Exercises: Cosmology (Masahiro Takada)
Due date: Aug 5th (Wed)
submitted to Report Box 1 at the Register Office of Physics Dept. (物理教務)

Answer the following problems (以下の設問に答えよ。日本語で解答しても良い。)

1. The Hubble expansion rate for the universe at sufficiently high redshift is given by

$$H(t)^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[\bar{\rho}_{\gamma 0}a^{-4} + \bar{\rho}_{\rm m0}a^{-3}\right], \tag{1}$$

where $\bar{\rho}_{\gamma 0}$ and $\bar{\rho}_{\rm m0}$ are the present-day energy densities of radiation (relativistic matter) and non-relativistic matter, respectively, and the notation '˙' denotes its derivative with respect to time (t). Derive the relation between the scale factor and time, $a(t)$, in the radiation and matter dominated era, respectively. Note that the contributions of the cosmological constant and the curvature are approximately ignored at high redshift.

2. For an Einstein de-Sitter universe ($\Omega_{\rm m0} = 1, \Omega_\Lambda = 0, \Omega_K = 0$), derive the distance-redshift relation for the angular diameter distance and the luminosity distance, respectively (express the distances $d_A(z)$ and $d_L(z)$ as a function of redshift z). In the derivation, ignore the radiation contribution to the cosmic expansion rate for simplicity.

3. Explain why the cosmological constant (Λ) causes an accelerated expansion of the universe, and explain why the ordinary matter (radiation and matter) cannot cause the acceleration.

4. Express the redshift of matter-radiation equality ($z_{\rm eq}$) in terms of $\Omega_{\rm m0}h^2$. In calculating $z_{\rm eq}$, adopt $T_0 = 2.725$K for the present-day temperature of CMB and take into account the neutrino contribution to the energy density of relativistic species. For simplicity, assume that the neutrinos are massless.

5. Neutrinos are massive as revealed by terrestrial experiments such as Kamiokande and Super-K. However, the absolute mass scale of neutrino hasn't yet been measured. In cosmology, the neutrinos behave as relativistic species at sufficiently high redshift, and then become non-relativistic. Discuss when (around which redshift) the neutrinos become relativistic and derive the expression for the present-day energy density of the neutrinos, $\Omega_{\nu 0}h^2 = m_{\nu,\rm tot}/(94.1\ \text{eV})$. Here $m_{\nu,\rm tot}$ is the sum of masses of the standard three-flavor neutrinos, in the units of [eV].

6. Discuss the horizon problem and explain why an inflationary scenario (the accelerated expansion in the early universe) can resolve the horizon problem.

7. Consider the matter dominated regime ($\Omega_{\rm m}(a) \simeq 1$). Here $\Omega_m(a)$ is the energy density parameter at epoch a, defined as $\Omega_m(a) \equiv \bar{\rho}_{\rm m}(a)/H^2(a) = \bar{\rho}_{\rm m0}a^{-3}/H^2(a)$. The differential equation to govern the time evolution of the total matter density fluctuation is given by

$$\ddot{\delta}_{\rm m} + 2H\dot{\delta}_{\rm m} - 4\pi G\bar{\rho}_{\rm m}\delta_{\rm m} = 0. \tag{2}$$

Derive the growth rate $D(a)$, where $\delta_{\rm m}(\mathbf{k}, a) \propto D(a)$.

8. In the presence of massive neutrinos, we need to consider the total matter density fluctuation defined by

$$\begin{aligned} \delta\rho_{\rm m}(\mathbf{x},t) &= \bar{\rho}_{\rm m}(t)\delta_{\rm m}(\mathbf{x},t) \\ &= \bar{\rho}_{\rm cdm}\delta_{\rm cdm} + \bar{\rho}_{\rm b}\delta_{\rm b} + \bar{\rho}_{\nu}\delta_{\nu} \\ &= \bar{\rho}_{\rm cb}(t)\delta_{\rm cb}(\mathbf{x},t) + \bar{\rho}_{\nu}(t)\delta_{\nu}(\mathbf{x},t), \end{aligned} \tag{3}$$

where we assumed that the CDM and baryon fluctuations are indistinguishable, $\delta_{\rm c} = \delta_{\rm b}$, and the subscript 'cb' denotes the CDM plus baryon. Hence, using the ratio of the present-day energy density of massive neutrinos to the total matter, defined as $f_\nu = \Omega_{\nu 0}/\Omega_{\rm m0}$, the total matter fluctuation field is

$$\delta_{\rm m} = (1 - f_\nu)\delta_{\rm cb} + f_\nu \delta_\nu. \tag{4}$$

Consider the fluctuations with scales much below the neutrino free-streaming scale, where the neutrino fluctuation is absent ($\delta_\nu \simeq 0$). In this case, the differential equation to govern the time evolution of CDM plus baryon fluctuation is given as

$$\ddot{\delta}_{\rm cb} + 2H\dot{\delta}_{\rm cb} - 4\pi G\bar{\rho}_{\rm m}(1 - f_\nu)\delta_{\rm cb} = 0. \tag{5}$$

For the Einstein de-Sitter model with $\Omega_{\rm m0} = \Omega_{\nu 0} + \Omega_{\rm cb0} = 1$, compare the growth of the total matter fluctuation ($\delta_{\rm m}$) from $z = 1000$ to $z = 0$, i.e. $D(z = 0)/D(z = 1000)$, for the two models with and without the massive neutrinos. In these calculations, assume that the neutrinos have been non-relativistic from $z = 1000$ to today, and then consider the effect up to the linear order of f_ν assuming $f_\nu \ll 1$. Also discuss the dependence of the effect on the neutrino mass.