

Gauge Theories and Macdonald Polynomials

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Exact Methods in Gauge/String Theories

PCTS, 11/11/11

4d $\mathcal{N} = 2$ susy theories of class $\mathcal{S}(ix)$

(Gaiotto, Gaiotto Moore Neitzke, ...)

“Partially twisted” compactification of the (2,0) 6d theory on a 2d surface \mathcal{C} with punctures
 $\implies \mathcal{N} = 2$ **superconformal theories** in four dimensions.

- Space of complex structures $\mathcal{C} \cong$ marginal gauge couplings of the 4d theory.
- Conformal factor of the metric on \mathcal{C} believed to be RG-irrelevant.

Recent check at large N .

Holographic RG equation (Φ is related to conformal factor)

$$\partial_\rho^2 e^\Phi + (\partial_x^2 + \partial_y^2)\Phi = e^\Phi.$$

Global existence proof of regular flows interpolating between arbitrary UV metric and canonical IR metric of constant negative curvature (for $g > 1$).

(Anderson Beem Bobev LR)

- Moore-Seiberg groupoid of $\mathcal{C} =$ (generalized) 4d S-duality
Vast generalization of “ $\mathcal{N} = 4$ S-duality as modular group of T^2 ”.

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$6=4+2$: beautiful and unexpected $4d/2d$ connections. Most famous example,

- Correlators of Liouville/Toda on \mathcal{C} compute the $4d$ partition functions (on S^4)
(Alday Gaiotto Tachikawa)

In this talk we focus on another surprising connection:

- The superconformal index $\mathcal{I}(q, p, t; , x_i)$ is computed by topological QFT¹ on \mathcal{C} .

Index = twisted partition function on $S^3 \times S^1$.

Encodes the protected spectrum of the $4d$ theory: independent of the gauge theory moduli.

Simpler $4d/2d$ relation (topological): may hope to derive it from $6d$.

Still very non-trivial. Index of generic theory unknown.

¹Term used loosely: infinite-dimensional state-space.

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The superconformal index

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- It is evaluated by a trace formula of the schematic form

$$\mathcal{I}(\mu_i) = \text{Tr}(-1)^F e^{-\sum_i \mu_i T_i} e^{-\beta \delta}, \quad \delta = 2 \{Q, Q^\dagger\} (\geq 0),$$

where Q is the supercharge “with respect to which” the index is calculated and $\{T_i\}$ a complete set of generators that commute with Q and with each other.

- The trace is over the states of the theory on S^3 (in radial quantization). States with $\delta \neq 0$ cancel pairwise, so the index counts states with $\delta = 0$ and it is independent of β .
- For a theory with a Lagrangian description one can compute the index in the free limit by counting the gauge-invariant operators in terms of a matrix integral. (Unlike the S^4 partition function, which is sensitive to non-perturbative physics.)

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$\mathcal{N} = 2$ index

- $\mathcal{N} = 2$ SCFTs have 8 supercharges (+8 superconformal charges): $Q_{l\alpha}$, $\tilde{Q}_{l\dot{\alpha}}$. Here $l = 1, 2$ are $SU(2)_R$ indices and $\alpha = \pm$, $\dot{\alpha} = \pm$ Lorentz indices. We choose to compute the index “with respect to” $\tilde{Q}_{1\dot{-}}$ (all other choices are equivalent). The commutant subalgebra to $\tilde{Q}_{1\dot{-}}$ and $(\tilde{Q}_{1\dot{-}})^\dagger$ has δ_{1-} , δ_{1+} , $\tilde{\delta}_{2\dot{-}}$ as Cartan generators,

$$\delta_{1-} \equiv 2 \{ Q_{1-}, (Q_{1-})^\dagger \} = E - 2j_1 - 2R - r,$$

$$\delta_{1+} \equiv 2 \{ Q_{1+}, (Q_{1+})^\dagger \} = E + 2j_1 - 2R - r,$$

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E is the conformal dimension, (j_1, j_2) the Cartan generators of the $SU(2)_1 \otimes SU(2)_2$ isometry group, and (R, r) , the Cartan generators of the $SU(2)_R \otimes U(1)_r$ R-symmetry.

