Instanton calculus in quiver gauge theories

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What are N=2 4d quiver theories?

$$G = \prod_{i=1}^{n} SU(N_i)$$

nodes - gauge groups
arrows - bifundamentals

What are asymptotically free or conformal N=2 quivers?

for i-th node beta-function is

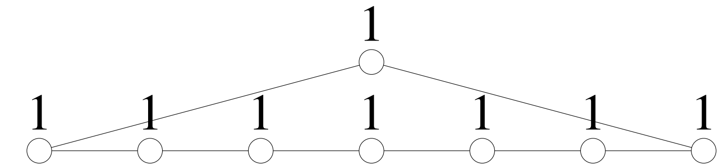
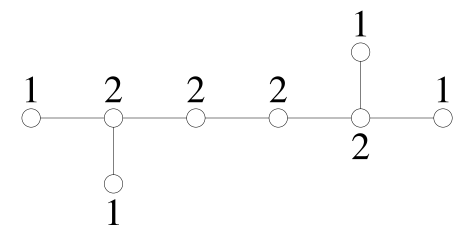
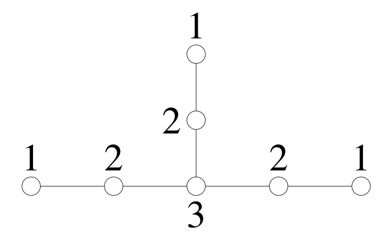
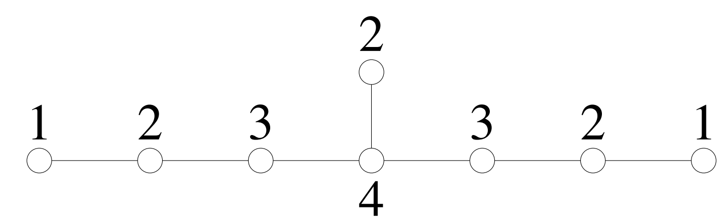
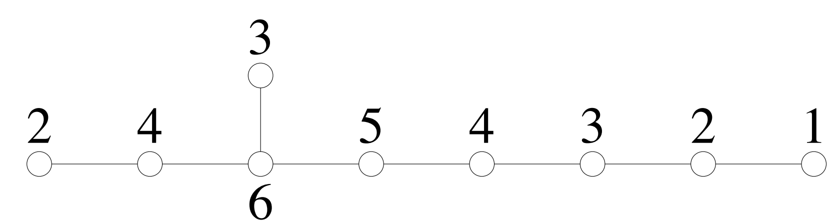
$$\beta_i \sim 2N_i - \sum_{\text{links } ij} N_j$$

$$a_{ii} = 2, \quad a_{ij} = -\text{number of arrows ij}$$

matrix (a_{ij}) is generalized Cartan matrix

$$\beta_i = a_{ij} N_j$$

beta non-negative => matrix (a) is of Fin or Aff ADE type

Finite subgroup of $SU(2)$		Affine simply laced Dynkin diagram	
$\mathbb{Z}/n\mathbb{Z}$	$\langle x \mid x^n = 1 \rangle$	$\widetilde{A}_{n-1}$	
$\mathbb{B}D_{2n}$	$\langle x, y, z \mid x^2 = y^2 = y^n = xyz \rangle$	$\widetilde{D}_{n-2}$	
$\mathbb{BT}$	$\langle x, y, z \mid x^2 = y^3 = z^3 = xyz \rangle$	$\widetilde{E}_6$	
$\mathbb{BO}$	$\langle x, y, z \mid x^2 = y^3 = z^4 = xyz \rangle$	$\widetilde{E}_7$	
$\mathbb{BD}$	$\langle x, y, z \mid x^2 = y^3 = z^5 = xyz \rangle$	$\widetilde{E}_8$	

