Braids, Walls and Mirrors

Based on joint work with

Sergio Cecotti and Clay Cordova

N=2, d=4 from M5 branes:

M5 brane on

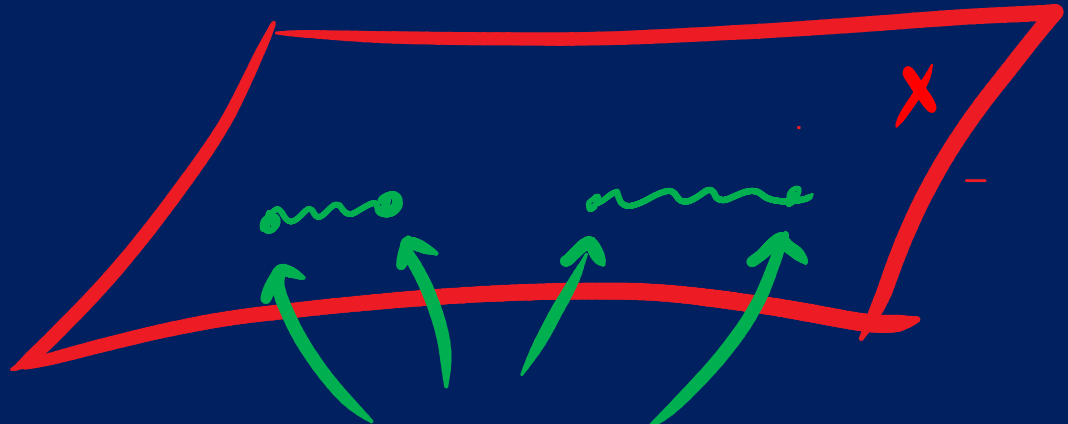
$$R^4 \times \Sigma$$

$$\Sigma \subset \mathbb{C}^2 : f(x,y) = 0$$

$$\lambda_{SW} = y\, dx$$

$$f(x,y) := y^2 - g(x)$$

$$y = \pm\sqrt{g(x)}$$



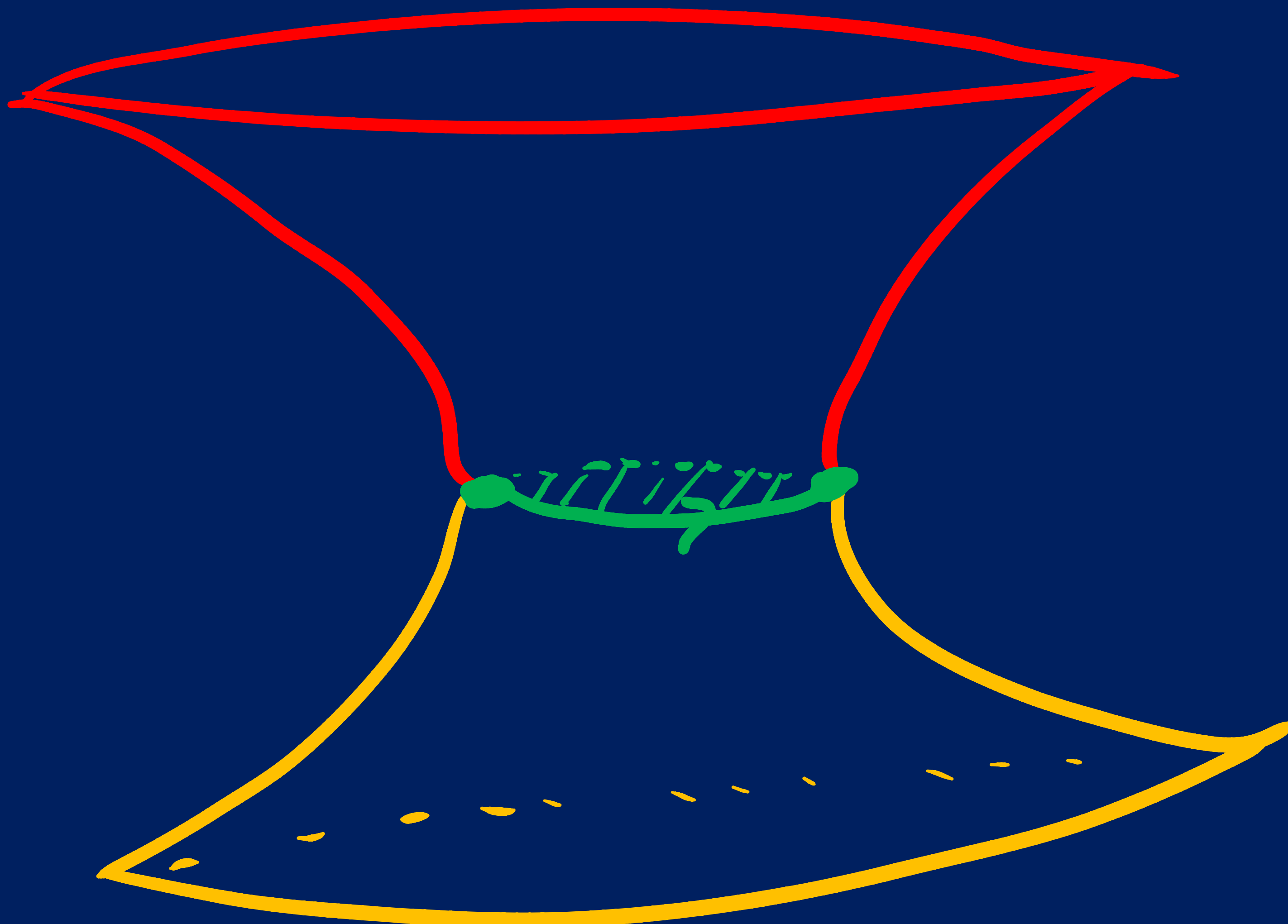
$g(x) = 0$

BPS states: M2 branes ending on M5 branes.

SUSY: ydx has the same phase: (Klemm et.al., Shapere,GMN)

$$y\,dx = \alpha\,dt$$

$$Z_\gamma = |Z_\gamma| e^{i\alpha}$$



$$Z = \oint y\, dx$$

$$y^2 = x^2 - \mu$$

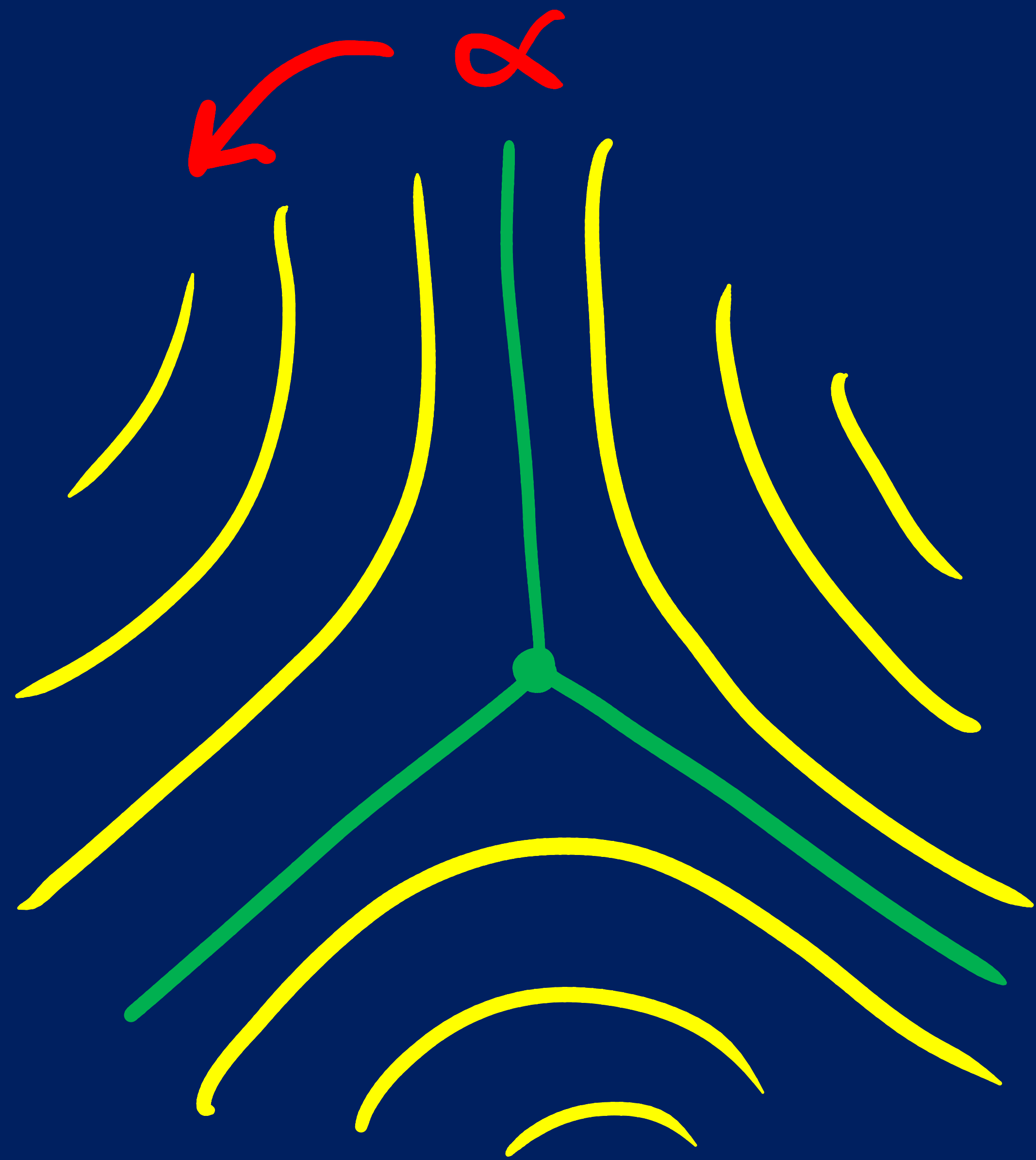
Near each branch point where y=0, the behaviour of the flow has a cubic structure:

$$y\,y^{2} \sim x\,x^2 - \mu \qquad x = \pm\sqrt{\mu}$$

$$y\,dx \simeq \alpha\,dt$$

$$\sqrt{x}\;dx \sim \alpha\,dt \Rightarrow \Delta x^{3/2} \sim \alpha t$$

$$x = (x_0^{3/2} + \alpha^{2/3} t)^{2/3}$$


α

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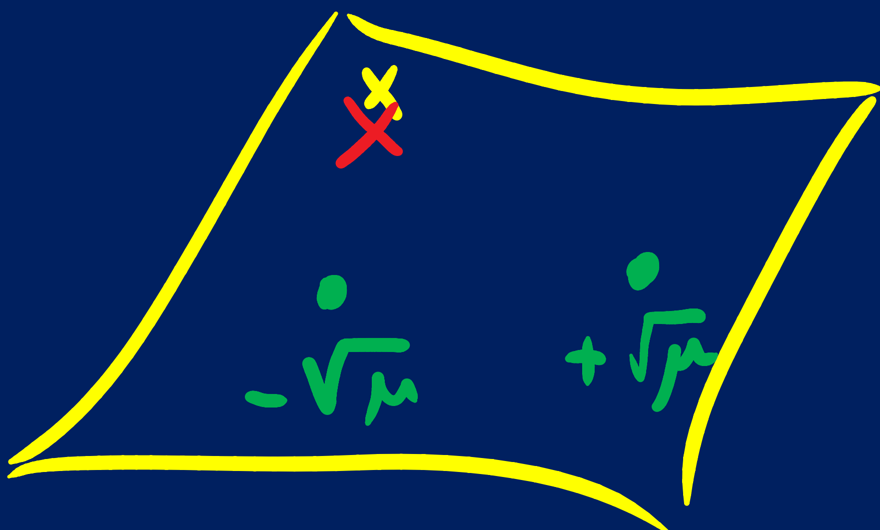