- The index is defined as

$$\mathcal{I}(\sigma, \rho, \tau, \dots) = \text{Tr}(-1)^F \sigma^{\frac{1}{2}\delta_{1+}} \rho^{\frac{1}{2}\delta_{1-}} \tau^{\frac{1}{2}\tilde{\delta}_{2\dot{-}}} e^{-\beta \tilde{\delta}_{1\dot{-}}} \dots$$

or equivalently

$$\mathcal{I}(\rho, q, t, \dots) = \text{Tr}(-1)^F \rho^{\frac{1}{2}\delta_{1+}} q^{\frac{1}{2}\delta_{1-}} t^{R+r} e^{-\beta' \tilde{\delta}_{1\dot{-}}} \dots$$

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$\mathcal{N} = 2$ SCFTs of class \mathcal{S} (of type A)

- Defined as the IR limit of the A_{k-1} $(2, 0)$ theory on $\mathbb{R}^4 \times \mathcal{C}$, where \mathcal{C} a Riemann surface with appropriate punctures (defects).
- Complex moduli of $\mathcal{C} \cong 4d$ gauge couplings
- Punctures are associated with flavor symmetries.
Flavor symmetries are classified by “auxiliary Young diagrams” with k boxes (embeddings of $SU(2)$ into $SU(k)$).
- Basic building blocks: theories corresponding to **spheres with three punctures** (no moduli=no tunable couplings)
 - ▶ **Free hypermultiplets** of $SU(k)$ theories correspond to spheres with two “maximal” punctures ($SU(k)$ flavor symmetry) and one “minimal” ($U(1)$ flavor symmetry).
 - ▶ All other three-punctured spheres **do not** have Lagrangian description.
Simplest example: $SU(3)$ theory with three maximal punctures $\cong E_6$ SCFT
- “Gluing” three-punctured spheres at two maximal punctures corresponds to gauging the diagonal $SU(k)$
- Different “pair-of-pants” decompositions correspond to different S-duality frames.

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TQFT structure

- The superconformal index is independent of the marginal couplings. For theories of class \mathcal{S} this means that the index does not depend on the complex moduli of \mathcal{C} : it must be computed by a $2d$ TQFT correlator on \mathcal{C} .
- The index of a generic theory of class \mathcal{S} can be written in terms of the index of the basic constituents: three-puncture spheres and propagators. We parametrize the indices of the three-punctured spheres as

$$\mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$$

where \mathbf{a}_i are fugacities dual to the Cartan subgroup of the flavor symmetry.

In general these are a priori unknown functions.

The propagators are known explicitly,

$$\eta(\mathbf{a}, \mathbf{b}) = \Delta(\mathbf{a}) \mathcal{I}^V(\mathbf{a}) \delta(\mathbf{a}, \mathbf{b}^{-1})$$

where $\Delta(\mathbf{a})$ is the Haar measure and $\mathcal{I}^V(\mathbf{a})$ the index of the vector multiplet.

- As the simplest example of gluing,

$$\begin{aligned} \mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) &= \oint [d\mathbf{a}] \oint [d\mathbf{b}] \mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}) \eta(\mathbf{a}, \mathbf{b}) \mathcal{I}(\mathbf{b}, \mathbf{a}_3, \mathbf{a}_4) \\ &= \oint [d\mathbf{a}] \Delta(\mathbf{a}) \mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}) \mathcal{I}^V(\mathbf{a}) \mathcal{I}(\mathbf{a}^{-1}, \mathbf{a}_3, \mathbf{a}_4), \end{aligned}$$

S-duality implies that this index is invariant under permutations of \mathbf{x}_i , which translates into associativity of the TQFT structure constants.

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Expanding in a convenient basis of functions $\{f^\alpha(\mathbf{a})\}$, labeled by $SU(k)$ representations $\{\alpha\}$,

$$\begin{aligned}\mathcal{I}(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= \sum_{\alpha, \beta, \gamma} C_{\alpha\beta\gamma} f^\alpha(\mathbf{a}) f^\beta(\mathbf{b}) f^\gamma(\mathbf{c}) \\ \eta^{\alpha\beta} &= \oint [d\mathbf{a}] \oint [d\mathbf{b}] \eta(\mathbf{a}, \mathbf{b}) f^\alpha(\mathbf{a}) f^\beta(\mathbf{b}).\end{aligned}$$

Invariance of the index under the different pairs-of-pants decomposition of \mathcal{C} is equivalent to the associativity of the structure constants,

$$C_{\alpha\beta\gamma} C^\gamma_{\delta\epsilon} = C_{\alpha\delta\gamma} C^\gamma_{\beta\epsilon},$$

where indices are raised with the metric $\eta^{\alpha\beta}$ and lowered with the inverse metric $\eta_{\alpha\beta}$.

