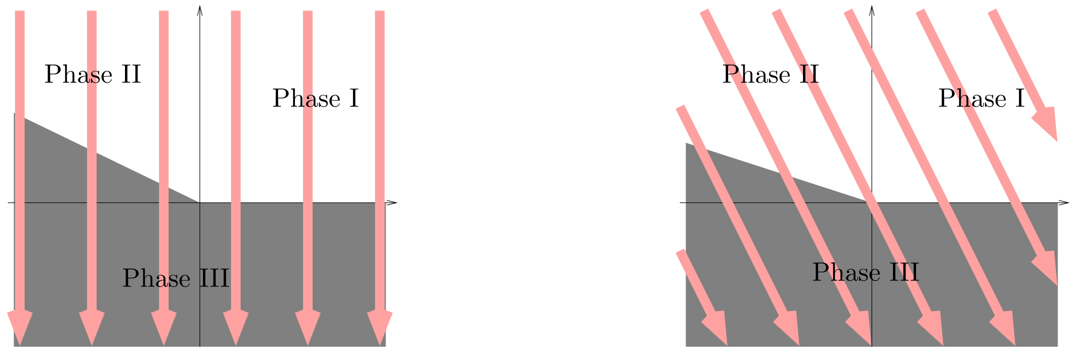

Figure 5: The RG flow for H_2 (left) and H_3 (right).

the flows go through Phase I. Let us discuss whether we can identify the $\mathbb{F}_n$ sigma model as a part of the model H_n. We write the FI-theta parameters as $q_1 = (\Lambda/\mu)^{2-n}$ and $q_2 = q(\Lambda/\mu)^2$ for RG invariant parameters q and Λ. The theory is in Phase I when $q_1 \ll 1$ and $q_2 \ll 1$, i.e. for the window of energy scales $q^{\frac{1}{2}}\Lambda \ll \mu \ll \Lambda$, which is possible only if $q \ll 1$. If we would like to keep the two Kähler parameters of $\mathbb{F}_n$, we should hold both q and Λ finite, but if we give up one or both, we can take some limits. For example, the limit $q \to 0$ holding Λ fixed (and looking at the field space with $|\phi_3|^2 \sim \zeta^2$) decouples the pair of ϕ_3 and the second $U(1)$ gauge group, reducing the model to $\mathrm{T}^{U(1)}_{N,d}(0)$ with $N = 2$ and $d = n$ — i.e. it focuses to the orbifold point and its resolution $\mathrm{tot}[\mathcal{O}_{\mathbb{P}^1}(-n)] \to \mathbb{C}^2/\mathbb{Z}_n$. The vacuum equation can be written in terms of $z := \sigma_2 - n\sigma_1$ as

$$(z^2 - q\Lambda^2)^2 = n^2\Lambda^{2-n}z^{n+2}; \qquad \sigma_1 = \frac{q\Lambda^2 - z^2}{nz}, \quad \sigma_2 = \frac{q\Lambda^2}{z}. \tag{4.38}$$

When $q \ll 1$, there are four solutions with $z^2 \sim q\Lambda^2$ for which $(\sigma_1, \sigma_2) \sim (\epsilon_1 q^{\frac{n}{4}}\Lambda, \epsilon_2 q^{\frac{1}{2}}\Lambda)$ with $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ and $(n-2)$ solutions with $1 \sim n^2\Lambda^{2-n}z^{n-2}$ for which $(\sigma_1, \sigma_2) \sim (-z/n, q\Lambda^2/z)$. In the limit $q \to 0$ holding Λ fixed, the former four go to $(\sigma_1, \sigma_2) \to (0, 0)$ while the latter $(n-2)$ go to $(\sigma_1, \sigma_2) \to (\widetilde{z}, 0)$ with $\widetilde{z}^{n-2} = (-n)^{-n}\Lambda^{n-2}$. In the model $\mathrm{T}^{U(1)}_{2,n}(0)$ (with $n > 2$), the vacua at $\sigma = 0$ correspond to the resuolution $\mathrm{tot}[\mathcal{O}_{\mathbb{P}^1}(-n)]$ while the $(n-2)$ vacua at $\sigma^{n-2} = (-n)^{-n}\Lambda^{n-2}$ are massive Coulomb vacua. This suggests that the former four solutions correspond to vacua of the $\mathbb{F}_n$ sigma model. However, there is no limit that select only those vacua and send the rest to infinity, if we would like to keep the two Kähler parameters of $\mathbb{F}_n$. Thus, we can identify the vacua corresponding to the $\mathbb{F}_n$ sigma model but cannot decouple the remaing sectors. I.e. we cannot isolate "purely $\mathbb{F}_n$" out of H_n

4.5 Moment Polytope and Toric Variety

Let us consider the general model with connected Abelian gauge group and no superpotential:

$$\begin{aligned} G &= U(1)^k, \\ V &= \bigoplus_{i=1}^{N} \mathbb{C}(Q_i) \ni \phi = (\phi_1, \dots, \phi_N), \\ W &= 0, \\ \widetilde{W} &= -\sum_{a=1}^{k} t^a \sigma_a, \end{aligned}$$

where $Q_i = (Q_i^1, \dots, Q_i^k) \in \mathbb{Z}^{\oplus k}$ is a weight of $G = U(1)^k$. We assume that the G action on V is faithful. The $\mathbb{CP}^{N-1}$ model, the models $\mathrm{T}_{N,d}^{U(1)}(0)$ and H_n are particular cases. By faithfulness, G may be regarded as a subgroup of $U(1)^N$ that acts on V with the diagonal weights. An important rôle will be played by the group

$$T_F := U(1)^N / G. \tag{4.39}$$

It is a group of global symmetries of the theory, which we call the (toroidal) flavor group. Being a compact connected Abelian Lie group of dimension $N - k$, it is isomorphic to $U(1)^{N-k}$.

Suppose there is a geometric or orbifold phase where the gauge group is broken to a finite subgroup at any classical supersymetric vacuum. Then, the theory reduces at low energies to the sigma model whose target space is the vacuum manifold, which is

$$X_\zeta = \mu^{-1}(\zeta)/G. \tag{4.40}$$

The flavor group T_F acts on X_ζ as a symmetry group of its Kähler structure, in particular, of its symplectic structure. In fact, there is a moment map $\mu_F : X_\zeta \to i\mathfrak{t}_F^*$. Let us choose an isomorphism $f : U(1)^{N-k} \to T_F$

$$f : (c_1, \dots, c_{N-k}) \in U(1)^{N-k} \longmapsto \left[\left(\prod_\alpha c_\alpha^{f_1^\alpha}, \dots, \prod_\alpha c_\alpha^{f_N^\alpha} \right) \right] \in T_F. \tag{4.41}$$

Then, the components of $\mu_F(\phi) \in i\mathfrak{t}_F^*$ are given by

$$\mu_F^\alpha(\phi) = \sum_{i=1}^{N} f_i^\alpha |\phi_i|^2, \qquad \alpha = 1, \dots, N-k. \tag{4.42}$$

A given $f : U(1)^{N-k} \to T_F$ does not determine the integers f_i^α uniquely — there is an ambiguity of shifting f_i^α by $\sum_a Q_i^a m_a^\alpha$ for some integers m_a^α. But that only shifts this $\mu_F^\alpha(\phi)$ by a constant $\sum_a \zeta^a m_a^\alpha$ thanks to the vacuum equation $\sum_{i=1}^N Q_i^a |\phi_i|^2 = \zeta^a$.