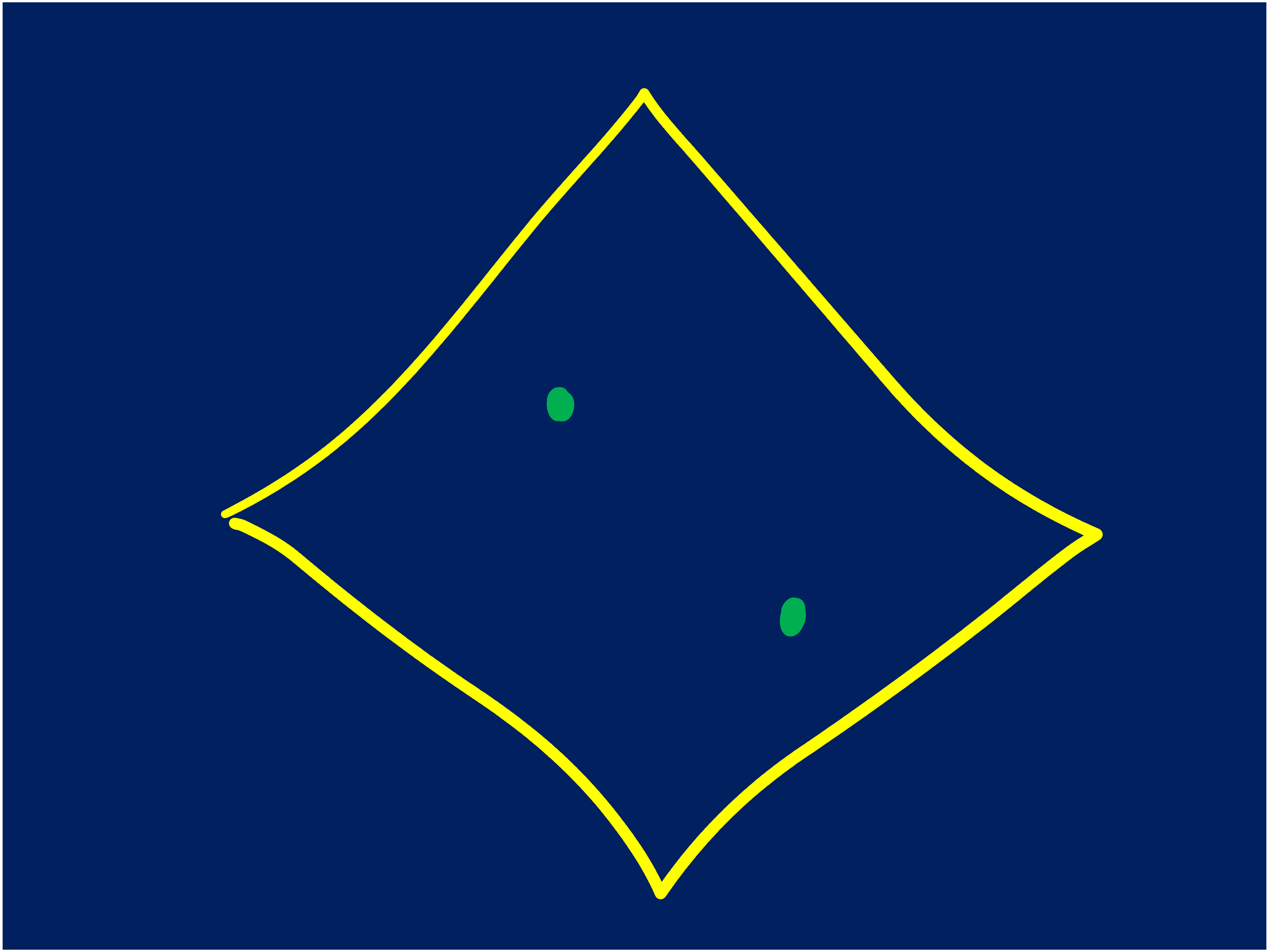

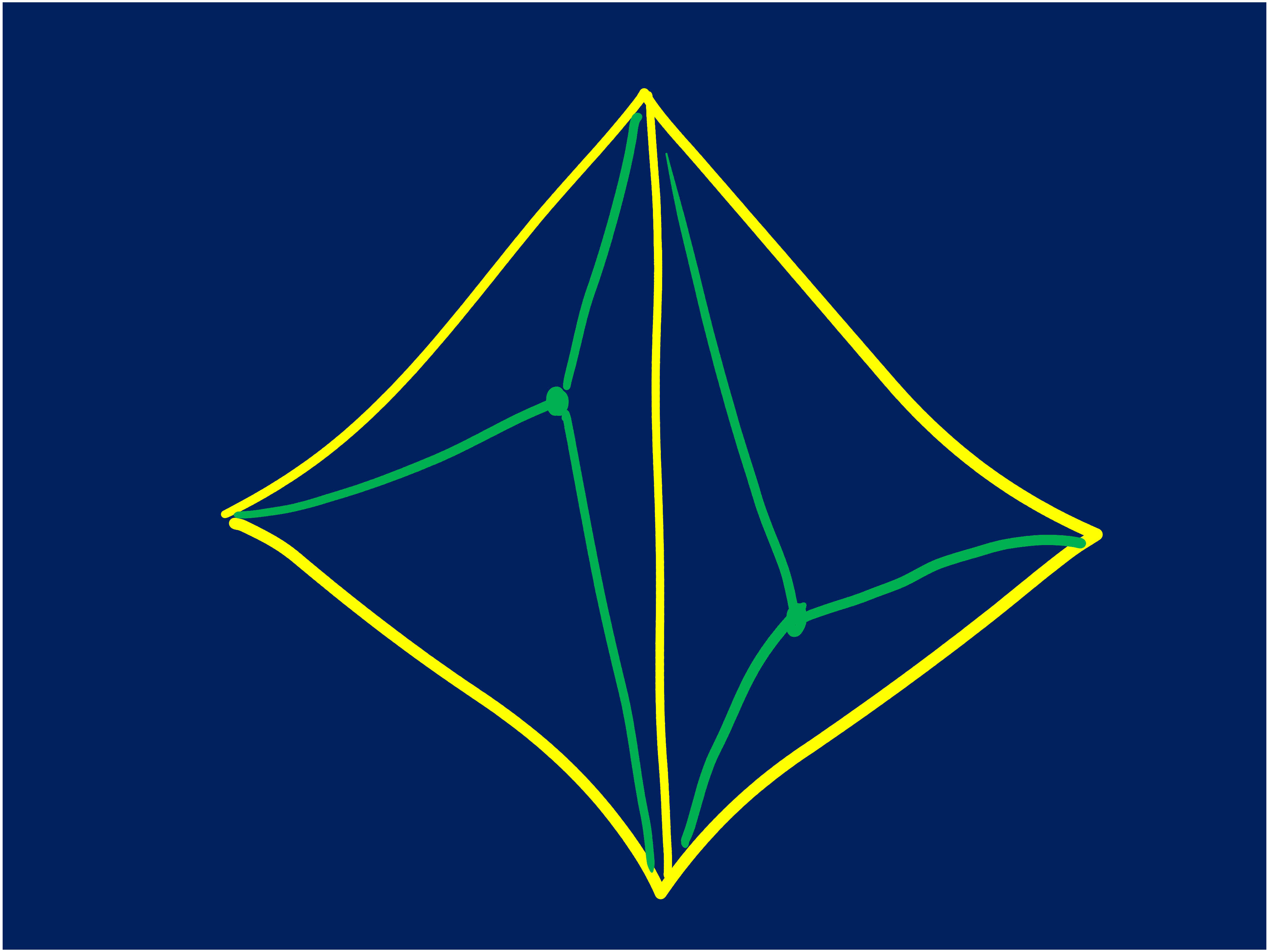

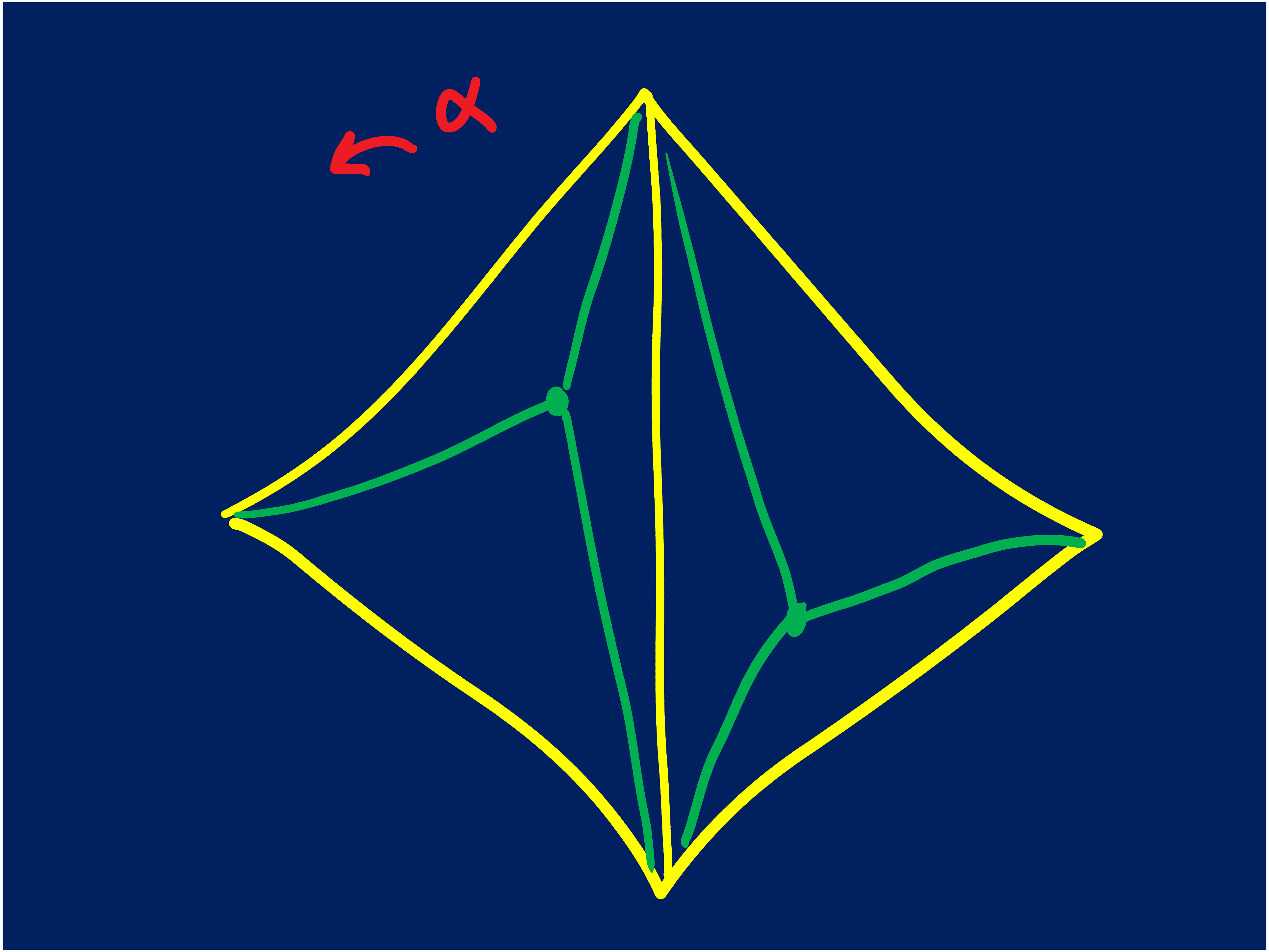
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