The full (“elliptic”) index for A_1 and A_2 theories

- The “full index” is elegantly expressed in terms of elliptic Gamma functions. For example, the index of a free hypermultiplet is given by a product of eight elliptic Gamma functions (Dolan-Osborn)

$$\mathcal{I}(a, b, c) = \Gamma(t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1}; p, q), \quad \Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - p^{i+1} q^{j+1} / z}{1 - p^i q^j z}.$$

- For A_1 quivers everything is explicit and associativity (S-duality) can be checked (Gadde-Pomoni-LR-Razamat) by recently-found non-trivial integral identities (van de Bult).
- One can in principle use dualities to obtain the indices of the strongly-coupled building blocks for higher rank theories.
- Indeed Argyres-Seiberg duality was used to compute the index of the E_6 SCFT (the structure constants of A_2 quivers with three maximal punctures) (Gadde-Razamat-LR-Yan),

$$\int \frac{da}{a} \tilde{\Delta}(a, c) \mathcal{I}_{E_6}((a, b), \mathbf{x}, \mathbf{y}) \sim \mathcal{I}_{N_f=6, SU(3)}(\mathbf{x}, \mathbf{y}, b, c).$$

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$$\int \frac{da}{a} \tilde{\Delta}(a, c) \mathcal{I}_{E_6}((a, b), \mathbf{x}, \mathbf{y}) \sim \mathcal{I}_{N_f=6, SU(3)}(\mathbf{x}, \mathbf{y}, b, c).$$

- This strategy is hard to generalize to A_n , $n > 2$.

The full (“elliptic”) index for A_1 and A_2 theories

- The “full index” is elegantly expressed in terms of elliptic Gamma functions. For example, the index of a free hypermultiplet is given by a product of eight elliptic Gamma functions (Dolan-Osborn)

$$\mathcal{I}(a, b, c) = \Gamma(t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1}; p, q), \quad \Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - p^{i+1} q^{j+1} / z}{1 - p^i q^j z}.$$

- For A_1 quivers everything is explicit and associativity (S-duality) can be checked (Gadde-Pomoni-LR-Razamat) by recently-found non-trivial integral identities (van de Bult).
- One can in principle use dualities to obtain the indices of the strongly-coupled building blocks for higher rank theories.
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- Find an algorithm to calculate the index for *all* class \mathcal{S} theories
- Identify explicitly the $2d$ TQFT

Strategy:

- “Bottom-up”, experimental approach.
- Extrapolate the results for Lagrangian theories (A_1 quivers) to higher rank.

Results:

We succeeded for a slice $(q, 0, t)$ of the (q, p, t) fugacity space.

- TQFT $\sim (q, t)$ -deformation of $2d$ Yang-Mills in the zero-area limit.
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Strategy

- Choose $\{f^\alpha(\mathbf{a})\}$ to be orthonormal under the propagator measure, that is,

$$\eta^{\alpha\beta} = \delta^{\alpha\beta}. \quad (1)$$

- Perform further orthogonal transformation to basis where structure constants are diagonal,

$$C_{\alpha\beta\gamma} \neq 0 \rightarrow \alpha = \beta = \gamma.$$

Associativity is then automatic.

Finding the diagonal basis in principle always possible: challenge is describe it concretely.

- Useful to consider the ansatz

$$f^\alpha(\mathbf{a}) = \mathcal{K}(\mathbf{a})P^\alpha(\mathbf{a}), \quad (2)$$

for some cleverly chosen $\mathcal{K}(\mathbf{a})$.