We want to compute the partition function

$$Z = \int [DA \ldots] e^{-S[A,\ldots]}$$

by direct evaluation of the 4d path integral
and see how SW geometry appears

Of course, the prepotential F is known from

Witten'97 (M-theory): A quivers
KMV'97 (geom. eng & top strings): all ADE quivers

but it still might be useful to solve the problem in another way

Losev, Moore, Nekrasov, Shatashvilli'97:
developed equivariant integration over 4d instantons moduli spaces

$$Z_k = \int_{\mathcal{M}_k} \mu \text{ = finite contour integral}$$

$$Z = \sum_{k=0}^{\infty} Z_k q^k$$

Nekrasov'02 claimed: $Z = e^{-\frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}_{SW}}$ as $\epsilon_1, \epsilon_2 \to 0$

Nekrasov, Okounkov' 02 proved (for SU(N))

Shadchin'05: SU(N)xSU(N)

$$Z_\Omega = \int_{\mathcal{A}/\mathcal{G}} e^{-S_{cl}} \, eu_T(\Omega^{2+} \otimes \mathrm{ad}\,\mathcal{E})\Big|_{MQ,\; F^+=0} \, eu_T((S^- \ominus S^+) \otimes R)\Big|_{MQ,\; D\!\!\!/ \psi = 0}$$

$$\int_M \alpha = \int_F \frac{i_F \alpha}{eu N_F}$$

$$SD: \quad \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2+} \quad \text{vector}$$

$$\not{D}: \quad S^- \xrightarrow{\not{D}} S^+ \quad \text{hyper}$$

$$Z_\Omega = \sum_{\substack{\text{fixed} \\ \text{points } \lambda}} e^{-S_\lambda} \, eu_T(SD \otimes ad\, \mathfrak{g}) \, eu_T(\not{D} \otimes R)$$

R - rep of G for hypers

$T = T_G \times T_F \times T_L$

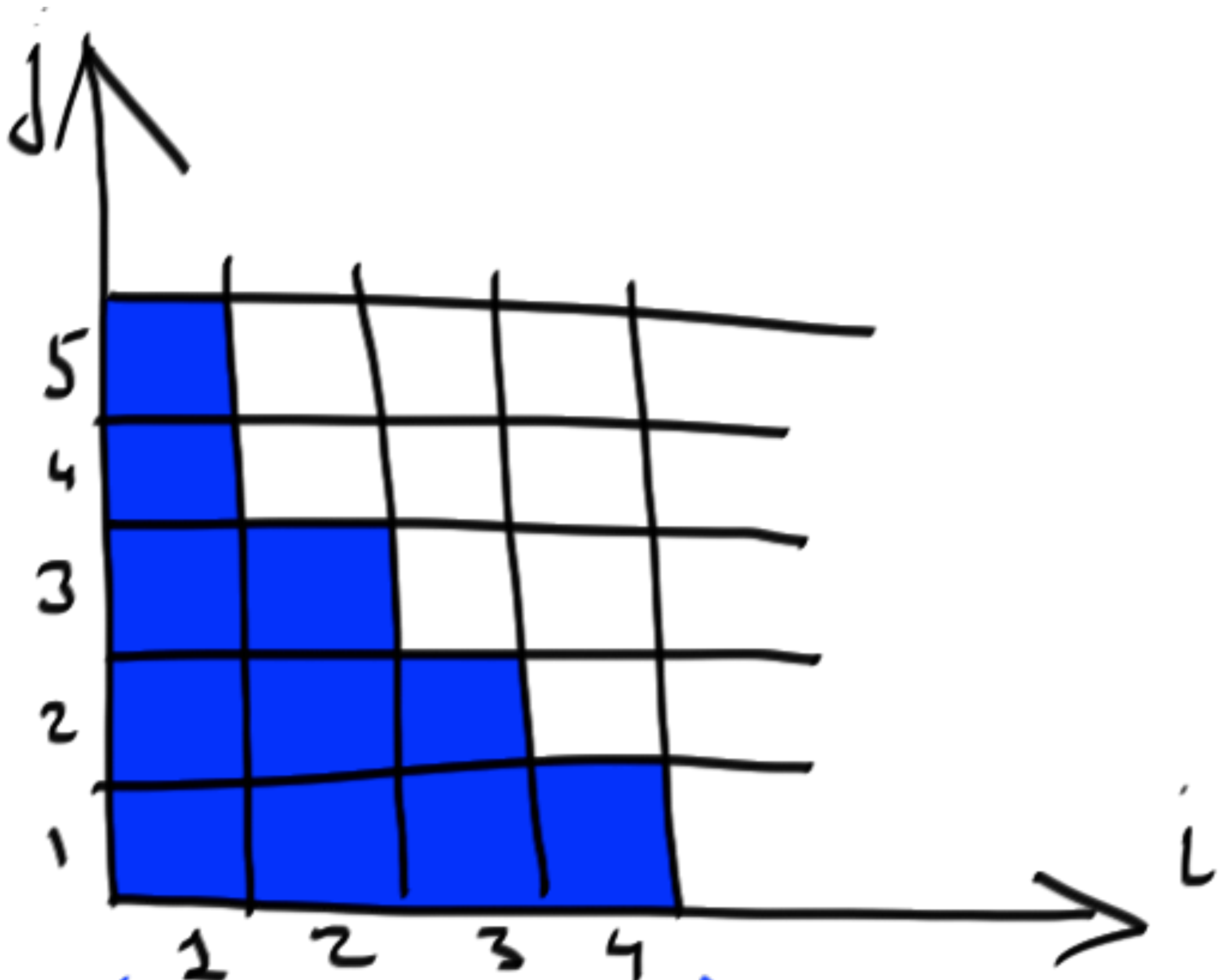
$T_G x T_L$ fixed points for $G = \Pi\ SU(N_i)$ are rank 1 torsion free sheaves