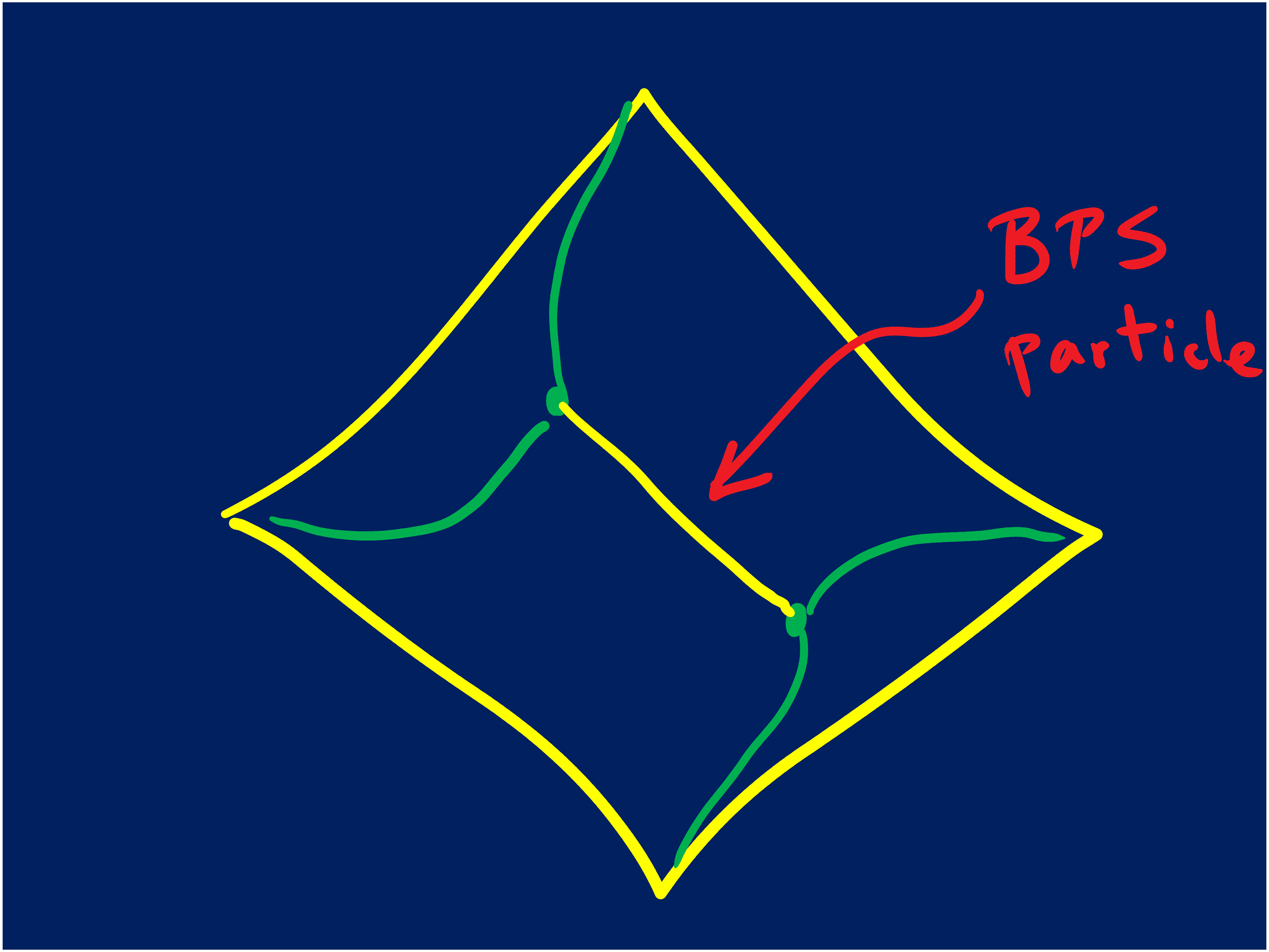
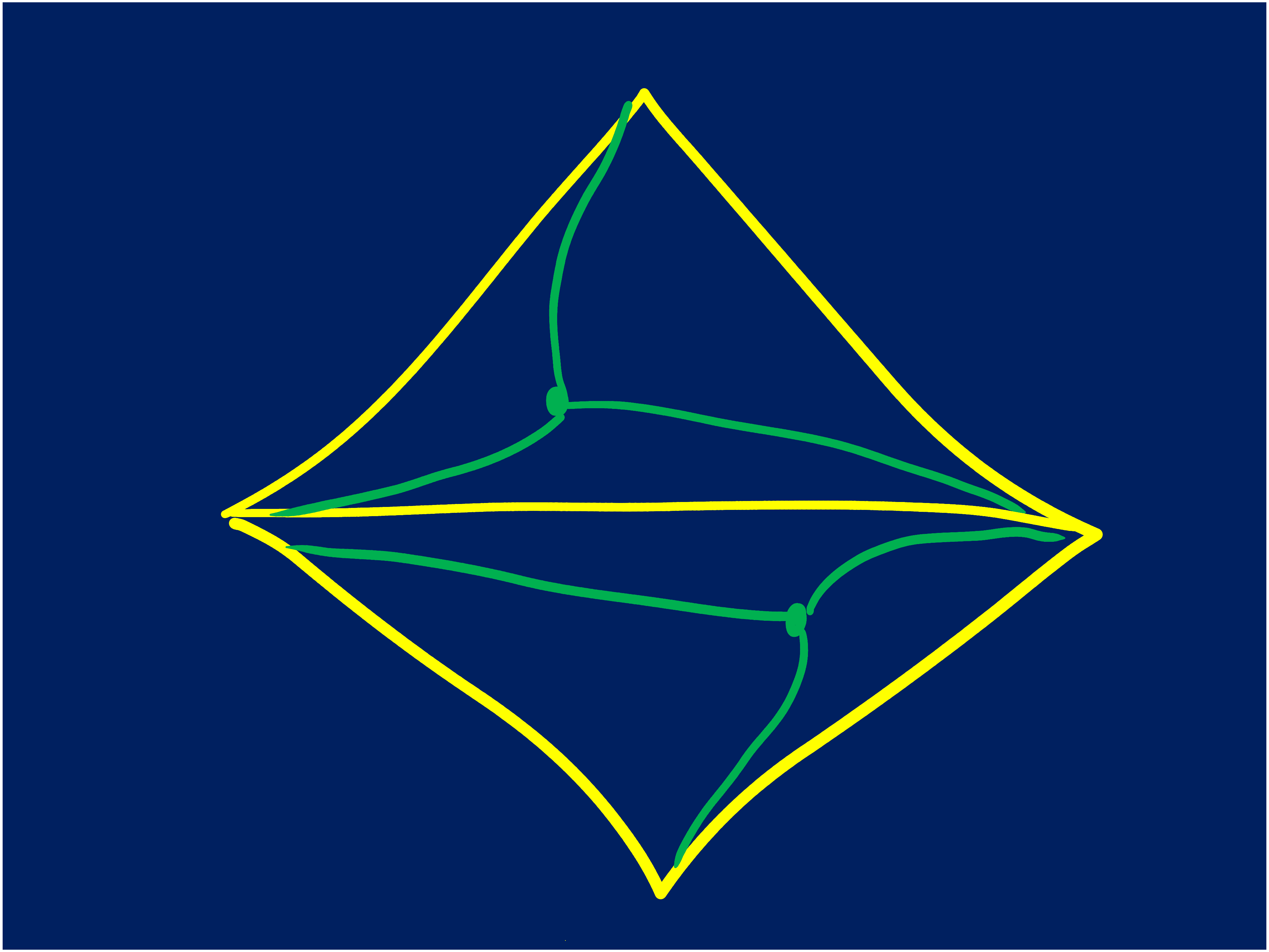
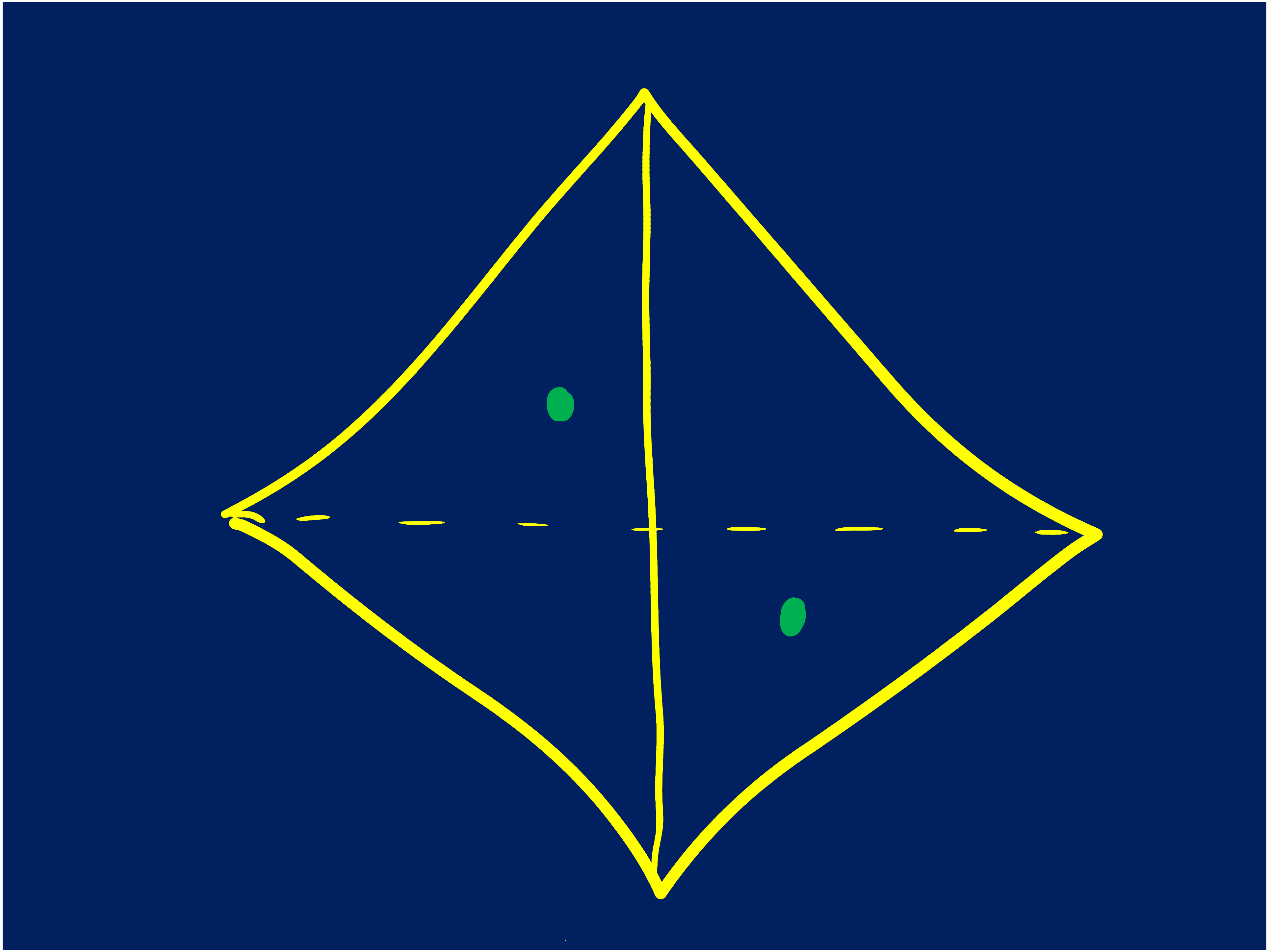
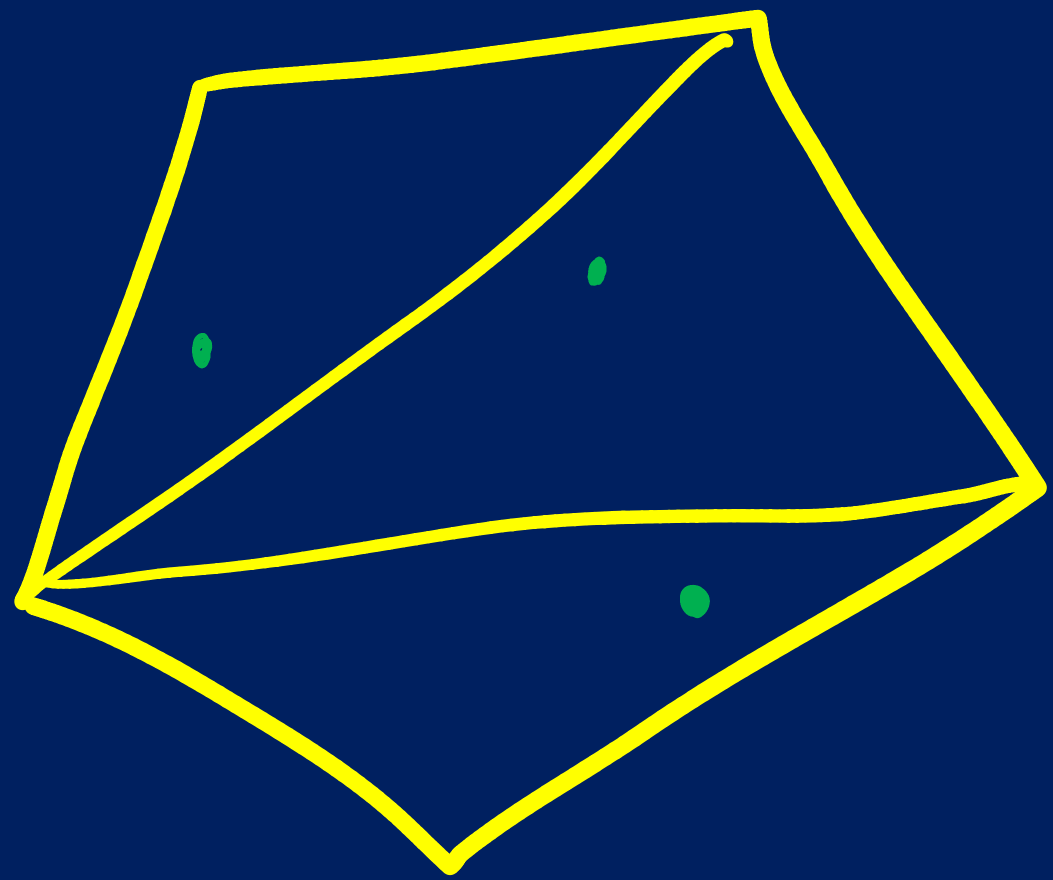