- Focus on A_1 quivers, which are Lagrangian. In special limits, able to diagonalize the structure constants: $\{P^\alpha(\mathbf{a})\}$ turn out to be well-known orthogonal **polynomials** (Macdonald, Schur, Hall-Littlewood)
- These polynomials are defined for **any** root system. Immediate to formulate compelling conjectures for A_n quiver. Test against expected dualities.
- Then immediate to evaluate index for genus g surface with s (maximal) punctures,

$$\mathcal{I}_{g,s}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s) = \sum_{\alpha} (C_{\alpha\alpha\alpha})^{2g-2+s} \prod_{j=1}^s f^\alpha(\mathbf{a}_j).$$

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Interesting limits

Use susy enhancement to select special limits

$$\mathcal{I}(p, q, t, \dots) = \text{Tr}(-1)^F p^{\frac{1}{2}\delta_{1+}} q^{\frac{1}{2}\delta_{1-}} t^{R+r} e^{-\beta' \tilde{\delta}_{1\dot{-}}} \dots$$

- $p \rightarrow 0$: Macdonald $\mathcal{I}_M(q, t)$ $\tilde{Q}_{1\dot{-}}, Q_{1+}$
- $p \rightarrow 0, q \rightarrow 0$: Hall-Littlewood $\mathcal{I}_{HL}(t)$ $\tilde{Q}_{1\dot{-}}, Q_{1+}, Q_{1-}$
- $q = t$ (independent of p it turns out): Schur $\mathcal{I}_S(q)$ $\tilde{Q}_{1\dot{-}}, Q_{1+}$

Macdonald index

- $p \rightarrow 0$ limit of the full index,

$$\mathcal{I}_M = \text{Tr}_M(-1)^F q^{-2j_1} t^{R+r},$$

where the trace is over states with $E + 2j_1 - 2R - r = 0$.

$\frac{1}{4}$ BPS: one chiral and one antichiral supercharge.

- Basic ansatz for complete set of functions that diagonalize the structure constants:

$$f_{q,t}^\lambda(\mathbf{a}) = \mathcal{K}_{q,t}(\mathbf{a}) P^\lambda(\mathbf{a}|q, t).$$

Macdonald polynomials $P_\lambda(\mathbf{a}; q, t)$ associated to the root system A_{k-1} are labeled by representations λ of $SU(k)$. They are orthogonal with respect to the measure

$$\Delta_{q,t}(\mathbf{a}) = \frac{1}{k!} \prod_{n=0}^{\infty} \prod_{i \neq j} \frac{1 - q^n a_i/a_j}{1 - t q^n a_i/a_j},$$

- $q = t$ gives Schur (still $\frac{1}{4}$ BPS) while $q = 0$ gives Hall-Littlewood ($\frac{3}{8}$ BPS)

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Symbol	Surface	Value
$C_{\alpha\beta\gamma}$		$\frac{\mathcal{A}(q,t)}{\dim_{q,t}(\alpha)} \delta_{\alpha\beta} \delta_{\alpha\gamma}$
V^α		$\frac{\dim_{q,t}(\alpha)}{\mathcal{A}(q,t)}$
$\eta^{\alpha\beta}$		$\delta^{\alpha\beta}$

Table: The structure constants, the cap, and the metric for the TQFT of the Macdonald index for A_{k-1} quivers.

$$\dim_{q,t}(\lambda) = P^\lambda(t^{\frac{k-1}{2}}, \dots, t^{\frac{1-k}{2}} \mid q, t)$$

$$\mathcal{A}(q, t) = PE \left[\frac{1}{2}(k-1) \frac{t-q}{1-q} \right] \prod_{j=2}^k (t^j; q).$$

The Macdonald index of the theory corresponding to a sphere with generic punctures is

$$\mathcal{I}_{\Lambda_1, \Lambda_2, \Lambda_3} = (t; q)^{k+2} \prod_{j=2}^k \frac{(t^j; q)}{(q; q)} \prod_{i=1}^3 \hat{\mathcal{K}}_{\Lambda_i}(\mathbf{a}_i) \sum_{\lambda} \frac{\prod_{i=1}^3 P_{\lambda}(\mathbf{a}_i(\Lambda_i) | q, t)}{P_{\lambda}(t^{\frac{k-1}{2}}, t^{\frac{k-3}{2}}, \dots, t^{\frac{1-k}{2}} | q, t)}.$$

There is well-defined rule $\mathbf{a}_i(\Lambda_i)$ to “partially-close” punctures by specializing the flavor fugacities.