$\updownarrow$

point U(1) instantantos

$\updownarrow$

ideals in $\mathbb{C}[[z_1, z_2]]$



$\lambda = (5, 3, 2, 1)$

$$I_\lambda = \bigoplus_{(i,j) \notin \lambda} \mathbb{C}\, Z_1^{i-1} Z_2^{j-1}$$

T_L action:

$$z_1 \to t_1^{-1} z_1 \qquad t_1 = e^{i\varepsilon_1}$$

$$z_2 \to t_2^{-1} z_2 \qquad t_2 = e^{i\varepsilon_2}$$

$$ch_T \mathcal{O} = \sum_{n_1, n_2 \geq 0} t_1^{n_1} t_2^{n_2} = \frac{1}{(1-t_1)(1-t_2)}$$

$$ch_T I_\lambda = ch_T \mathcal{O} - V_\lambda$$

$$S^- - S^+ = -K^{1/2} \otimes \left(\Omega^{0,0} \xrightarrow{\bar\partial} \Omega^{0,1} \xrightarrow{\bar\partial} \Omega^{0,2}\right)$$

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2+} = \frac{1}{2}(1+K) \otimes \left(\Omega^{0,0} \xrightarrow{\bar\partial} \Omega^{0,1} \xrightarrow{\bar\partial} \Omega^{0,2}\right)$$

$$\mathcal{O} = \Omega^{0,0} \xrightarrow{\bar\partial} \Omega^{0,1} \xrightarrow{\bar\partial} \Omega^{0,2}$$

for empty partition

$$\mathrm{ch}_T(S^- - S^+) = -\frac{t_1^{1/2} t_2^{1/2}}{(1-t_1)(1-t_2)}$$

for non-empty partition and fundamental representation just subtract missing sections

$$\mathrm{ch}_T(S^- - S^+) \otimes_\lambda W = -t_1^{1/2} t_2^{1/2} \left(\frac{W}{(1-t_1)(1-t_2)} - V_\lambda \right)$$

for adjoint representation

$$\mathrm{ch}_T\left(H'(\mathrm{ad}\,W)\right) = \mathrm{ch}\left(\Theta_\lambda(w) \otimes_\Theta \Theta_\lambda^*(w)\right) =$$

$$= \frac{\mathrm{ch}\,\Theta_\lambda(w) \cdot \mathrm{ch}\,\Theta_\lambda^*(w)}{\mathrm{ch}\,\Theta} =$$

$$= \frac{\mathcal{E}\mathcal{E}^*}{(1-t_1)(1-t_2)}, \quad \mathcal{E} = W - (1-t_1)(1-t_2)V$$