BPS
particle





$$y^2 = x^3 - \alpha x + \beta$$



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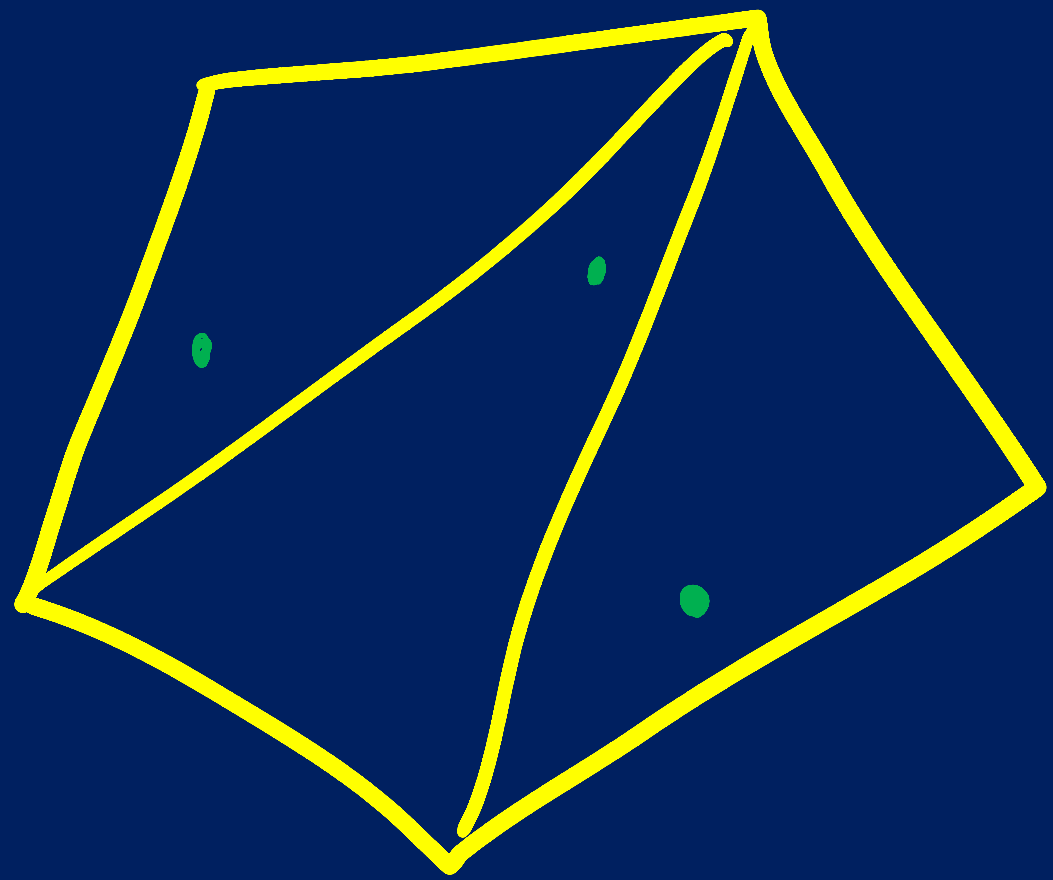

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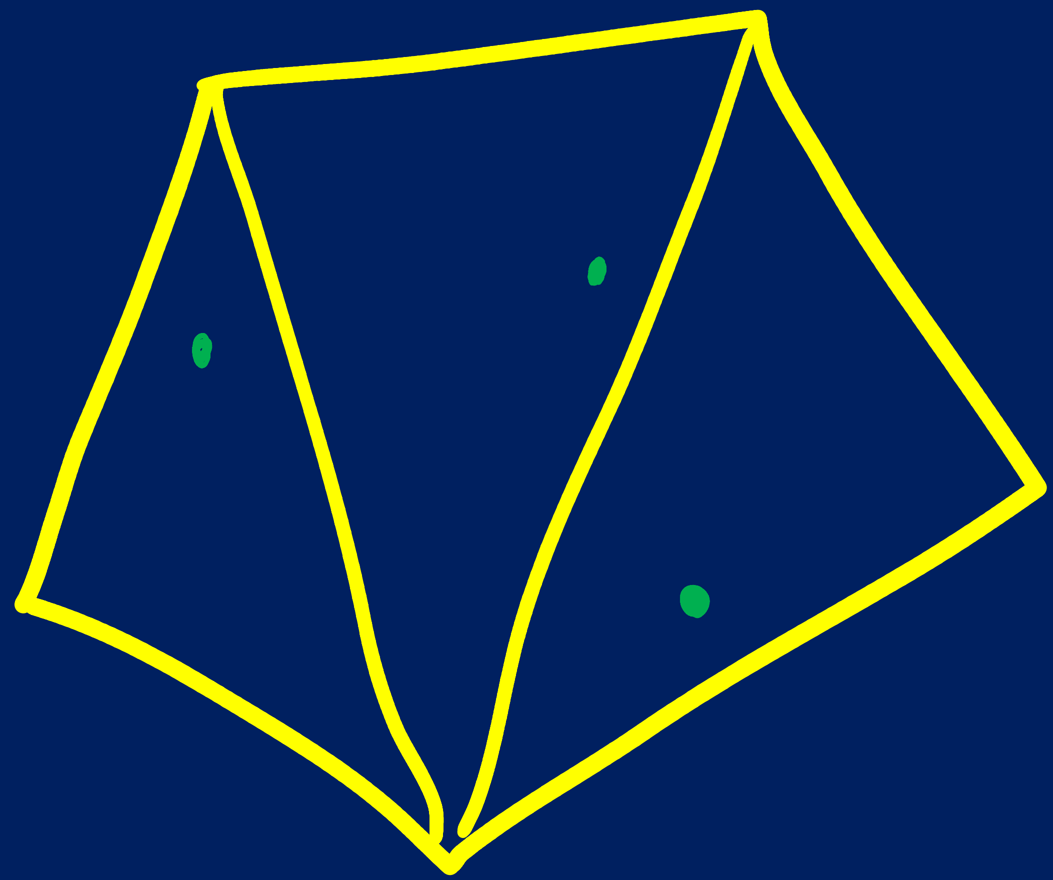

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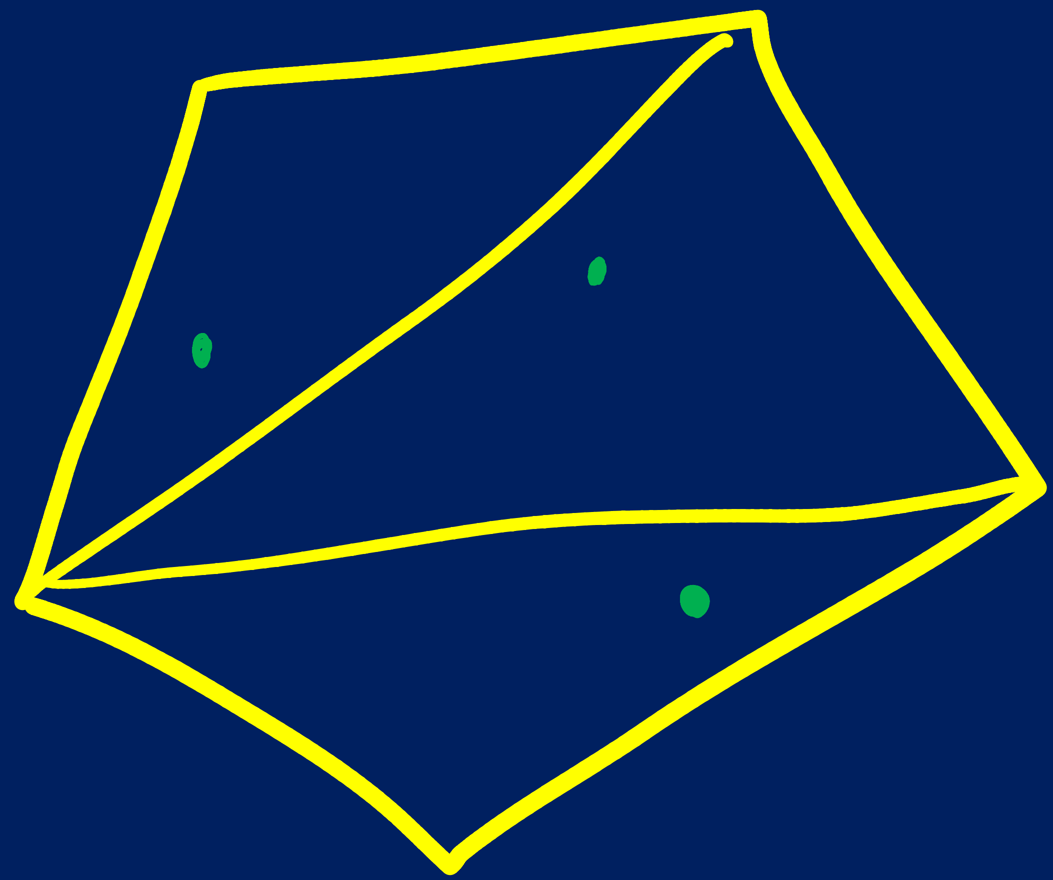