The μ_F-image of X_ζ is a convex polytope in $i\mathfrak{t}_F^*$ [7], which we shall call the *moment polytope* P_ζ of X_ζ. The interior points of P_ζ correspond to $[\phi] \in X_\zeta$ with all components ϕ_i non-vanishing while the boundary points correspond to those where some of ϕ_i's vanish. For example, let us consider the $\zeta > 0$ phase of the $\mathbb{CP}^{N-1}$ model. As the isomorophism $f : U(1)^{N-1} \to T_F = U(1)^N/G$ we may choose $(c_1, \ldots, c_{N-1}) \mapsto [(c_1, \ldots, c_{N-1}, 1)]$. Then, the moment map is $\mu_F(\phi) = (|\phi_1|^2, \ldots, |\phi_{N-1}|^2)$ for ϕ obeying $\sum_{i=1}^N |\phi_i|^2 = \zeta$. Since each $|\phi_i|^2$ is non-negative, we see that $(\eta^1, \ldots, \eta^{N-1}) \in i\mathfrak{t}_F^* \cong \mathbb{R}^{N-1}$ is in P_ζ if and only if $\eta^\alpha \geq 0$ and $\eta^1 + \cdots + \eta^{N-1} \leq \zeta$. That is, P_ζ is the standard $(N-1)$-simplex in $\mathbb{R}^{N-1}$ scaled by ζ. See Fig. 6 for the cases $N = 2$ and 3, where we put the fields that vanish on the

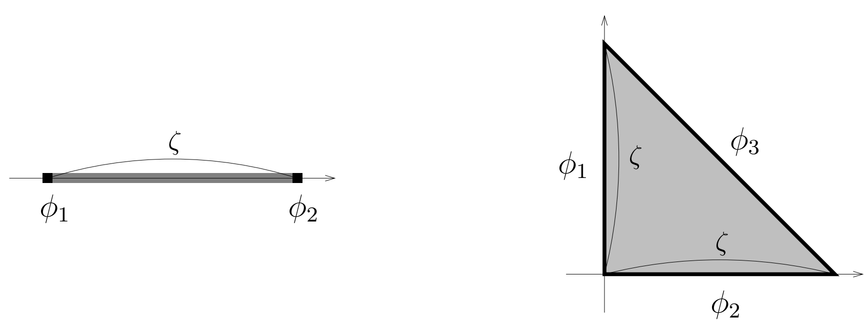

Figure 6: The moment polytope of $\mathbb{CP}^1$ (left) and $\mathbb{CP}^2$ (right).

boundary hyperplanes. Let us also describe the moment polytopes in the other examples. For $\mathrm{T}_{N,d}^{U(1)}(0)$,

$$\begin{aligned} &f : (c_0, c_1, \ldots, c_{N-1}) \in U(1)^N \mapsto [(c_0, c_1, \ldots, c_{N-1}, 1)] \in U(1)^{N+1}/G, \\ &P_\zeta = \left\{ (\eta^0, \eta^1, \ldots, \eta^{N-1}) \;\middle|\; \eta^\alpha \geq 0,\ \eta^1 + \cdots + \eta^{N-1} - d\eta^0 \leq \zeta \right\}, \end{aligned} \tag{4.43}$$

and for H_n,

$$\begin{aligned} &f : (c_1, c_2) \in U(1)^2 \mapsto [(c_1, 1, c_2, 1)] \in U(1)^4/G, \\ &P_\zeta = \left\{ (\eta^1, \eta^2) \;\middle|\; \eta^1 \geq 0,\ 0 \leq \eta^2 \leq \zeta^2,\ \eta^1 + n\eta^2 \leq \zeta^1 + n\zeta^2 \right\}. \end{aligned} \tag{4.44}$$

In both examples, the expression for P_ζ is valid in either of the the orbifold phase ($\zeta < 0$ for $\mathrm{T}_{N,d}^{U(1)}$ and Phase II for H_n) and the geometric phase ($\zeta > 0$ for $\mathrm{T}_{N,d}^{U(1)}$ and Phase I for H_n). See Fig. 7 and Fig. 8, where we again put the fields that vanish on the boundary hyperplanes.

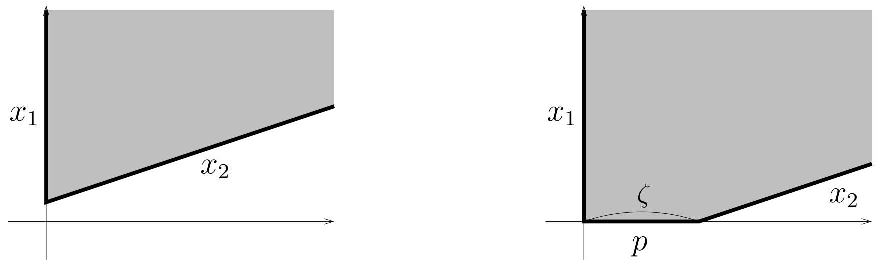

Figure 7: The moment polytope of $\mathbb{C}^2/\mathbb{Z}_d$ (left) and $\mathrm{tot}[\mathcal{O}(-d) \to \mathbb{CP}^1]$ (right).

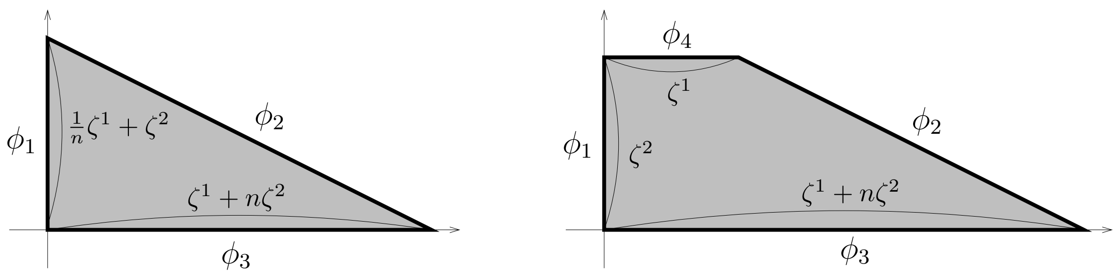

Figure 8: The moment polytope of $\mathrm{W}\mathbb{P}^2_{1,1,n}$ (left) and $\mathbb{F}_n$ (right).

The most important point for us is that $\mu_F : X_\zeta \to P_\zeta$ is a Lagrangian torus fibration. Over the interior of P_ζ, it is a smooth torus fibration with T_F fibers — in fact it is a principal T_F bundle — since any of ϕ_i's are non-zero. At the boundary of P_ζ, a part of the T_F fibers degenerate as some of the ϕ_i's vanish. This is nothing but what we have seen for $\mathbb{CP}^1$ (see Fig. 1), and the GLSM provides a systematic generalization of that. As in the case of $\mathbb{CP}^1$, we would like to ask what happens when we perform T-duality on the torus fibers and what is the quantum correction from the singular fibers. This will be the subject of the next section.