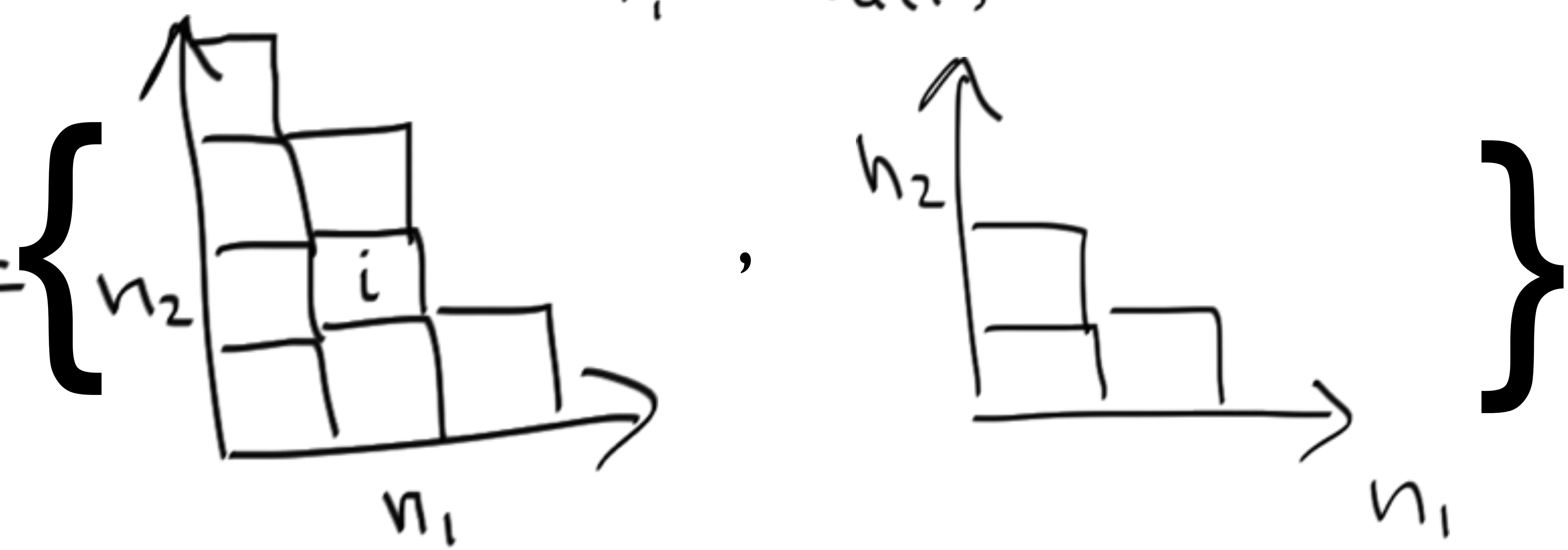
$$W = \sum_{\alpha=1}^{N} e^{i a_\alpha}$$

$$V = \sum_{i=1}^{K} e^{i \varphi_i}$$

for colored partition $\vec{\lambda} = (\lambda_\alpha) = (\lambda_1, \dots \lambda_N)$

i runs over boxes $\varphi_i = a_{\alpha(i)} - (n_1(i) - 1)\epsilon_1 - (n_2(i) - 1)\epsilon_2$