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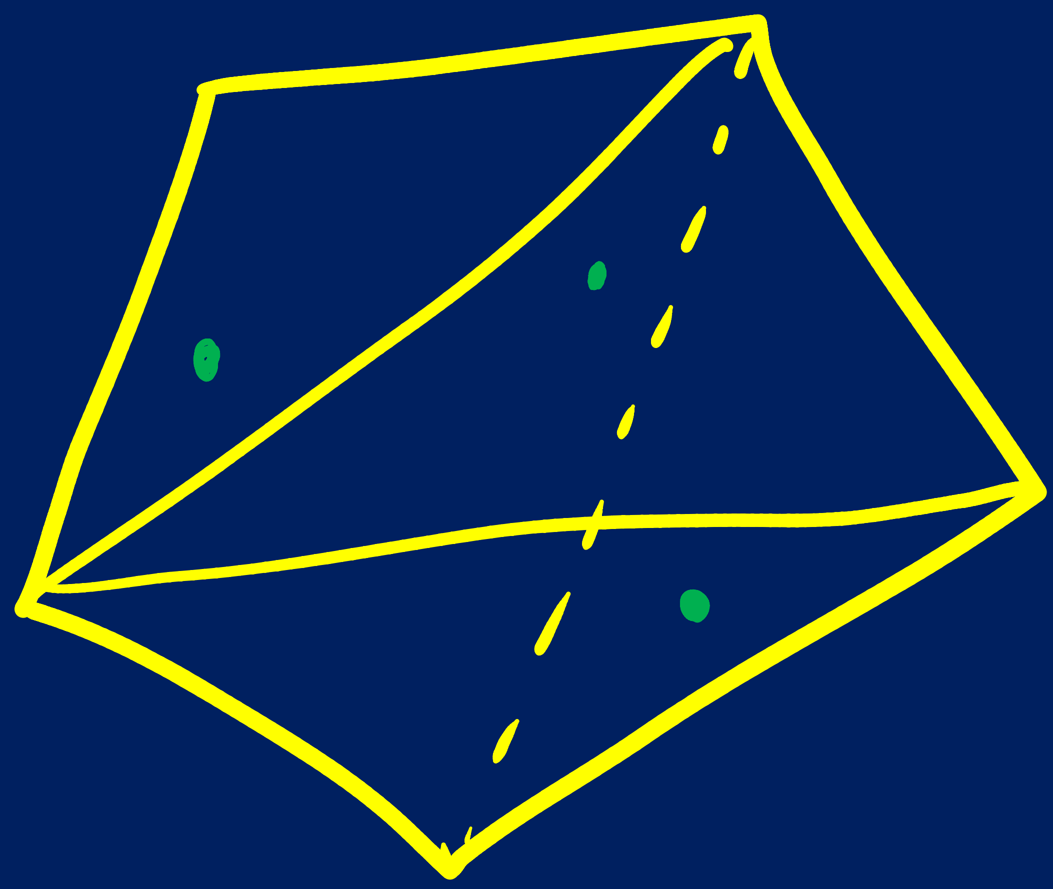

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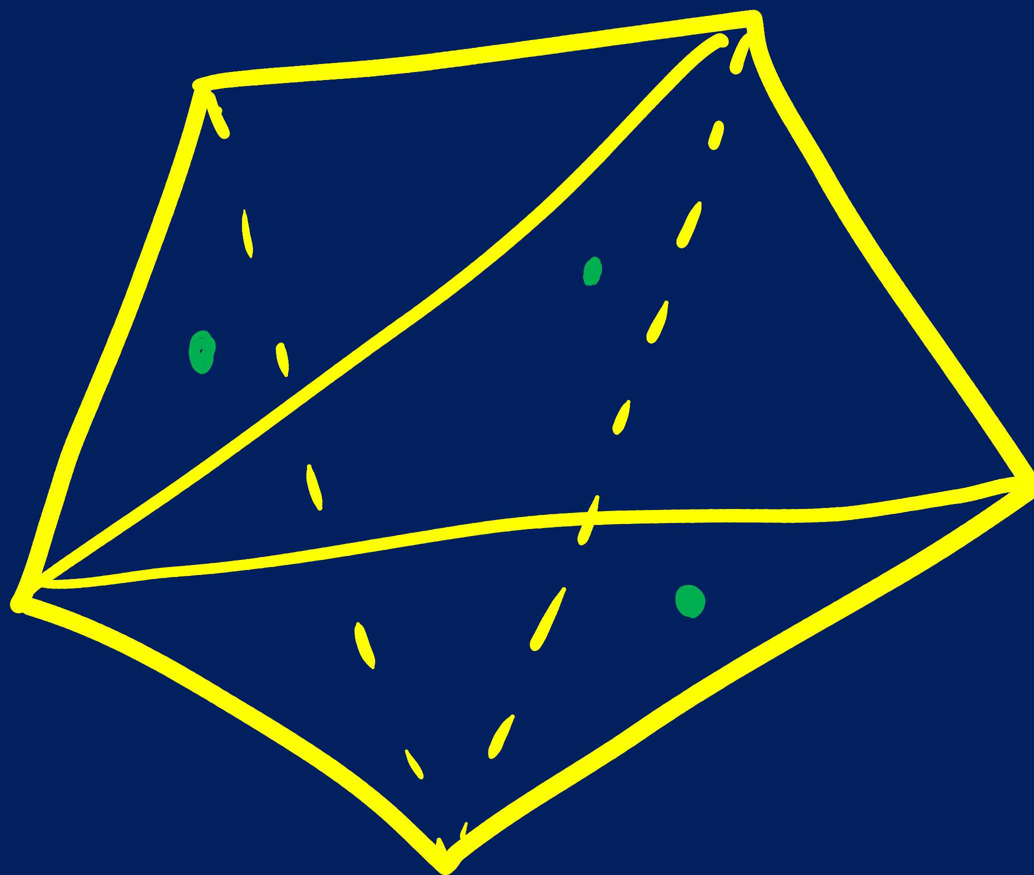
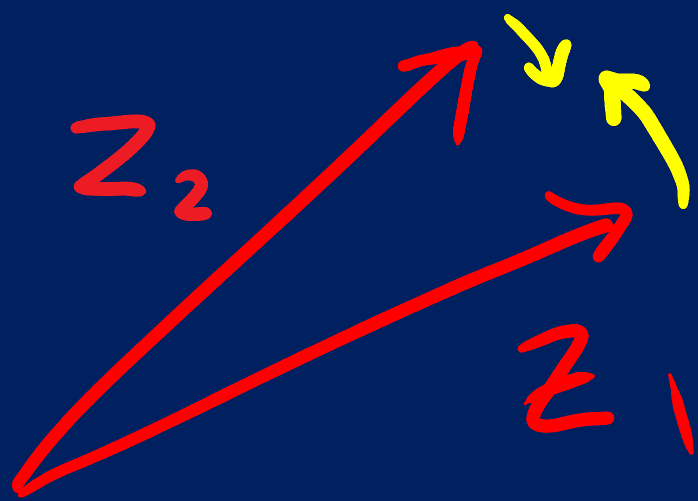


By wall-crossing, we can go to a 3-particle chamber



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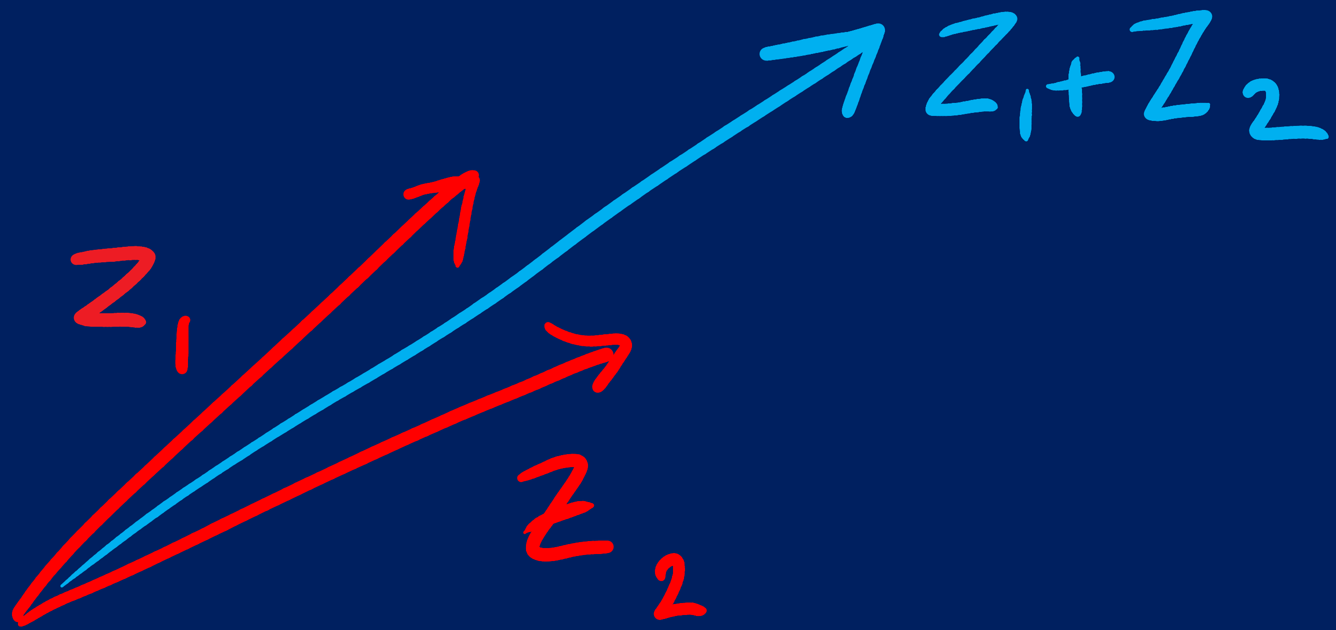

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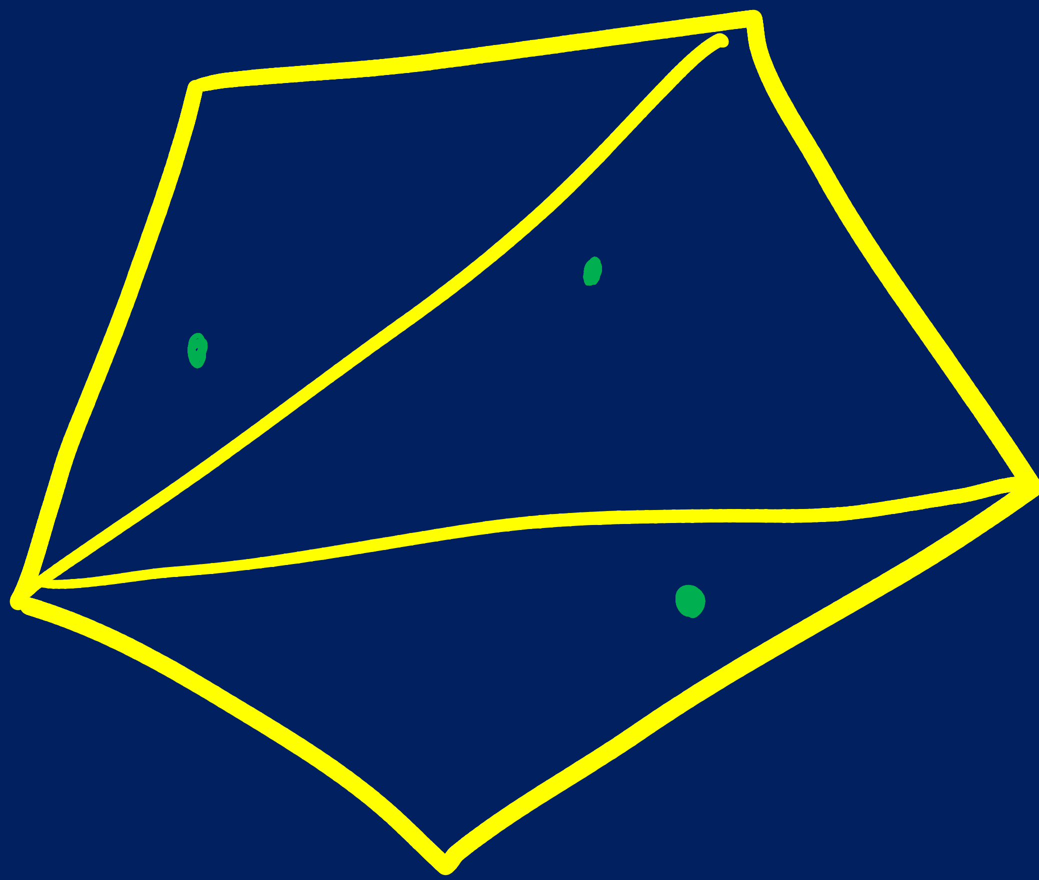