As a complex manifold, or as an algebraic variety, X_ζ can be described as the geometric invariant theory quotient of V by the complexified gauge group $G_\mathbb{C}$ with respect to the stability condition determined by ζ. Set theoretically, it is the quotient by $G_\mathbb{C}$ of the complement of the unstable locus F_ζ:

$$X_\zeta = (V \setminus F_\zeta)/G_\mathbb{C}. \tag{4.45}$$

The locus F_ζ is the union of linear subspaces

$$F_\zeta = \bigcup_{I \in \mathcal{I}_\zeta} \{\, \phi_i = 0, \quad \forall i \in I \,\} \tag{4.46}$$

where $\mathcal{I}_\zeta$ is a collection of non-empty subsets of $\{1, \ldots, N\}$. For $v \in i\mathfrak{g}$, let $I_v \subset \{1, \ldots, N\}$

be the set of i's for which $Q_i(v) > 0$. Then,

$$\mathcal{I}_\zeta := \left\{ \emptyset \neq I \subset \{1, \ldots, N\} \;\middle|\; I_v \subset I \;\text{ for some } v \in i\mathfrak{g} \text{ such that } \zeta(v) > 0 \right\}. \tag{4.47}$$

This description makes it clear that the action of the toroidal flavor symmetry group T_F can be extended as the holomorphic action of its complexification $T_{F\mathbb{C}}$. In fact, points with $\phi_i \neq 0 \; \forall i$ (i.e. the μ_F pre-image of the interior points of P_ζ) form a single free orbit of $T_{F\mathbb{C}}$ which is dense open in X_ζ. An algebraic variety with such a torus action is called a *toric variety*.

The standard description of a toric variety is in terms of a *Fan*. From the data given to us, it is given as follows. Let $\mathtt{M}$ and $\mathtt{N}$ be the weight lattice and the cocharacter lattice of T_F respectively:

$$\begin{aligned} \mathtt{M} &= \mathrm{Hom}(T_F, U(1)), & (4.48)\\ \mathtt{N} &= \mathrm{Ker}\,(v \in i\mathfrak{t}_F \mapsto \exp(2\pi i v) \in T_F). & (4.49) \end{aligned}$$

They are naturally dual to each other, and $\mathtt{M} \otimes \mathbb{R} = i\mathfrak{t}_F^*$, $\mathtt{N} \otimes \mathbb{R} = i\mathfrak{t}_F$. Let us define $v^1, \ldots, v^N \in \mathtt{N}$ by

$$v^i = [(0, \ldots, 0, 1, 0, \ldots, 0)] \qquad (1 \text{ at the } i\text{-th intry}), \tag{4.50}$$

so that $\exp(i \sum_{i=1}^N \theta_i v^i) = [(\,\mathrm{e}^{i\theta_1}, \ldots, \,\mathrm{e}^{i\theta_N})]$. Let v^i_α be the components of v^i with respect to the basis determined by (4.41), that is, $v^i = f(v^i_1, \ldots, v^i_{N-k})$. Since $t \mapsto \exp(itv^i)$ has periodicity 2π, the components v^i_α must be all integers. In fact, they define the inverse of (4.41),

$$f^{-1} : [(\omega_1, \ldots, \omega_N)] \in T_F \longmapsto \left(\prod_i \omega_i^{v^i_1}, \ldots, \prod_i \omega_i^{v^i_{N-k}} \right) \in U(1)^{N-k}. \tag{4.51}$$

Since it must be a well-defined map, we have

$$\sum_{i=1}^N v^i_\alpha Q^a_i = 0. \tag{4.52}$$

Since (4.41) and (4.51) are inverse to each other, we have

$$\sum_{i=1}^N v^i_\beta f^\alpha_i = \delta^\alpha_\beta, \qquad \sum_{\alpha=1}^{N-k} f^\alpha_j v^i_\alpha = \delta^i_j + \sum_{a=1}^k Q^a_j \xi^i_a \;\; (\text{for some } \xi^i_a). \tag{4.53}$$

Note that, if $\phi \in V$ satisfies the vacuum equation $\sum_i Q^a_i |\phi_i|^2 = \zeta^a$, we have

$$\begin{aligned} \langle \mu_F(\phi), v^i \rangle &= \sum_\alpha \left(\sum_j f^\alpha_j |\phi_j|^2 \right) v^i_\alpha \\ &= \sum_j \left(\delta^i_j + \sum_a Q^a_j \xi^i_a \right) |\phi_j|^2 = |\phi_i|^2 + \sum_a \zeta^a \xi^i_a. \end{aligned} \tag{4.54}$$

This means that v^i is an inward normal to the polytope $P_\zeta \subset i\mathfrak{t}_F^*$ at the boundary hyperplane corresponding to $\phi_i = 0$, if the latter is present. For a subset $I \subset \{1, \ldots, N\}$, let σ_I be the cone spanned by $\{v^i\}_{i\in I}$. We put $\sigma_\emptyset := \{0\}$. Then, X_ζ is the toric variety corresponding to the Fan

$$\Delta_\zeta = \left\{ \sigma_I \;\middle|\; I \subset \{1, \ldots, N\},\;\; I \notin \mathcal{I}_\zeta \right\}. \tag{4.55}$$

In particular, the one dimensional cones in Δ_ζ are

$$\Delta_\zeta^{(1)} = \left\{ \mathbb{R}_{\geq 0} v^i \;\middle|\; \exists \text{ boundary of } P_\zeta \text{ corresponding to } \phi_i = 0 \right\}. \tag{4.56}$$

5 Mirror Symmetry

5.1 T-duality of Charged Fields

The basic idea is to apply T-duality on the phase of the scalar component of each matter multiplet. To this end, let us first consider the theory is a single scalar ϕ which has charge 1 under a $U(1)$ gauge symmetry. In terms of the polar variables (ρ, φ) defined by $\phi = \rho\, \mathrm{e}^{i\varphi}$, the scalar kinetic term $|D_0\phi|^2 - |D_1\phi|^2$ reads $(\partial_0\rho)^2 - (\partial_1\rho)^2$ plus

$$\mathcal{L}_\varphi = \rho^2(\partial_0\varphi + v_0)^2 - \rho^2(\partial_1\varphi + v_1)^2. \tag{5.1}$$

We now apply T-duality on φ employing the path-integral argument. We consider the system of $\varphi : \Sigma \to \mathbb{R}/2\pi\mathbb{Z}$ and a one-form $J \in \Omega^1(\Sigma, \mathbb{R})$ with Lagrangian

$$\mathcal{L}' = \frac{1}{4\rho^2}\left(J_0^2 - J_1^2\right) + J_1(\partial_0\varphi + v_0) - J_0(\partial_1\varphi + v_1). \tag{5.2}$$