$\vec{\lambda} =$



Now convert Chern character to Euler character

$$\mathrm{ch}\, M = \sum e^{iw_i}$$

$$\mathrm{ch}_t\, M = \sum e^{tw_i}$$

$$\mathrm{eu}\, M = \prod w_i$$

That is the integral transform

$$\int_0^\infty dt\, t^{s-1} e^{-tw} = \Gamma(s) w^{-s}$$

$$\mathrm{eu}\, M = \exp(-\frac{d}{ds}\Gamma(s)^{-1} \int_0^\infty dt\, t^{s-1}\, \mathrm{ch}\, M)|_{s=0}$$

$$\mathrm{eu(vect)} = \exp\left(-\frac{d}{ds}\Gamma(s)^{-1}\int_0^\infty dt\, t^{s-1}\frac{E_t E_{-t}}{(1-e^{-\epsilon_1 t})(1-e^{-\epsilon_2 t})}\right)$$

where

$$E_t = W_t - (1-e^{-\epsilon_1 t})(1-e^{-\epsilon_2 t})V$$

in terms of the 2-gamma function

$$\gamma_2(x|\epsilon_1,\epsilon_2) = \frac{d}{ds}\Gamma(s)^{-1}\int_0^\infty dt\, t^{s-1}\frac{e^{-tx}}{(1-e^{-t\epsilon_1})(1-e^{-t\epsilon_2})}$$

and the Chern root densities

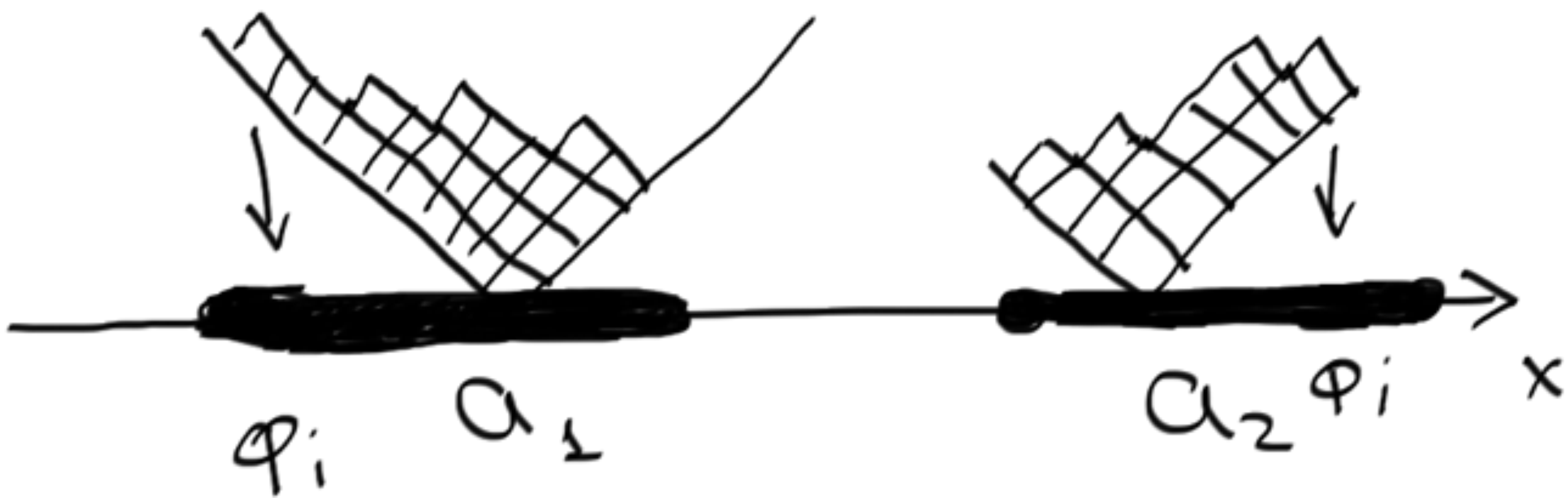
$$E_t = \int \rho(x) e^{-tx} dx$$

we get

$$\mathrm{eu(vect)} = \exp(-\int dx\, dx'\, \rho(x)\gamma_2(x-x')\rho(x'))$$

densities $\rho(x)$:

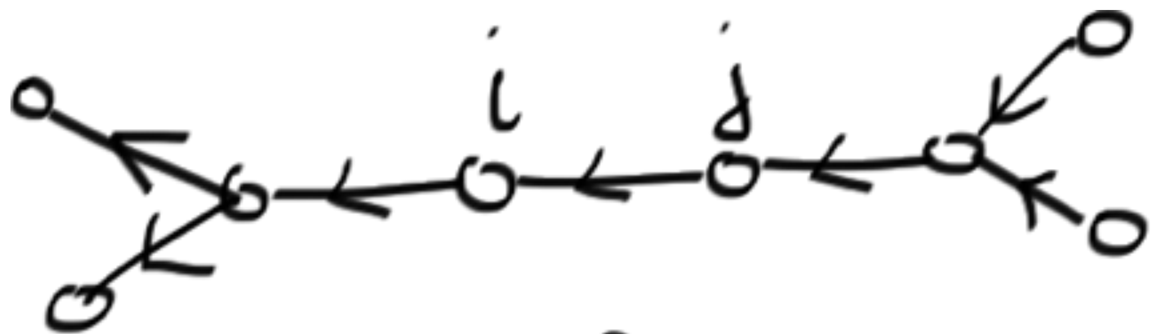
$$\rho(x) = \sum \delta(x - a_\alpha) - \sum_i \Big(\delta(x - \varphi_i) + \delta(x - \varphi_i + \epsilon_1 + \epsilon_2) - \delta(x - \varphi_i + \epsilon_1) - \delta(x - \varphi_i + \epsilon_2) \Big)$$



$$\int_{I_{i\alpha}} \rho_i(x)\,dx = 1$$

$$\int_{I_{i\alpha}} x\rho_i(x)\,dx = a_{i\alpha}$$

$$\int x^2 \rho_i(x)\,dx = \sum_\alpha a_{i\alpha}^2 - 2\epsilon_1\epsilon_2 K_i$$



each bifund contributes:

$$\int \rho_i(x)\, \delta_2(x + M_{ij} - x')\, \rho_j(x')\; dx\, dx'$$

assume we can find M_i | $M_{ij} = M_i - M_j$

(no cycles or $\sum M_{ij} = 0$)

$$\tilde{\rho}(x) = \rho(x - M_i)$$

In $\tilde{\rho}(x)$

$$Z = \sum_{\substack{\text{fixed} \\ \text{points} \\ \rho(\lambda)}} e^{\frac{1}{\varepsilon_1 \varepsilon_2} \mathcal{E}[\tilde{\rho}(x)]}$$

$$\mathcal{E}[\tilde{\rho}(x)] = \frac{1}{2} \int dx\, \tilde{\rho}_i(x) K(x-x') a_{ij} \tilde{\rho}_j(x')\, dx' - \frac{1}{2} \log q_i \int \hat{\rho}(x) (x - \mu_i)^2 dx$$

$\mathcal{E} \to \min$

w/ constraints

$$\int_{\tilde{I}_{i\alpha}} \rho(x)\, dx = 1$$

$$\int_{\tilde{I}_{i\alpha}} x \rho(x)\, dx = a_{i\alpha} - \mu_i$$

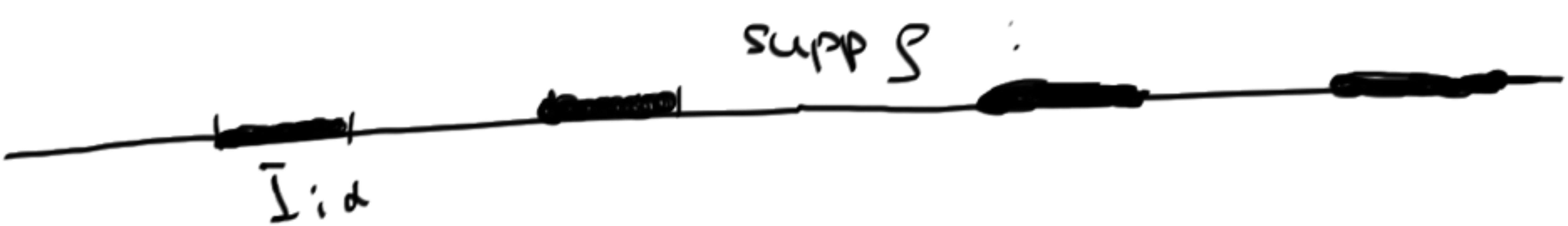
Kernel $K(x) = \frac{x^2}{2}\log(x - \frac{3}{2})$