The index and 2d Yang-Mills

- Consider the genus g partition function in the Schur limit, $q = t$,

$$\mathcal{I}_g(q) = \left[(q; q)^{2g-2} \right]^{k-1} S_{00}(q)^{2-2g} \sum_{\lambda} \frac{1}{[\dim_q(\lambda)]^{2g-2}}.$$

where S_{00} is the partition function of $SU(k)$ level ℓ Chern-Simons theory on S^3 if one formally identifies $q = e^{\frac{2\pi i}{\ell+k}}$,

$$S_{00}(q) = \prod_{j=2}^k \frac{(q; q)}{(q^j; q)}. \quad (3)$$

Up to a simple prefactor, this is the genus g partition function of a **q-deformed 2d Yang-Mills in the zero area limit** (equivalently, the analytic continuation of CS partition function on $C_g \times S^1$)

- In the more general case of $q \neq t$

$$\mathcal{I}_g(q, t) = \left[(t; q)^{g-1} (q; q)^{g-1} \right]^{k-1} \hat{S}_{00}(q, t)^{2-2g} \sum_{\lambda} \frac{1}{[\dim_{q,t}(\lambda)]^{2g-2}}, \quad (4)$$

where

$$\hat{S}_{00}(q, t) = \prod_{j=2}^k \frac{(t; q)}{(t^j; q)}.$$

Closely related to the "refinement" of Chern-Simons theory recently discussed by [Aganagic and Shakirov](#)
Possible 2d interpretation: 2d YM with modified (Macdonald) measure. Such modification might arise by integrating out degrees of freedom.

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Hall-Littlewood index

- We take the limit $\sigma, \rho \rightarrow 0$ of the full index

$$\mathcal{I} = \text{Tr}(-1)^F \sigma^{\frac{1}{2}\delta_{1+}} \rho^{\frac{1}{2}\delta_{1-}} \tau^{\frac{1}{2}\bar{\delta}_{2+}} e^{-\frac{1}{2}\beta\bar{\delta}_{1-}} .$$

- Alternatively can state that it is given by

$$\mathcal{I} = \text{Tr}_{HL}(-1)^F \tau^{2E-2R} ,$$

where the trace is over states satisfying $j_1 = 0$ and $E - 2R - r = 0$.

- The states contributing to this index are annihilated by **three** supercharges, two chiral and one anti-chiral.
- For Lagrangian theories the only “letters” contributing to this index are a scalar q (τ) from the hypermultiplet and a gaugino $\bar{\lambda}_{1+}$ ($-\tau^2$) from the vector multiplet.
- For genus-zero quivers, HL index \cong Hanany’s “Hilbert series of the Higgs branch”.

HL index - $SU(2)$ quivers

- All these theories have a **Lagrangian** description.
- The basic building block (sphere with three punctures) is a trifundamental **free** hypermultiplet.
- The HL index of the free hypermultiplet is given by

$$\mathcal{I}(a_1, a_2, a_3) = \frac{1}{\prod_{\pm 1} (1 - \tau a_1^{\pm 1} a_2^{\pm 1} a_3^{\pm 1})}.$$

- By explicit diagonalization, it can be rewritten

$$\begin{aligned}\mathcal{I}(a_1, a_2, a_3) &= \frac{1 + \tau^2}{1 - \tau^2} \prod_{i=1}^3 \frac{1}{(1 - \tau^2 a_i^2)(1 - \tau^2/a_i^2)} \sum_{\lambda=0}^{\infty} \frac{1}{P_{\lambda}^{HL}(\tau, \tau^{-1} | \tau)} \prod_{i=1}^3 P_{\lambda}^{HL}(a_i, a_i^{-1} | \tau) \\ &= \mathcal{N}(\tau) \prod_{i=1}^3 \mathcal{K}(a_i) \sum_{\lambda=0}^{\infty} C_{\lambda\lambda\lambda} \prod_{i=1}^3 f^{\lambda}(a_i).\end{aligned}$$

where

$$P_{\lambda}^{HL}(a, a^{-1} | \tau) = \mathcal{N}_{\lambda}(\tau) (\chi_{\lambda}(a) - \tau^2 \chi_{\lambda-2}(a))$$

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HL index - higher rank generalization

