Higher-order moments

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What are higher-order moments?

\[ \langle \gamma \gamma \rangle \]
\[ \langle \gamma \gamma \gamma \rangle \]
\[ \langle \gamma \gamma h \rangle \]

- Anything beyond 2pt statistics \( \langle \gamma \gamma \rangle \) or \( \langle \gamma h \rangle \) ...
- Note: the peak statistics may carry complementary information to the 2pt statistics, but I don’t cover here
• We know the seed fluctuations are Gaussian
• The power spectrum carries the full information contained in a Gaussian field
\[
\langle xx \rangle = 0
\]
\[
\langle xxx \rangle \rightarrow \langle xx \rangle \langle xx \rangle
\]
The error of band power measurement for a Gaussian field (shot noise ignored)

\[
\frac{\sigma^2(C_l)}{C^2_l} = \frac{2}{N_l} \propto \frac{1}{l \Delta f_{\text{sky}}}
\]

The information content of power spectrum = the sum of the number of Fourier modes

\[
\left( \frac{S}{N} \right)^2 \equiv \sum_{l=l_{\text{min}}}^{l_{\text{max}}} \left( \frac{C_l}{\sigma(C_l)} \right)^2 \propto l_{\text{max}}^2 f_{\text{sky}}
\]

WMAP: Imax~1000
Planck: Imax~3000 (the improvement equivalent to 9 different universes)
Weak lensing case

- The information content of WL power spectrum is (significantly) smaller than the Gaussian expectation (also see Lee & Pen 08; MT & Jain 09; Yu+09)
- The power spectrum is not enough in WL case
- Where is the information contained in the initial field gone? The initial information is lost?

Information lost is more significant for lower source redshifts
Cosmological complementarity

• Let’s consider convergence field
\[ \kappa(\theta) = \int d\chi W(\chi)\delta(\theta) \]
where \( W(\chi) \propto \Omega_{m0} \)

• Consider the 3-point function of convergence, which is the lowest-order function to extract the non-Gaussian signal
  – The lensing 3pt function arises from the 3pt function of density fluctuation field
\[ \langle \kappa \kappa \kappa \rangle = \int d\chi d\chi' d\chi''W(\chi)W(\chi')W(\chi'')\langle \delta \delta \delta \rangle \]

• The perturbation theory for nonlinear structure formation (e.g., Bernardeau et al. 02 Phys. Rep.) predicts
\[ \delta = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \ldots \]
\[ \left( \delta^{(1)} \right)^2 \quad \left( \delta^{(1)} \right)^3 \]
• The 3pt function (bispectrum) of density fluctuation field is found to be

\[
\langle \delta \delta \delta \rangle = \left\langle \left( \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \ldots \right) \left( \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \ldots \right) \left( \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \ldots \right) \right\rangle \\
= \left\langle \delta^{(1)} \delta^{(1)} \delta^{(1)} \right\rangle + \left\langle \delta^{(1)} \delta^{(1)} \delta^{(2)} \right\rangle + \ldots + \left\langle \delta^{(1)} \delta^{(2)} \delta^{(2)} \right\rangle + \ldots + \left\langle \delta^{(2)} \delta^{(2)} \delta^{(2)} \right\rangle + \ldots
\]

non-vanishing lowest-order term

\[
\left\langle \tilde{\delta}(k_1) \tilde{\delta}(k_2) \tilde{\delta}(k_3) \right\rangle = (2\pi)^3 B_\delta(k_1, k_2, k_3) \delta_D^3(k_1 + k_2 + k_3)
\]

\[
B_\delta(k_1, k_2, k_3) = F(k_1, k_2) P(k_1) P(k_2) + \text{(cyc.)}
\]

where \( F(k_1, k_2) \equiv \frac{10}{7} + \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) (k_1 \cdot k_2) + \frac{4}{7} \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2} \)

• Hence, using the Limber’s approximation, the lensing 3pt function can be given in terms of the mass bispectrum as

\[
B_\kappa(l_1, l_2, l_3) = \int d\chi W^3(\chi) \chi^{-4} B_\delta(k_1, k_2, k_3) \bigg|_{k_i = l_i/\chi}
\]

where \( \left\langle \tilde{\kappa}(l_1) \tilde{\kappa}(l_1) \tilde{\kappa}(l_1) \right\rangle \equiv (2\pi)^2 B_\kappa(l_1, l_2, l_3) \delta_D^2(l_1 + l_2 + l_3) \)
• The lensing power spectrum (2pt) and bispectrum (3pt) depend on the lensing efficiency function and the mass power spectrum amplitude

\[ P_\kappa(l) = \int d\chi W^2(\chi)\chi^{-2}P_\delta(k = l/\chi) \propto W^2(\chi)P_\delta \propto \Omega_{m0}^2\sigma_8^2 \]

\[ B_\kappa(l_1,l_2,l_3) = \int d\chi W^3(\chi)\chi^{-4}B_\delta(k_1,k_2,k_3)\bigg|_{k_i = l_i/\chi} \propto W^3P_\delta^2 \propto \Omega_{m0}^3\sigma_8^4 \]