If we integrate out the J-field first, with the center $J_0 = 2\rho^2(\partial_1\varphi + v_1)$ and $J_1 = 2\rho^2(\partial_0\varphi + v_0)$, we get back the Lagrangian $\mathcal{L}_\varphi$ in (5.1). On the other hand, if we integrate out φ first, we obtain the constraint $J_\mu = \partial_\mu\widetilde{\varphi}$ for some map $\widetilde{\varphi} : \Sigma \to \mathbb{R}/2\pi\mathbb{Z}$. Inserting this to the rest of the Lagrangian, we obtain

$$\begin{aligned}
\mathcal{L}_{\widetilde{\varphi}} \;&=\; \frac{1}{4\rho^2}\left((\partial_0\widetilde{\varphi})^2 - (\partial_1\widetilde{\varphi})^2\right) + \partial_1\widetilde{\varphi}\, v_0 - \partial_0\widetilde{\varphi}\, v_1 \\
&\simeq\; \frac{1}{4\rho^2}\left((\partial_0\widetilde{\varphi})^2 - (\partial_1\widetilde{\varphi})^2\right) + \widetilde{\varphi}\, v_{01},
\end{aligned} \tag{5.3}$$

where $\simeq$ means equality up to total derivative. We see that the dual variable $\widetilde{\varphi}$ enters into the action as a dynamical theta angle. Note that the metric for $(\rho, \widetilde{\varphi})$ is given by

$$\mathrm{d}\widetilde{s}^2 = \mathrm{d}\rho^2 + \frac{1}{4\rho^2}\mathrm{d}\widetilde{\varphi}^2 = \frac{|\mathrm{d}(\rho^2) + i\mathrm{d}\widetilde{\varphi}|^2}{4\rho^2} = \frac{|\mathrm{d}y|^2}{2(y + \overline{y})}, \tag{5.4}$$

where $y = \rho^2 + i\widetilde{\varphi}$. If we compare the two expressions for J, we find the relation between the original and the dual variables. Written interms of y and ϕ, it is $(\partial_0 + \partial_1)y = 2\overline{\phi}(D_0 + D_1)\phi$ and $(\partial_0 - \partial_1)y = 2\phi(D_0 - D_1)\overline{\phi}$. Note also that $\mathrm{Re}(y) = |\phi|^2$.

The same can be done for (2,2) supersymmetric version of the theory, that is, the GLSM with $G = U(1)$, $V = \mathbb{C}(1) \ni \phi$, $W = 0$ and $\widetilde{W} = -t\sigma$ (i.e. "the $\mathbb{CP}^0$ model"). After dualization of the phase of ϕ,[1] the matter multiplet (ϕ, ψ, F) turns into a supermultiplet (y, η, G) which transforms under the supersymmetry as

$$\begin{aligned} \delta y &= -\overline{\epsilon}_+\eta_- - \epsilon_-\eta_+ \\ \delta\eta_+ &= i\overline{\epsilon}_-(\partial_0 + \partial_1)y - \overline{\epsilon}_+ G, \qquad \delta\eta_- = i\epsilon_+(\partial_0 - \partial_1)y + \epsilon_- G, \\ \delta G &= i\epsilon_+(\partial_0 - \partial_1)\eta_+ - i\overline{\epsilon}_-(\partial_0 + \partial_1)\eta_-. \end{aligned} \tag{5.5}$$

As in the bosonic case, $\mathrm{Im}(y)$ is the T-dual image $\widetilde{\varphi}$ of the phase φ of ϕ. In particular, y is periodic in the imaginary direction,

$$y \equiv y + 2\pi i. \tag{5.6}$$

Note that y is A-chiral

$$\overline{Q}_+ y = Q_- y = 0. \tag{5.7}$$

As we have seen in Section 3.2, the dualization turns a B-chiral multiplet (ϕ, ψ, F) to an A-chiral multiplet (y, η, G). Therefore, it will be a mirror symmetry. The Lagrangian of the dualized theory is given by

$$\widetilde{\mathcal{L}} = Q_+Q_-\overline{Q}_+\overline{Q}_- \left(-\frac{1}{2e^2}|\sigma|^2 - \mathrm{Re}(y)\log\mathrm{Re}(y)\right) + \mathrm{Re}\, Q_+\overline{Q}_- \Big((y - t)\sigma\Big). \tag{5.8}$$

Note that the dualized theory has the twisted superpotential

$$\widetilde{W}(\sigma, y) = (y - t)\sigma, \tag{5.9}$$

which corresponds to the fact that the dual circle variable $\widetilde{\varphi} = \mathrm{Im}(y)$ plays the rôle of a dynamical theta angle (5.3). Note also that $K = \mathrm{Re}(y)\log\mathrm{Re}(y)$ is indeed the Kähler potential for the metric (5.4). The relationship between the original and the dual variables is

$$\mathrm{Re}(y) = |\phi|^2, \tag{5.10}$$

and others that follow from this by supersymmetry. For example,

$$\begin{aligned} \eta_+ = 2\overline{\phi}\psi_+, &\quad \eta_- = 2\overline{\psi}_-\phi, \\ (\partial_0 + \partial_1)y &= 2\overline{\phi}(D_0 + D_1)\phi - 2i\overline{\psi}_+\psi_+, \\ (\partial_0 - \partial_1)y &= 2\phi(D_0 - D_1)\overline{\phi} + 2i\overline{\psi}_-\psi_-. \end{aligned} \tag{5.11}$$

[1] There is a manifestly (2,2) supersymmetric way to do so [8, 10].

Note that the relation reduces to the bosonic case obtained above when we set the fermionic fields zero.

The relation (5.10) implies that $\mathrm{Re}(y)$ can take only non-genative values, that is, y spans the half-cylinder that ends at $\mathrm{Re}(y) = 0$ where the metric (5.4) blows up. This theory might look strange. However, according to the running of the FI parameter $\zeta' = \zeta + \log(\mu'/\mu)$, we renormalize the dual variable as $y' = y + \log(\mu'/\mu)$ so that the twisted superpotential (5.9) takes the same form at every energy scale. In particular, the variable y_0 at the cut-off scale Λ_0 is related to the variable y at a finite scale μ via

$$y_0 = \log(\Lambda_0/\mu) + y. \tag{5.12}$$

Dualization is performed at the cut-off scale Λ_0, and hence the bound and the metric are $\mathrm{Re}(y_0) \geq 0$ and $\mathrm{d}\widetilde{s}^2 = |\mathrm{d}y_0|^2/(2(y_0 + \overline{y}_0))$, that is,

$$\mathrm{Re}(y) \geq -\log(\Lambda_0/\mu), \tag{5.13}$$

and

$$\mathrm{d}\widetilde{s}^2 = \frac{|\mathrm{d}y|^2}{2(2\log(\Lambda_0/\mu) + y + \overline{y})}. \tag{5.14}$$

For $\Lambda_0/\mu \gg 1$, the end is far away and the metric is approximately that of the flat cylinder.

Similarly to the renormalization, y also transforms non-trivially under the (anomalous) $U(1)$ axial rotations:

$$\mathrm{e}^{i\beta F_A} : y \longrightarrow y - 2i\beta. \tag{5.15}$$

This can be read by computing the operator product expansion of the axial current and $(\partial_0 \pm \partial_1)y$ given in (5.11). Note also that the axial shift of the theta parameter (4.22), which is $\theta \to \theta + 2\beta$ in the present case, is effectively realized by (5.15) through the twisted superpotential (5.9): $\widetilde{W} = (y - \zeta + i\theta)\sigma$. One can also show that y is invariant under the vector $U(1)$ R-symmetry.