- The HL polynomials can be defined for $U(k)$ groups

$$P_{\lambda}^{HL}(x_1, \dots, x_k | \tau) = \mathcal{N}_{\lambda}(\tau) \sum_{\sigma \in S_k} \sigma \left(x_1^{\lambda_1} \dots x_k^{\lambda_k} \prod_{i < j} \frac{x_i - \tau^2 x_j}{x_i - x_j} \right).$$

and thus for higher rank building blocks, the T_k theories, we conjecture

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- For arbitrary punctures

$$\mathcal{I}_{\Lambda_1, \Lambda_2, \Lambda_3}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \mathcal{N}_k(\tau) \prod_{l=1}^3 \mathcal{K}_{\Lambda_l}(\mathbf{a}_l) \sum_{\lambda} \frac{1}{P_{\lambda}^{HL}(\tau^{k-1}, \dots, \tau^{1-k})} \prod_{l=1}^3 P_{\lambda}^{HL}(\mathbf{a}_l(\Lambda_l)).$$

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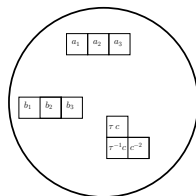
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HL index - $SU(3)$ quivers - Closing punctures

The index of the $SU(3) \times SU(3) \times U(1)$ free hypermultiplet is given by

$$\mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, c) = \prod_{i,j=1}^3 \frac{1}{1 - \tau a_i b_j c} \frac{1}{1 - \tau \frac{1}{a_i b_j c}}.$$



- It can be rewritten as

$$\mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, c) = \frac{1 - \tau^6}{(1 - \tau^2)^3} \frac{\mathcal{K}(\mathbf{a}_1)\mathcal{K}(\mathbf{a}_2)}{(1 - \tau^3 c^3)(1 - \tau^3 c^{-3})} \sum_{\lambda_1, \lambda_2} \frac{P_{\lambda_1, \lambda_2}^{HL}(\tau c, \tau^{-1}c, c^{-2} | \tau)}{P_{\lambda_1, \lambda_2}^{HL}(\tau^2, \tau^{-2}, 1 | \tau)} \prod_{i=1}^2 P_{\lambda_1, \lambda_2}^{HL}(\mathbf{a}_i | \tau).$$

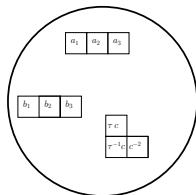
- Further setting $c = \tau$ we completely close one puncture to obtain a cylinder (propagator)

$$\delta^{HL}(\mathbf{a}, \mathbf{b}) \sim \sum_{\lambda_1, \lambda_2} P_{\lambda_1, \lambda_2}^{HL}(\mathbf{a} | \tau) P_{\lambda_1, \lambda_2}^{HL}(\mathbf{b} | \tau).$$

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$$\mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, c) = \prod_{i,j=1}^3 \frac{1}{1 - \tau a_i b_j c} \frac{1}{1 - \tau \frac{1}{a_i b_j c}}.$$



- It can be rewritten as

$$\mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, c) = \frac{1 - \tau^6}{(1 - \tau^2)^3} \frac{\mathcal{K}(\mathbf{a}_1)\mathcal{K}(\mathbf{a}_2)}{(1 - \tau^3 c^3)(1 - \tau^3 c^{-3})} \sum_{\lambda_1, \lambda_2} \frac{P_{\lambda_1, \lambda_2}^{HL}(\tau c, \tau^{-1}c, c^{-2} | \tau)}{P_{\lambda_1, \lambda_2}^{HL}(\tau^2, \tau^{-2}, 1 | \tau)} \prod_{i=1}^2 P_{\lambda_1, \lambda_2}^{HL}(\mathbf{a}_i | \tau).$$

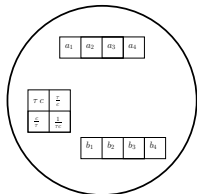
- Further setting $c = \tau$ we completely **close** one puncture to obtain a cylinder (propagator)

$$\delta^{HL}(\mathbf{a}, \mathbf{b}) \sim \sum_{\lambda_1, \lambda_2} P_{\lambda_1, \lambda_2}^{HL}(\mathbf{a} | \tau) P_{\lambda_1, \lambda_2}^{HL}(\mathbf{b} | \tau).$$