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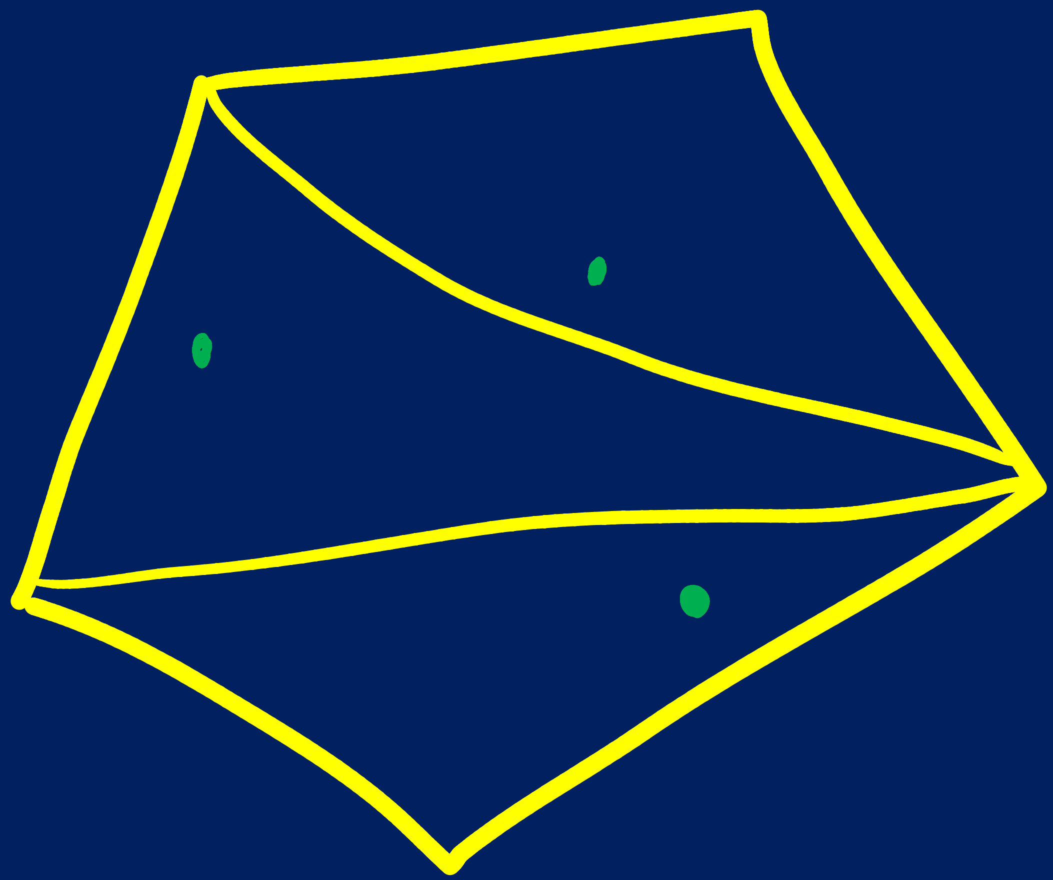

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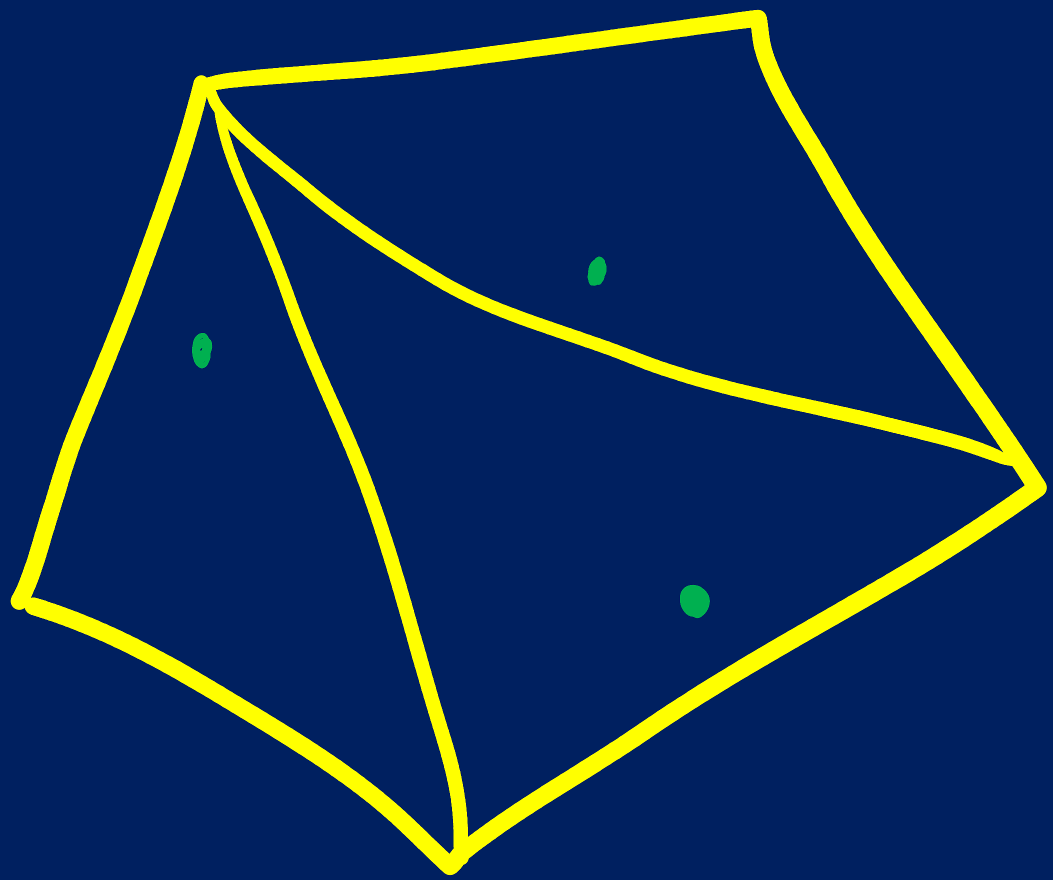

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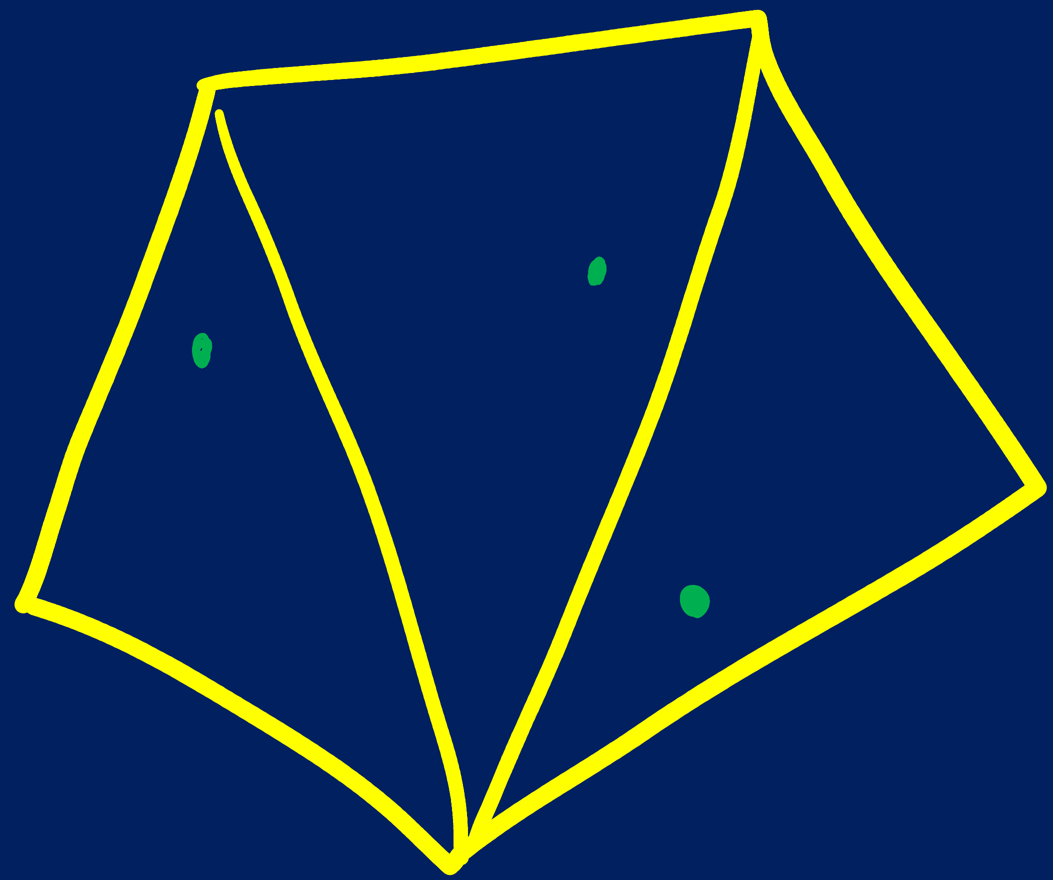

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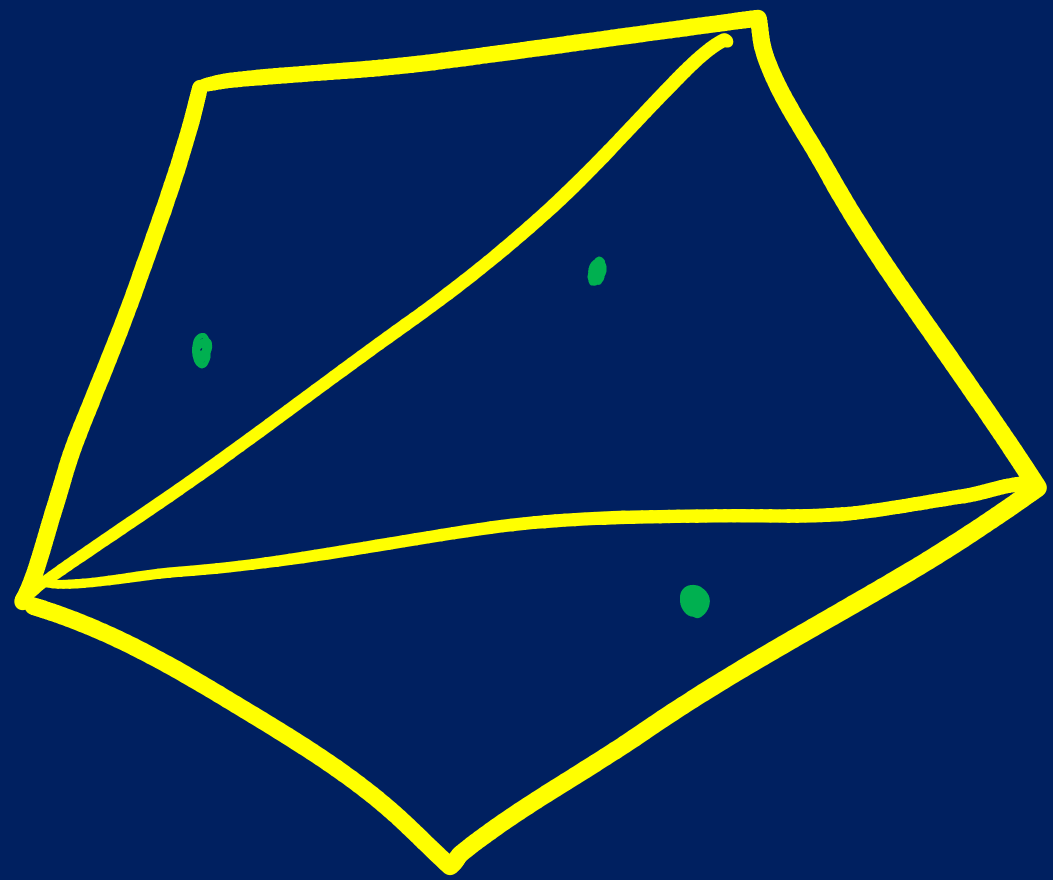