$$K'(x) = x(\log x - 1)$$

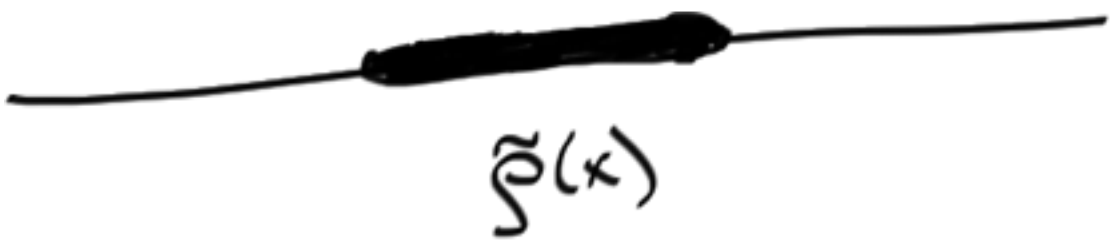
$$K''(x) = \log x$$

crit. points of $\mathcal{E}[\tilde{\rho}]$:

$$\int dx' \, K(x-x') \, a_{ij} \tilde{\rho}_j(x') = \log q_i \quad , \quad x \in \operatorname{supp} \tilde{\rho}_i$$



Like 2d electrons restricted to intervals,
(matrix model eigenvalues, etc)



$$\hat{\rho}_i(x) = \int_{\mathrm{supp}\,\tilde{\rho}} dx' \log(x-x')\,\tilde{\rho}_i(x')$$

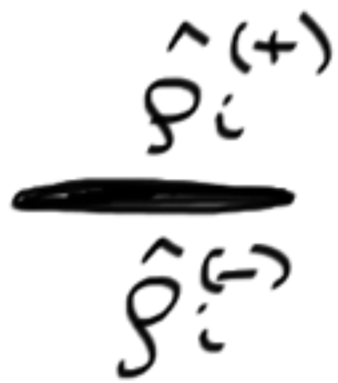
$\hat{\rho}_i(x)$ analytic function on $\mathbb{C} \setminus \mathrm{supp}\,\tilde{\rho}$

$x \to \infty \quad \hat{\rho}_i(x) \sim N_i \log(x)$

$$\hat{\rho}_i(x) = \int \log(x - x')\, \tilde{\rho}_i(x')$$

$\hat{\rho}_i(x)$ – holo on $\mathbb{C} \setminus \mathrm{supp}\, \tilde{\rho}_i$

$\hat{\rho}_i(x) \sim x^{N_i}, \quad x \to \infty$



crit. point equations $\Rightarrow$ cut crossing equation

$$\hat{\rho}_i^{(-)} = \hat{\rho}_i^{(+)} - a_{ij}\hat{\rho}_j + \log q_i$$

Quiver ADE Weyl reflections!

$\mathfrak{h} \in$ Cartan of quiver ADE Lie alg.

Let $h = \sum \hat{\rho}_i \alpha_i^\vee - \sum \log q_i \Lambda_i^\vee$

$\alpha_i^\vee$ - coroots

$\Lambda_i^\vee$ - fund. coweights

$\langle \Lambda_i^\vee, \alpha_j \rangle = \delta_{ij}$

α_i reflection:

$$h \to h - \alpha_i^\vee \langle \alpha_i h \rangle$$

$$\hat{\rho}_i \to \hat{\rho}_i - a_{ij} \hat{\rho}_j + \log q_i$$

Cut crossing are isomorphic
to simple reflections generating quiver Weyl group

Consider Weyl invariant functions of $h(\hat{\rho})$
We take characters of irreps with heighest weight Λ_i