The generalization of the dualization to the theories having gauge groups with multiple $U(1)$ factors and multiple matter multiplets is straightforward. Let us consider the general model with connected Abelian gauge group and no superpotential

$$\begin{aligned} G &= U(1)^k, \\ V &= \bigoplus_{i=1}^{N} \mathbb{C}(Q_i) \ni \phi = (\phi_1, \ldots, \phi_N), \\ W &= 0, \\ \widetilde{W} &= -\sum_{a=1}^{k} t^a \sigma_a, \end{aligned}$$

where we again assume that the G action on V is faithful. Dualization of the phase turns the B-chiral multiplet (ϕ_i, ψ_i, F_i) into an A-chiral multiplet (y_i, η_i, G_i), where y_i is periodic in the imaginary part,

$$y_i \equiv y_i + 2\pi i, \tag{5.16}$$

and the Lagrangian of the dualized theory is

$$\widetilde{\mathcal{L}} = Q_+Q_-\overline{Q}_+\overline{Q}_-(-K) + \mathrm{Re}\, Q_+\overline{Q}_-\widetilde{W}, \tag{5.17}$$

where

$$K = \sum_{a=1}^{k} \frac{1}{e_a^2}|\sigma_a|^2 + \sum_{i=1}^{N} \mathrm{Re}(y_i^0) \log \mathrm{Re}(y_i^0), \tag{5.18}$$

$$\widetilde{W} = \sum_{a=1}^{k} \left(\sum_{i=1}^{N} Q_i^a y_i - t^a \right) \sigma_a. \tag{5.19}$$

Here y_i^0 is the dual variable at the cut-off scale Λ_0 which is related to y_i at a physical scale μ by $y_i^0 = \log(\Lambda_0/\mu) + y_i$. The variables y_i transform as $y_i \to y_i - 2i\beta$ under the (possibly anomalous) axial $U(1)$ R-rotation and are invariant under the vector $U(1)$ R-symmetry.

Let us pause for a moment and see what you get in the $\mathbb{CP}^1$ model. In this case, $K = |\sigma|^2/(2e^2) + \mathrm{Re}(y_1^0)\log\mathrm{Re}(y_1^0) + \mathrm{Re}(y_2^0)\log\mathrm{Re}(y_2^0)$ and $\widetilde{W} = (y_1 + y_2 - t)\sigma$. Recall that the GLSM can be regarded as the $\mathbb{CP}^1$ sigma model when $e \gg |\Lambda| = \mathrm{e}^{-\zeta/2}\mu$, for example when $e \to \infty$. We may integrate out the gauge multiplet first, obtaining a constraint $y_1 + y_2 = t$, and we are left with the sigma model with taget space parametrized by $y = y_1$ with metric

$$\mathrm{d}\widetilde{s}^2 = \frac{|\mathrm{d}y|^2}{2(2\log(\Lambda_0/\mu) + y + \overline{y})} + \frac{|\mathrm{d}y|^2}{2(2\log(\Lambda_0/\mu) + 2\zeta - y - \overline{y})}. \tag{5.20}$$

Note that the real part of $y^0 = y + \log(\Lambda_0/\mu)$, is bounded as $0 \leq \mathrm{Re}(y^0) \leq \zeta_0$ where $\zeta_0 = \zeta + 2\log(\Lambda_0/\mu)$ is the FI parameter at the cut-off scale Λ_0. This is essentially the same geometry as depicted in Fig. 2, which we know to have some serious problems as the dual to $\mathbb{CP}^1$. Thus, the above cannot be the whole story.

Similarly, in a general model, after integrating out the gauge multiplet, we obtain the sigma model to the naïve dual of the torus fibration $\mu_F : X_{\zeta_0} \to P_{\zeta_0}$ where ζ_0 is the FI parameter at the cut-off scale. The space of $\mathrm{Re}(y_i^0)$'s obeying the bound $\mathrm{Re}(y_i^0) \geq 0$ and the constraints $\sum_i Q_i^a \mathrm{Re}(y_i^0) = \zeta_0^a$ is projected isomorphically onto the moment polytope P_{ζ_0} by $(\mathrm{Re}(y_i^0)) \mapsto (\sum_i f_i^\alpha \mathrm{Re}(y_i^0))$, and $\mathrm{Im}(y_i^0)$'s obeying $\sum_i Q_i^a \mathrm{Im}(y_i^0) \equiv -\theta^a \pmod{2\pi\mathbb{Z}}$ parametrize the dual fiber. This has the same problem as in the $\mathbb{CP}^1$ model, and again cannot be the whole story.

5.2 Superpotential Generation

The Lagrangian of the dual system, (5.17)-(5.19), is certainly not exact. We have ignored many things in the dualization procedure. For example, when we integrated over the J-field in (5.2), we pretended that ρ is a fixed non-zero constant, but that is actually a fluctuating field. A usual way to incorporate the correction is to expand the field around a classical value, say at a large value of $\rho_i^2 = \mathrm{Re}(y_i)$, and then perform the path-integral for each term of the expansion. Such corrections are called *perturbative corrections* and takes the form of power series in the "coupling constant", which is $1/\mathrm{Re}(y_i)$ in the present case. There will be full of such corrections to the Kähler potential (5.18). However, for the twisted superpotential (5.19), there is no room for perturbative corrections — it is simply not possible to write down a possible correction term that is compatible with holomorphy, (possibly anomalous) R-symmetry, and periodicity $y_i \equiv y_i + 2\pi i$. That is, (5.19) is perturbatively exact.

However, there can be *non-perturbative* corrections to the twisted superpotential. A typical non-perturbative effect is generated by instantons. In a two-diemsional Abelian gauge theory with broken gauge symmetry like our GLSM, instantons are *vortices*. A vortex is a configuration of a charged scalar ϕ and a gauge potential v minimizing the action, with the following feature: ϕ vanishes at a point p; $|\phi|$ is nearly equal to the (non-zero) vacuum value away from a small disc D around p but the phase of ϕ has a non-trivial winding number along a circle around D; the gauge potential is nearly flat $\mathrm{d}v = 0$ away from D and ϕ is nearly parallel with repect to v there. A single vortex may be regarded as something that creates one unit of winding number for the phase φ of ϕ. Since T-duality exchanges the winding number and the momentum, it is dual to something that creates one unit of momentum, and that is provided by $\mathrm{e}^{\pm i\widetilde{\varphi}}$. The A-chiral operator $\mathrm{e}^{\pm y}$ is its supersymmetric completion, and one of them,

$$\mathrm{e}^{-y}, \tag{5.21}$$

has vector R-charge 0 and axial R-charge 2 (see (5.15)) so that it can be added to the twisted superpotential. We shall show that a correction of this form is indeed generated by a non-perturbative effect. In fact, we will show that the correction is simply the sum of such terms, one for each ϕ_i, and the exact twisted superpotential is

$$\widetilde{W} = \sum_{a=1}^{k} \left(\sum_{i=1}^{N} Q_i^a y_i - t^a \right) \sigma_a + \mu \sum_{i=1}^{N} \mathrm{e}^{-y_i}. \tag{5.22}$$