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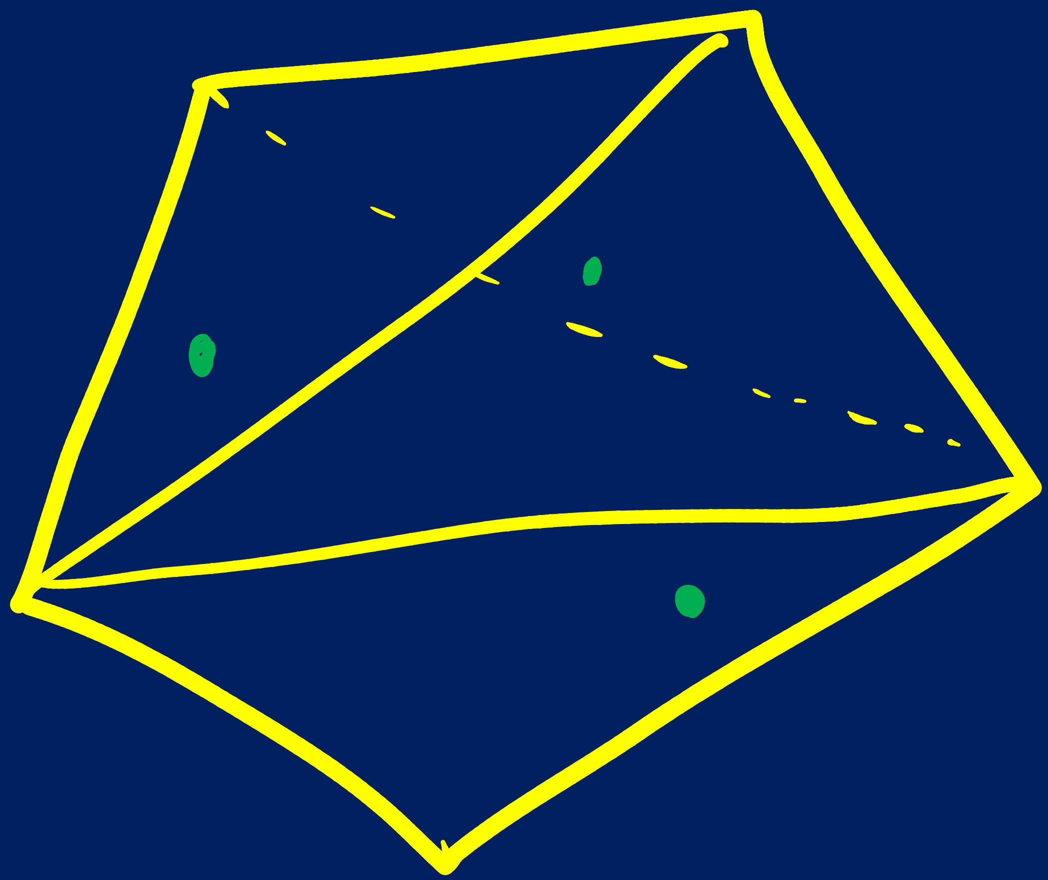

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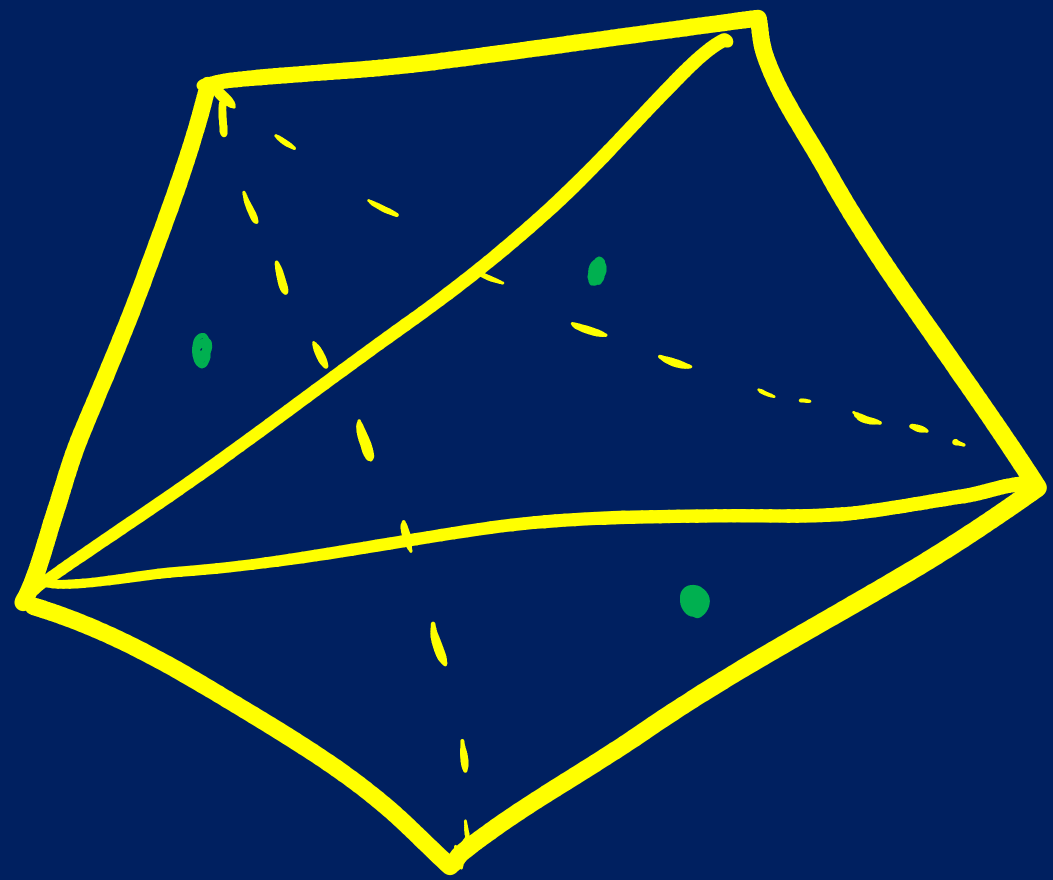

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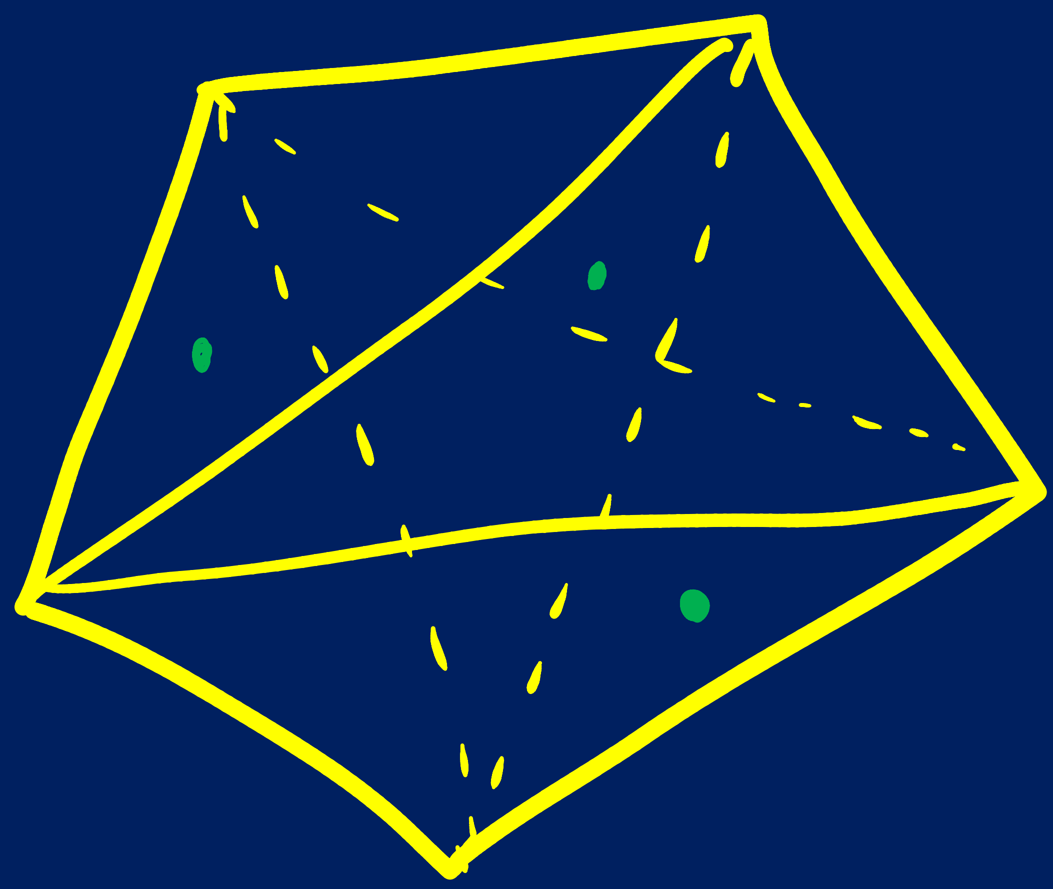

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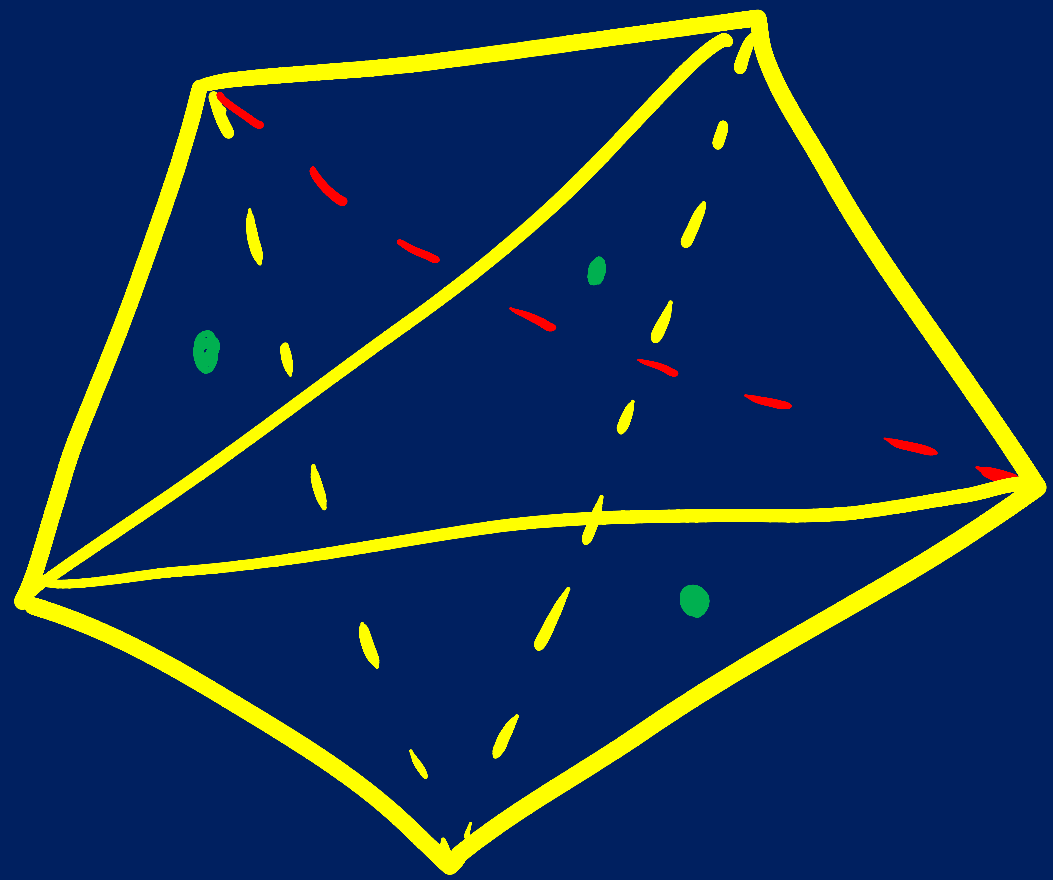