HL index - E_7 and E_8 SCFTs

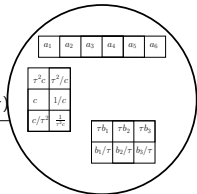
The index of the E_7 SCFT is given by

$$\begin{aligned} \mathcal{I}_{E_7}(\mathbf{a}_1, \mathbf{a}_2, c) &= \frac{(1 + \tau^2 + \tau^4)(1 + \tau^4)}{(1 - \tau^2)^3} \frac{\mathcal{K}(\mathbf{a}_1)\mathcal{K}(\mathbf{a}_2)}{(1 - \tau^2 c \pm 2)(1 - \tau^4 c \pm 2)} \times \\ &\sum_{\lambda_1, \lambda_2, \lambda_3} \frac{P_{\lambda_1, \lambda_2, \lambda_3}^{HL}(\tau c, \frac{c}{\tau}, \frac{\tau}{c}, \frac{1}{\tau c} | \tau)}{P_{\lambda_1, \lambda_2, \lambda_3}^{HL}(\tau^3, \tau, \tau^{-1}, \tau^{-3} | \tau)} \prod_{i=1}^2 P_{\lambda_1, \lambda_2, \lambda_3}^{HL}(\mathbf{a}_i | \tau) \cdot \\ &= \sum_{k=0}^{\infty} [k, 0, 0, 0, 0, 0, 0]_{\mathbf{z}} \tau^{2k}. \end{aligned}$$



The index of the E_8 SCFT is given by

$$\begin{aligned} \mathcal{I}_{E_8}(\mathbf{a}, (b_1, b_2), c) &= \frac{(1 - \tau^8)(1 - \tau^{10})(1 - \tau^{12})}{(1 - \tau^2)^3(1 - \tau^4)^4(1 - \tau^6)} \times \\ &\frac{\mathcal{K}(\mathbf{a})}{(1 - \tau^2 c \pm 2)(1 - \tau^4 c \pm 2)(1 - \tau^6 c \pm 2) \prod_{i \neq j} (1 - \tau^2 b_i / b_j)(1 - \tau^4 b_i / b_j)} \times \\ &\sum_{\lambda_1, \dots, \lambda_5 \equiv \lambda} \frac{P_{\lambda}^{HL}(\tau b_1, \tau b_2, \tau b_3, \frac{b_1}{\tau}, \frac{b_2}{\tau}, \frac{b_3}{\tau} | \tau) P_{\lambda}^{HL}(\tau^2 c, c, \frac{c}{\tau^2}, \frac{\tau^2}{c}, \frac{1}{c}, \frac{1}{\tau^2 c} | \tau) P_{\lambda}^{HL}(\mathbf{a} | \tau)}{P_{\lambda}^{HL}(\tau^5, \tau^3, \tau, \tau^{-1}, \tau^{-3}, \tau^{-5} | \tau)} \\ &= \sum_{k=0}^{\infty} [k, 0, 0, 0, 0, 0, 0, 0]_{\mathbf{z}} \tau^{2k}. \end{aligned}$$



Outlook

- Possible to include line and surface operators (in progress)
 - Macdonald index closely related to refined CS: hint of a relation to topological strings? $(2,0)$ theory on $S^3 \times S^1 \times \mathcal{C}$ versus $(\mathbb{C} \times S^1 \times M_3)_{q,t}$ with $M_3 = S^1 \times \mathcal{C} \dots$
 - Full three-parameter index?
Natural to try elliptic generalizations of Macdonald polynomials.
This idea can be sharpened by making recalling the relations between $2d$ YM, Macdonald polynomials an a family of integrable models:
 - ▶ The reduction of $2d$ YM on a cylinder gives the Calogero-Moser model (Gorsky-Nekrasov).
 - ▶ Macdonald polynomials are eigenfunctions of a relativistic version of the trigonometric Calogero-Moser model.
- The relevant elliptic generalizations of Macdonald polynomials should be eigenfunctions of the relativistic elliptic Calogero-Moser model. Not much is explicitly known about them ...
- Microscopic derivation of the $2d$ TQFT from the $(2,0)$ theory?
Perhaps easiest to derive quantum-mechanical model obtained by reduction of $2d$ TQFT to a graph \mathcal{G} . (From $5d$ SYM on $S^3 \times S^1 \times \mathcal{G}$?)

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