$$x_i(\hat\rho) = \mathrm{tr}_{\Lambda_i}\, e^{h(\hat\rho)}$$

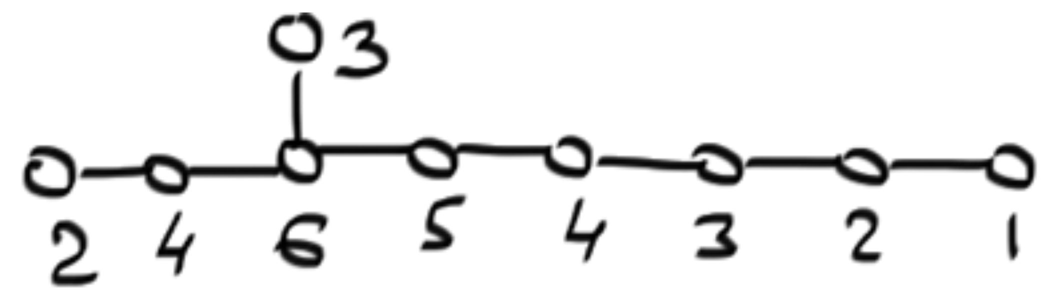
as $x \to \infty$

$$x_i(\hat\rho) \to e^{\langle \Lambda_i, \hat\rho_j \check\alpha_j - \log q_j \check\Lambda_j \rangle} \simeq x^{N_j} \prod q_j^{-\langle \Lambda_i, \check\Lambda_j \rangle} \simeq$$

$x_i(\hat\rho(x))$ are bounded holo fun on $\mathbb{C}$

$\Rightarrow$ $x_i(\hat\rho(x))$ are polynoms of deg N_j

Finally



$\hat{\rho}_i(x)$ are determined by system of n equations

$$\left\{ \mathrm{tr}_{\Lambda_i} \, e^{\alpha_i^\vee \hat{\rho}_i - \Lambda_i^\vee \log q_i} = P_i(x) \right\}$$

$$P_i(x) = \prod q_j^{\langle -\Lambda_i, \Lambda_j^\vee \rangle} \, x^{N_i} + \dots$$

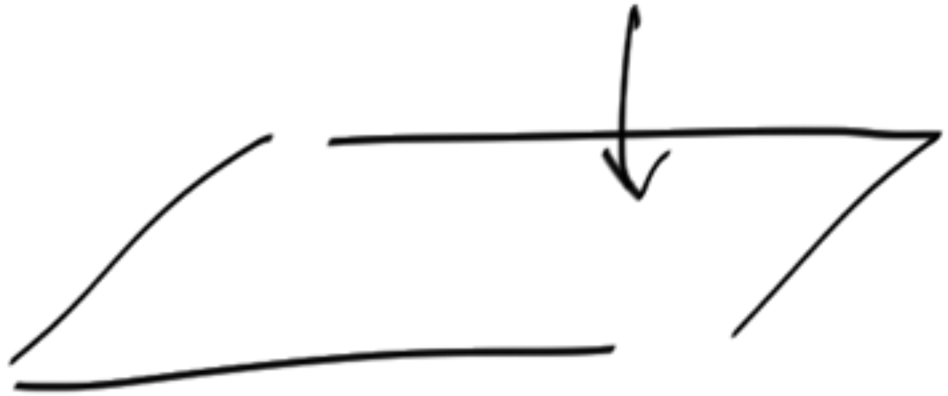
$\sum N_i$ coeffs of $P_i \Leftrightarrow \sum N_i$ Coulomb moduli $a_{i\alpha}$

(on this page G denotes quiver group)

<u>Aff(G)</u> $\{x_i\}$ = conj. class of Kac-Moody loop group $\hat{\mathcal{L}}(G)$

$\|$

holo G-bundle over E_q



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$$\delta(\alpha_i^\vee \hat{\rho}_i - \Lambda_i^\vee q_i) = - a_i \log q_i$$

elliptic modulus $Q = e^{-\delta(h)} = \prod_i q_i^{a_i}$

holo G-bundle on E_q

(in $\tilde{A}_n$: n points on dual curve)

×

Maps$(\mathbb{C}, \mathcal{M}_{G \text{ on } E_Q})$