Step 1 — Reduction to the $\mathbb{CP}^0$ model

First, we show that if the assertion that (5.22) is the exact twisted superpotential is true for the $\mathbb{CP}^0$ model, then, it must be ture for the general model. Let us start with the direct product of N-copies of the $\mathbb{CP}^0$ model. Applying the T-duality in each of them, assuming that the assertion is true, we obtain the direct product of N-copies of the dual theory. In particular, we have $\widetilde{K} = \sum_{i=1}^N \widetilde{K}_i$ and $W = \sum_{i=1}^N W_i$ where $\widetilde{K}_i$ is some complicated function of σ_i and y_i and $W_i = (y_i - t_i)\sigma_i + \mu\,\mathrm{e}^{-y_i}$. Next, we deform the original theory by changing the kinetic term of the gauge multiplet, by deforming $K_{\text{gauge}} = \sum_i |\sigma_i|^2/(2e_i^2)$. For this, we take a decomposition $U(1)^N \cong G \times U(1)^{N-k}$ and accordingly write $\sigma_i = \sum_{a=1}^k Q_i^a \sigma_a + \sum_{f=k+1}^N P_i^f \sigma_f$. And then we deform K_{gauge} to

$$K'_{\text{gauge}} = \sum_{a=1}^{k} \frac{1}{2e_a^2}|\sigma_a|^2 + \sum_{f=k+1}^{N} \frac{1}{2e_f^2}|\sigma_f|^2. \tag{5.23}$$

This defomation will certainly change $\widetilde{K}$ but cannot change $\widetilde{W}$ because the deformation parameter is not A-chiral. That is, $\widetilde{W}$ remains to be the sum $\sum_{i=1}^N \widetilde{W}_i$. At this stage, we take the limit where $e_f \to 0$ for $f = k+1, \ldots, N$. This freezes the components of the gauge multiplet corresponding to $U(1)^{N-k}$, and the original system reduces to the system with gauge group G under consideration. The FI-theta parameter at the scale μ is $t^a = \sum_{i=1}^N Q_i^a t_i$ as one can see by matching the effective twisted superpotentials. On the dual side, $\widetilde{W}$ remains the same except that σ_i is now constrained, $\sigma_i = \sum_{a=1}^k Q_i^a \sigma_a$, and this is nothing but the one in (5.19).

Step 2 — Exclusion of other corrections

Second, in the $\mathbb{CP}^0$ model, we show that

$$\Delta\widetilde{W} = C \cdot \mu\,\mathrm{e}^{-y} \tag{5.24}$$

for some constant C is the only possible correction to the twisted superpotential. The twisted superpotential $\widetilde{W}$ must be a holomorphic function of RG invariant A-chiral operators and parameters, which are σ, $\widetilde{y} = y - t$ and $\Lambda = \mathrm{e}^{-t}\mu$. Since $\widetilde{y}$ has periodicity $\widetilde{y} \equiv \widetilde{y} + 2\pi i$, it must depend on $\widetilde{y}$ through $\mathrm{e}^{-\widetilde{y}}$, except for the classical term $\widetilde{y}\sigma$ which is admissible since 2π shift of the theta parameter is admissible. Also, it must have axial R-charge 2 provided the FI-theta parameter transforms as $t \to t - 2i\beta$. Note that σ, $\mathrm{e}^{-\widetilde{y}}$ and Λ has axial R-charge 2, 0 and 2 respectively. By these conditions, a possible correction $\Delta\widetilde{W}$ must be of the form $\Lambda f(\sigma/\Lambda,\, \mathrm{e}^{-\widetilde{y}})$ for some holomorphic function $f(z, w)$. Since the gauge symmetry is broken by $\phi \neq 0$, the twisted superpotential must be regular at $\sigma = 0$. Also, the correction is expected to be small at large values of $\mathrm{Re}(y)$ that corresponds to large values of $|\phi|$. Thus, the correction cannot grow as $\mathrm{Re}(\widetilde{y}) \to +\infty$. Therefore the function $f(z, w)$ must be analytic at $(z, w) = (0, 0)$ and can be expanded as a power series in z and w,

$$\Delta\widetilde{W} = \Lambda \sum_{n,m\geq 0} C_{n,m}(\sigma/\Lambda)^n \,\mathrm{e}^{-m\widetilde{y}} = \sum_{n,m\geq 0} C_{n,m}\sigma^n \mu^{1-n}\,\mathrm{e}^{-t+(n+m)t}\,\mathrm{e}^{-my}. \tag{5.25}$$

Furtheremore, the correction is expected to be small at large values of $\zeta = \mathrm{Re}(t)$, and hence only the terms with $n+m \leq 1$ can be genertated. $(n, m) = (0, 0)$ is a constant and can be ignored. $(n, m) = (1, 0)$ is of the same form as the classical term and hence must vanish. Thus, we can only have $(n, m) = (0, 1)$ which is (5.24).

Step 3 — Instanton calculus

Now, we show that (5.24) is indeed generated with a non-zero coefficient C.

Step 4 — Coefficient

Finally, we show that the coefficient C is 1. Let us integrate out the y field at large values of σ. This is done by solving the equation $\partial_y\widetilde{W} = 0$ for y and then inserting the answer back to $\widetilde{W}$. The equation is $\sigma - C\mu\,\mathrm{e}^{-y} = 0$ and the solution is $y = -\log(\sigma/C\mu)$. Inserting it back to $\widetilde{W}$, we obtain

$$\widetilde{W} = (-\log(\sigma/C\mu) - t)\,\sigma + \sigma = -t\sigma - \sigma\,(\log(\sigma/C\mu) - 1)\,. \tag{5.26}$$

This is nothing but the effective twisted superpotential of the $\mathbb{CP}^0$ model at large σ, see (4.16), provided that C is 1. This shows that $C = 1$. We can also start with (5.22) and obtain the correct effective twisted superpotential (4.16) in the general model.

5.3 Derivation of Mirror Symmetry 1 — Theory Without Superpotential

Let us integrate out the $U(1)^k$ gauge multiplet. This yields the constraint

$$\sum_{i=1}^{N} Q_i^a y_i - t^a = 0, \qquad a = 1, \ldots, k, \tag{5.27}$$

which defines an algebraic torus $\mathbf{T}_{Q,t}^{N-k} \subset (\mathbb{C}^\times)^N$ isomorphic to $(\mathbb{C}^\times)^{N-k}$. And we are left with the twisted superpotential

$$\widetilde{W} = \mu \sum_{i=1}^{N} \mathrm{e}^{-y_i}. \tag{5.28}$$

That is, the mirror of the toric sigma model is the Landau-Ginzburg model with target $\mathbf{T}_{Q,t}^{N-k}$ and the superpotential $\widetilde{W} = \mu \sum_{i=1}^{N} \mathrm{e}^{-y_i}$.

For example, the mirror of the $\mathbb{CP}^1$ model is the Landau-Ginzburg model with target $\mathbf{T}^1_{(1,1),t}$, which is isomorphic to $\mathbb{C}^\times \ni y$ via $(y_1, y_2) = (y, t-y)$, and the superpotential $\widetilde{W} = \mu\,\mathrm{e}^{-y} + \mu\,\mathrm{e}^{-t+y}$. In terms of an RG-invariant variable $\widetilde{y} = y - t/2$ and the RG-invariant scale parameter $\Lambda = \mathrm{e}^{-t/2}\mu$, it is

$$\widetilde{W} = \Lambda \left(\mathrm{e}^{-\widetilde{y}} + \mathrm{e}^{\widetilde{y}} \right). \tag{5.29}$$

This is the mirror symmetry claimed in Section 3.3. Similarly, the mirror of the $\mathbb{CP}^{N-1}$ model is the Landau-Ginzburg model with target $(\mathbb{C}^\times)^{N-1}$ and superpotential $\widetilde{W} = \mu\left(\mathrm{e}^{-y_1} + \cdots + \mathrm{e}^{-y_{N-1}} + \mathrm{e}^{-t+y_1+\cdots+y_{N-1}}\right)$, or

$$\widetilde{W} = \Lambda \left(\mathrm{e}^{-\widetilde{y}_1} + \cdots + \mathrm{e}^{-\widetilde{y}_{N-1}} + \mathrm{e}^{\widetilde{y}_1+\cdots+\widetilde{y}_{N-1}} \right), \tag{5.30}$$

in terms of RG-invariant variables and parameter.