• Hence combining the 2pt and 3pt lensing correlations can break parameter degeneracies (Hui 99; MT & Jain 04): e.g.,

\[ \frac{B_\kappa}{P_\kappa^2} \propto \Omega_{m0}^{-1} \]
WL 2pt+3pt tomography
WL 2pt+3pt tomography

A factor 2-3 improvement (by recovering the initial information content via 3pt function)

This forecast includes all the triangle configurations for a given lmax=3000

Non-Gaussian errors need to be included
Prospect for WL 3pt measurement

Error bars = 1000 sq. deg. (no shot noise)

Estimate the lensing bispectrum as a function of triangle configurations, using 1000 ray-tracing simulations.

The left panel shows the example for equilateral triangle configuration.

What is the S/N?

Kayo & MT in prep.
Self-calibration of systematic errors

\[ P_κ \propto W^2(\chi)P_δ \]

\[ B_κ \propto W^3(\chi)P_δ^2 \]

- The lensing PS and Bisp depend on cosmological parameters as well as on the WL systematic errors, if exist, in a different way.
- The theory tells the cosmological dependences.
- Hence combining the 2pt and 3pt allows for a self-calibration of the systematic errors from the same data set.
Another usefulness of 3pt

- The bispectrum is measured via triangle configuration.
- Can explore a shape of large-scale structure.
- One characteristic feature in CDM-dominated structure formation is appearance of filamentary structures or void.
- Also for primordial non-Gaussianity beyond the local type.
Status of the measurement

- Pen et al. (03) reported the first detection of the WL skewness from CFHT data with 16 sq. deg.
- The skewness contains the collapsed information of the bispectrum

\[
\text{Skewness} \equiv \left( \kappa \otimes F(r_0) \right)^3
\]

- The next search is from CFHT 170 sq. deg. data?
Issues need to be studied (possible subjects for students)

• The E/B mode decomposition of lensing 3pt function
  – Doable by applying the CMB-used methods to lensing fields (e.g., see Hikage+11 for the E/B-mode decomposition of power spectrum)

• An optimal, reasonably fast, measurement method of measuring the 3pt functions of all the configurations
  – Even for the SDSS, the galaxy bispectrum hasn’t been fully explored to improve the cosmological constraints (BAO and cosmo paras)

• Develop an accurate model of the theoretical predictions
  – Need to use a sufficient number of simulation realizations

• The error covariances
  – The bispectrum covariance arises from up to the 6-pt correlation functions

• What about the 4pt function or the higher?
These are not the full story...

• Other various n-point functions are available from the same data; halo-shear-shear, halo-halo-shear, halo-halo-halo, ...
Simon et al (08) used the CFHT data of 34 sq. deg., and then reported a detection of the galaxy-galaxy-shear correlations.
Hope

2pt functions

\[ \langle \gamma \gamma \rangle \propto W_{gl}^2 P_\delta(k) \]
\[ \langle \gamma h \rangle \propto b W_{gl} P_\delta(k) \]
\[ \langle h h \rangle \propto b^2 P_\delta(k) \]

3pt functions

\[ \langle \gamma \gamma h \rangle \propto W_{gl}^3 P_\delta^2(k) \]
\[ \langle h h \gamma \rangle \propto b W_{gl}^2 P_\delta^2(k) \]
\[ \langle h h h \rangle \propto b^3 P_\delta^2(k) \]

• The different correlation functions depend on halo bias, matter power spectrum and lensing efficiency in different ways
• The comprehensive 2pt analysis of WL (Bernstein 09)
• A comprehensive joint analysis of 2pt and 3pt functions for a survey (not yet done)
• This combination is more powerful? If so, how much improvement?
Summary

• For any large-scale structure probe, the power spectrum is not sufficient to extract the full information.
• Definitely should explore the higher-order moments for coming/future surveys that are very expensive surveys.
• Combining the higher-order functions with the 2pt information can be useful for improving the constraining power or calibrating the systematic errors.
• Open up a new window to study the primordial non-Gaussianity.
• The full use of the higher-order moments haven’t yet been fully explored; projects for you!