5.4 Derivation of Mirror Symmetry 2 — Theory With Superpotential

Let us consider the model $\mathrm{T}^{U(1)}_{N,d}$. Suppose, for now, that the superpotential is zero, $W = 0$. Then, we know that it is equivalent to the system with twisted superpotential

$$\widetilde{W} = \mu \left(\mathrm{e}^{-y_1} + \cdots + \mathrm{e}^{-y_N} + \mathrm{e}^{-y_P} \right). \tag{5.31}$$

where $y_1, \ldots, y_N, y_P$ are $\mathbb{C}/2\pi i\mathbb{Z}$-valued A-chiral variables obeying the constraints $y_1 + \cdots + y_N - dy_P = t$, that is,

$$\mathrm{e}^{-dy_P} = \mathrm{e}^{t}\,\mathrm{e}^{-y_1} \cdots \mathrm{e}^{-y_N}. \tag{5.32}$$

Let us now recover the superpotential $W = pf(x)$ in the original theory. This cannot alter the twisted superpotential (5.31) because B-chiral parameter cannot enter there. However, this does change the Kähler potential in a rather drastic way, and that forces us to change the field variables. There is a way to find the good variables using something that is known as the D-brane central charge. According to that, at a special point in $\mathfrak{M}_W$, the good variables are $\mathbb{C}$ valued variables $\widetilde{x}_1, \ldots, \widetilde{x}_N$ which are related to y's via

$$\mathrm{e}^{-y_1} = \widetilde{x}_1^d, \quad \ldots \quad \mathrm{e}^{-y_N} = \widetilde{x}_N^d, \quad \mathrm{e}^{-y_P} = \mathrm{e}^{t/d}\widetilde{x}_1 \cdots \widetilde{x}_N. \tag{5.33}$$

Note that these indeed satisfy the constraint (5.32). The twisted superpotential (5.31) is written in terms of the new variables as

$$\widetilde{W} = \mu \left(\, \widetilde{x}_1^d + \cdots + \widetilde{x}_N^d + \mathrm{e}^{t/d} \widetilde{x}_1 \cdots \widetilde{x}_N \, \right). \tag{5.34}$$

Note that the map from $\widetilde{x}$'s to y's is not one to one. To fix this, we need to identify $(\widetilde{x}_1, \ldots, \widetilde{x}_N)$ and $(\omega_1 \widetilde{x}_1, \ldots, \omega_N \widetilde{x}_N)$ where $\omega_1^d = \cdots = \omega_N^d = \omega_1 \cdots \omega_N = 1$. That is, the mirror of the model $\mathrm{T}_{N,d}^{U(1)}$ is the Landau-Ginzburg orbifold $(\mathbb{C}^N/(\mathbb{Z}_d)^{N-1}, (5.34))$.

5.5 Examples

Let us discuss some physics of the mirror theories in some examples.

The $\mathbb{CP}^{N-1}$ Model

The Model $\mathbf{H}_n$

The mirror of the model H_n is the Landau-Ginzburg model of $\mathbb{C}/2\pi i\mathbb{Z}$ variables $y_1, \ldots, y_4$ subject to $y_1 + y_2 - ny_4 = t^1$ and $y_3 + y_4 = t^2$ with superpotential $\widetilde{W} = \mu \sum_{i=1}^4 \mathrm{e}^{-y_i}$. Solving the constanits, say, as $y_2 = t^1 + nt^2 - y_1 - ny_3$ and $y_4 = t^2 - y_3$, and writing $x_1 := \mathrm{e}^{-y_1}$ and $x_2 := \mathrm{e}^{-y_3}$, we see that it is the model with target $\{(x_1, x_2)\} = (\mathbb{C}^\times)^2$ and superpotential

$$\widetilde{W} = \mu \left(x_1 + \frac{q_1 q_2^n}{x_1 x_2^n} + x_2 + \frac{q_2}{x_2} \right), \tag{5.35}$$

where $q_a := \mathrm{e}^{-t_a}$. As discussed in Section 4.4, the model H_n is identified as the sigma model with target $\mathbb{F}_n$ (for $n = 0, 1, 2$) or $\mathrm{W}\mathbb{P}^2_{1,1,n}$ (for $n \geq 2$). Therefore, the above LG model can be regarded as the mirror of these sigma models. The vacuum equation $\mathrm{d}\widetilde{W} = 0$ reads

$$x_1 - \frac{q_1 q_2^n}{x_1 x_2^n} = 0, \qquad -n \frac{q_1 q_2^n}{x_1 x_2^n} + x_2 - \frac{q_2}{x_2} = 0. \tag{5.36}$$

It agrees with (4.35) under $\mu x_1 = \sigma_1$ and $\mu x_2 = \sigma_2$; no surprize since the x-σ relation comes out of $\partial_{y_1} \widetilde{W} = \partial_{y_3} \widetilde{W} = 0$ for $\widetilde{W}$ in (5.22) that involves both σ's and y's.

$\underline{n=0}$ The model H_0 is identified as the sigma model with target $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. The mirror superpotential (5.35) is indeed the sum of two copies of the mirror superpotential for $\mathbb{CP}^1$.

$\underline{n=1}$ The model H_1 is identified as the sigma model with target $\mathbb{F}_1 =$ the one point blow up of $\mathbb{CP}^2$. We recall that the theory having $q_1 = \Lambda/\mu$ and $q_2 = q(\Lambda/\mu)^2$ with

$q \ll 1$ is in the Phase II (corresponding to $\mathbb{CP}^2$) for $q^{\frac{1}{3}}\Lambda \ll \mu \ll \Lambda$. Let us see how things look in the mirror description. Translating the earlier computation, we see that there are three critical points with $x_1 \sim x_2$, $x_1^3 \sim q(\Lambda/\mu)^3$, and a single critical points with $x_1 \sim -\Lambda/\mu$, $x_2 \sim q\Lambda/\mu$. For the window of scales under consideration, we see that $|x_1| \ll 1$, $|x_2| \ll 1$, $|q_1q_2|/|x_1x_2| \ll 1$ at the former three critical points but $|x_1| \gg 1$, $|x_2| \ll 1$, $|q_1q_2|/|x_1x_2| \gg 1$ at the last single critical point. That is, $(\mathrm{Re}(y_1), \mathrm{Re}(y_3))$ is in the momentum polytope for $\mathbb{CP}^2$ with $\zeta = \zeta^1 + \zeta^2$ for the former three but outside it for the last single (see Fig. 6). This makes it clear that the former three critical points correspond to the vacua of the $\mathbb{CP}^2$ sector of the model. If we take the limit $q \to 0$ holding $\Lambda_{\mathbb{P}^2}^3 = q\Lambda^3$ fixed, which takes out purely $\mathbb{CP}^2$ out of H_1, the superpotential becomes

$$\widetilde{W} = \mu\left(x_1 + \frac{(\Lambda_{\mathbb{P}^2}/\mu)^3}{x_1x_2} + x_2 + \frac{q^{\frac{1}{3}}(\Lambda_{\mathbb{P}^2}/\mu)^2}{x_2} \right) \longrightarrow \mu\left(x_1 + \frac{(\Lambda_{\mathbb{P}^2}/\mu)^3}{x_1x_2} + x_2 \right). \tag{5.37}$$

Indeed this is nothing but the mirror superpotential for $\mathbb{CP}^2$.

$\underline{n=2}$ The model H_2 is identified with the sigma model with target $\mathbb{F}_2$ for $q_1 \ll 1$ and $\mathrm{W}\mathbb{P}^2_{1,1,n}$ for $q_1 \gg 1$. With $q_2 = (\Lambda/\mu)^2$, the mirror superpotential is written as

$$\widetilde{W} = \mu\left(x_1 + \frac{q_1(\Lambda/\mu)^4}{x_1x_2^2} + x_2 + \frac{(\Lambda/\mu)^2}{x_2} \right). \tag{5.38}$$

The critical points are

$$x_1 = \frac{\epsilon_2\epsilon_1 q_1^{\frac{1}{2}}\Lambda/\mu}{(1+2\epsilon_1 q_1^{\frac{1}{2}})^{\frac{1}{2}}}, \qquad x_2 = \epsilon_2(1+2\epsilon_1 q_1^{\frac{1}{2}})^{\frac{1}{2}}\Lambda/\mu, \qquad \epsilon_1, \epsilon_2 \in \{\pm 1\}. \tag{5.39}$$

At high energies $\mu \gg \Lambda$ with $q_1 \ll 1$, they satisfy $|x_1| \ll 1$, $|x_2| \ll 1$, $|q_1q_2^2|/|x_1x_2^2| \ll 1$, $|q_2|/|x_2| \ll 1$, that is, $(\mathrm{Re}(y_1), \mathrm{Re}(y_3))$ is in the momentum polytope for $\mathbb{F}_2$ (Fig. 8-right). At high energies $\mu \gg \Lambda$ with $q_1 \gg 1$, they satisfy $|x_1| \ll 1$, $|x_2| \ll 1$, $|q_1q_2^2|/|x_1x_2^2| \ll 1$, that is, $(\mathrm{Re}(y_1), \mathrm{Re}(y_3))$ is in the momentum polytope for $\mathrm{W}\mathbb{P}^2_{1,1,2}$ (Fig. 8-left). In fact, this is a consequence of the general fact we have already seen — at high energies, $(\mathrm{Re}(y_1), \mathrm{Re}(y_3))$ ranges over the momentum polytope of the UV target space, provided that the critical points are at finite points in the renormalized variables.

$\underline{n \geq 3}$ The model H_n is identified as the sigma model with target $\mathrm{W}\mathbb{P}^2_{1,1,n}$. We recall that the theory having $q_1 = (\Lambda/\mu)^{2-n}$ and $q_2 = q(\Lambda/\mu)^2$ with $q \ll 1$ is in the Phase I (corresponding to $\mathbb{F}_n$) for $q^{\frac{1}{2}}\Lambda \ll \mu \ll \Lambda$. Let us see how things look in the mirror description. Translating the earlier computation, we see that there are four critical points with $x_1 \sim \epsilon_1 q^{\frac{n}{4}}\Lambda/\mu$, $x_2 \sim \epsilon_2 q^{\frac{1}{2}}\Lambda/\mu$, and $(n-2)$ critical points with $x_1^{n-2} \sim (-n)^{-n}(\Lambda/\mu)^{n-2}$, $x_2 \sim -q(\Lambda/\mu)^2/x_1$. For the window of scales under consideration, we see that $|x_1| \ll 1$,

$|x_2| \ll 1$, $|q_1 q_2^n|/|x_1 x_2^n| \ll 1$, $|q_2|/|x_2| \ll 1$ for the former four critical points while $|x_1| \gg 1$, $|x_2| \ll 1$, $|q_1 q_2^n|/|x_1 x_2^n| \gg 1$, $|q_2|/|x_2| \gg 1$ for the latter $(n-2)$ critical points. That is, $(\mathrm{Re}(y_1), \mathrm{Re}(y_3))$ is in the momentum polytope for $\mathbb{F}_n$ for the former four but outside it for the latter $(n-2)$ (see Fig. 8-right). This make it certain that the former four critical points correspond to the vacua of the $\mathbb{F}_n$ sector of the model. However, as we have discussed earlier, there is no limit that isolate purely $\mathbb{F}_n$ out of H_n. Instead, let us see what happens to the superpotential in the limit $q \to 0$ holding Λ fixed, that reduces the theory with $|\phi_3|^2 \sim \zeta^2$ to $\mathrm{T}^{U(1)}_{2,n}(0)$:

$$\begin{aligned} \widetilde{W} &= \mu \left(x_1 + \frac{q_1 q_2^n}{x_1 x_2^n} + x_2 + \frac{q_2}{x_2} \right) = \mu \left(x_1 + \frac{(\Lambda/\mu)^{2-n}}{x_1 \widetilde{x}_2^n} + q(\Lambda/\mu)^2 \widetilde{x}_2 + \frac{1}{\widetilde{x}_2} \right) \\ &\longrightarrow \mu \left(x_1 + \frac{(\Lambda/\mu)^{2-n}}{x_1 \widetilde{x}_2^n} + \frac{1}{\widetilde{x}_2} \right). \end{aligned} \tag{5.40}$$

Here we use the variables x_1 and $\widetilde{x}_2 := x_2/q_2$ since we are looking at the region in the field space with $|\phi_3|^2 \sim \zeta^2$. This is nothing but the mirror superpotential for the model $\mathrm{T}^{U(1)}_{2,n}(0)$.

The Model $\mathbf{T}^{U(1)}_{N,d}$

As a test, let us look for the critical points of (5.34). When $d = N$ (CY case), there is a unique critical point at the origin, except in the case where $\mathrm{e}^t = (-N)^N$, which is the discriminant locus, see (4.33). When $d \neq N$, the origin is an isolated critical points but there are also $|N-d|$ critical points away from the origin (counted after dividing out by the orbifold group). These other critical points correspond to the massive Coulomb vacua. At the critical point at the origin, for $d < N$ (*resp.* $d > N$), the last term of (5.34) is of higher (*resp.* lower) order compared to the first N terms and hence is irrelevant (*resp.* relevant). Therefore, in the deep IR (*resp.* UV), the theory at the origin is the LG orbifold

$$\widetilde{W} = \widetilde{x}_1^d + \cdots + \widetilde{x}_N^d \,/(\mathbb{Z}_d)^{N-1}. \tag{5.41}$$

This is the mirror to the corresponding theory in the original model, that is, the LG orbifold $W = f(x_1, \ldots, x_N)/\mathbb{Z}_d$. Indeed, this is true when f is Fermat, $f(x) = x_1^d + \cdots + x_N^d$, the well-known Greene-Plesser mirror symmetry (an example of LG-LG mirror symmetry of Berglund-Hubsch-...).

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