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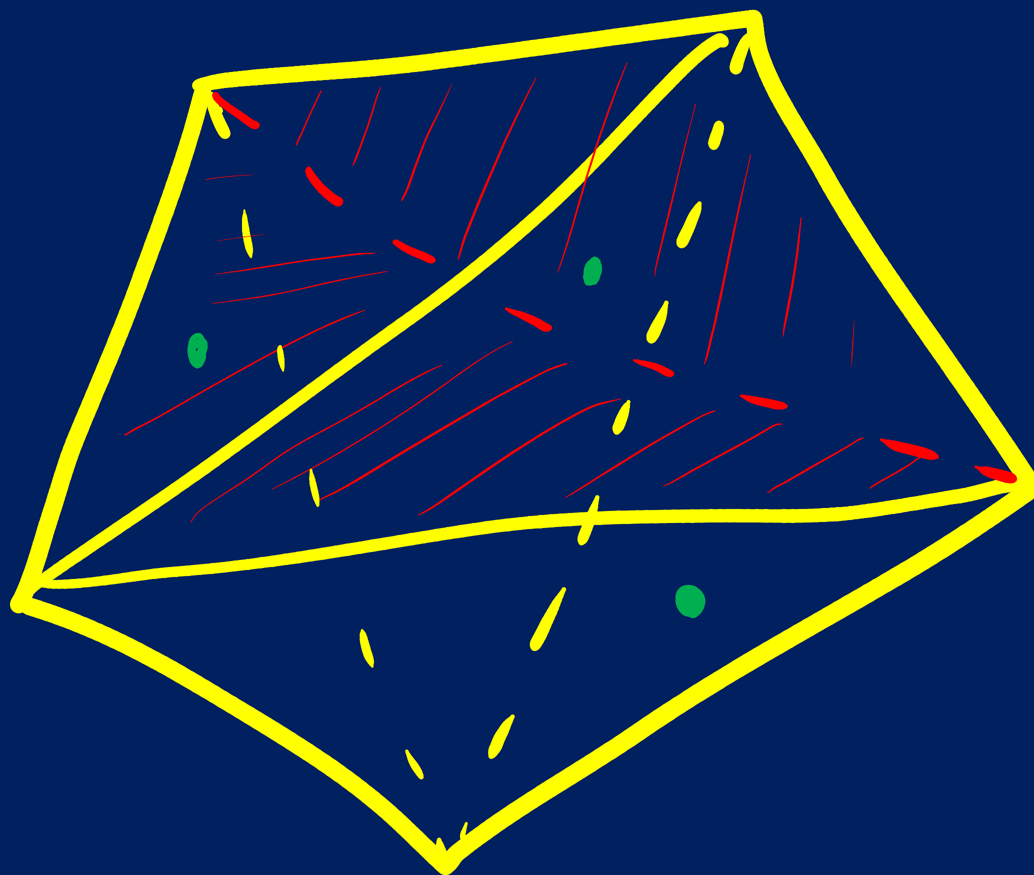

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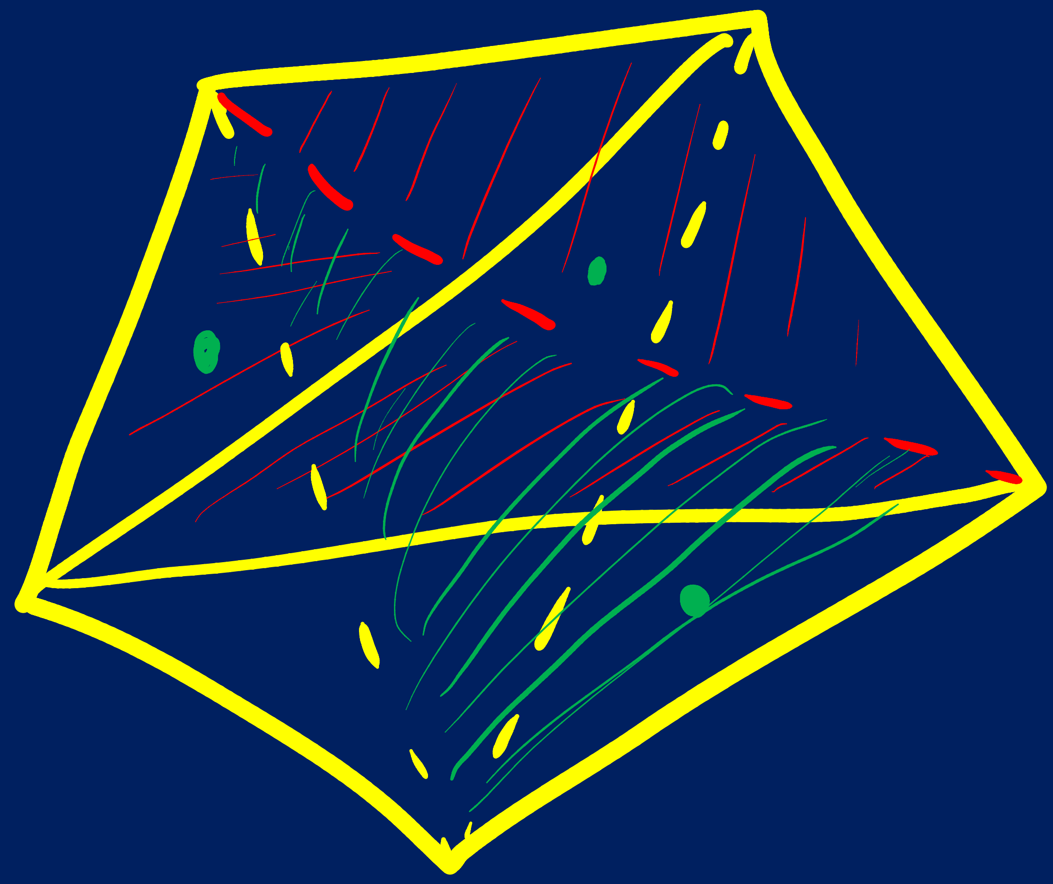

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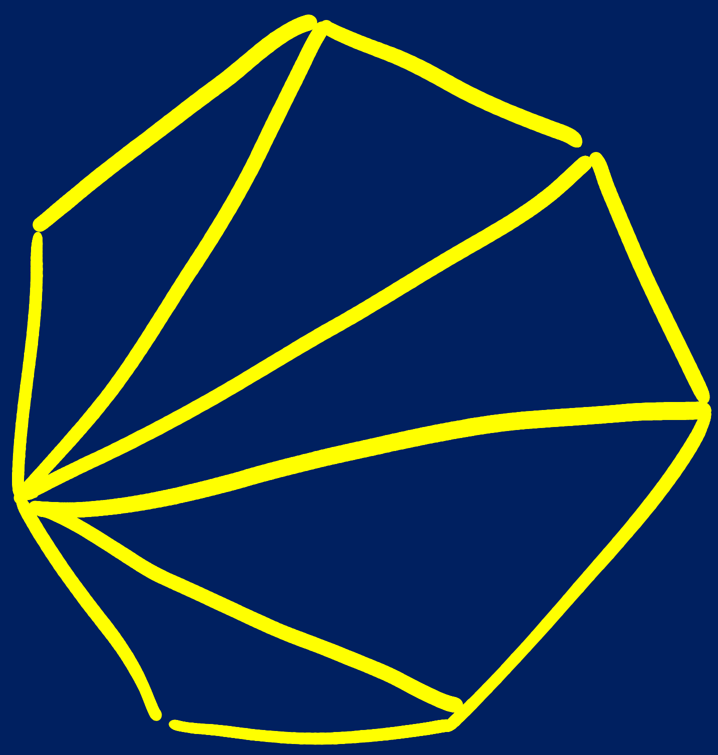

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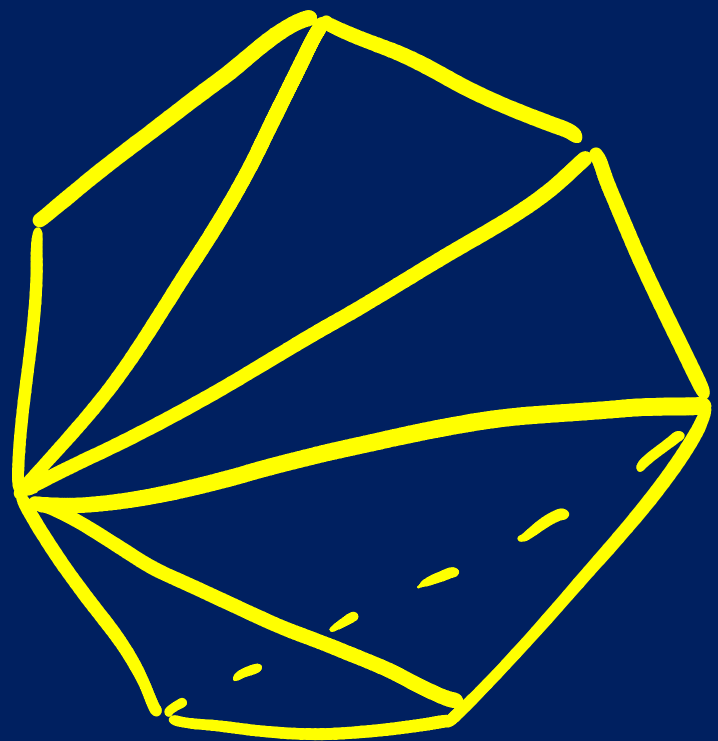

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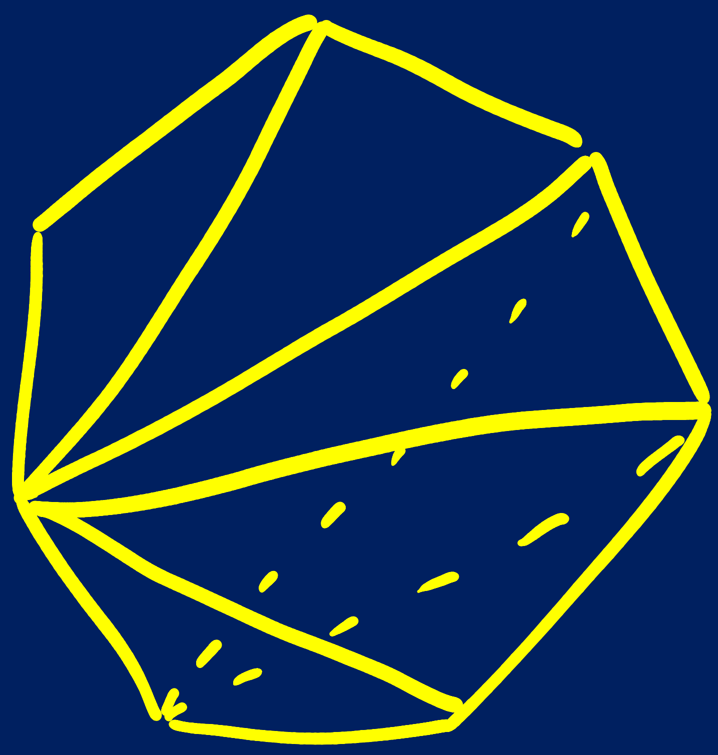

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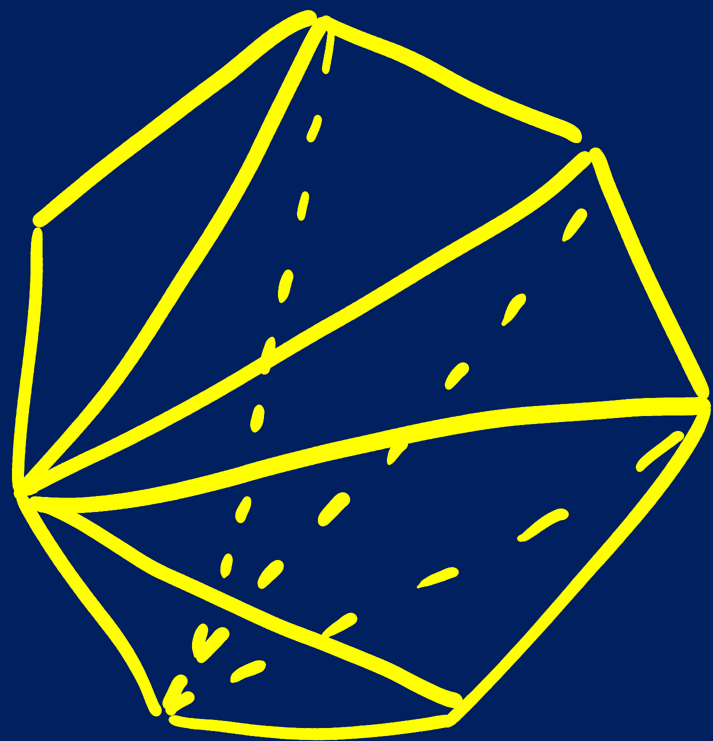

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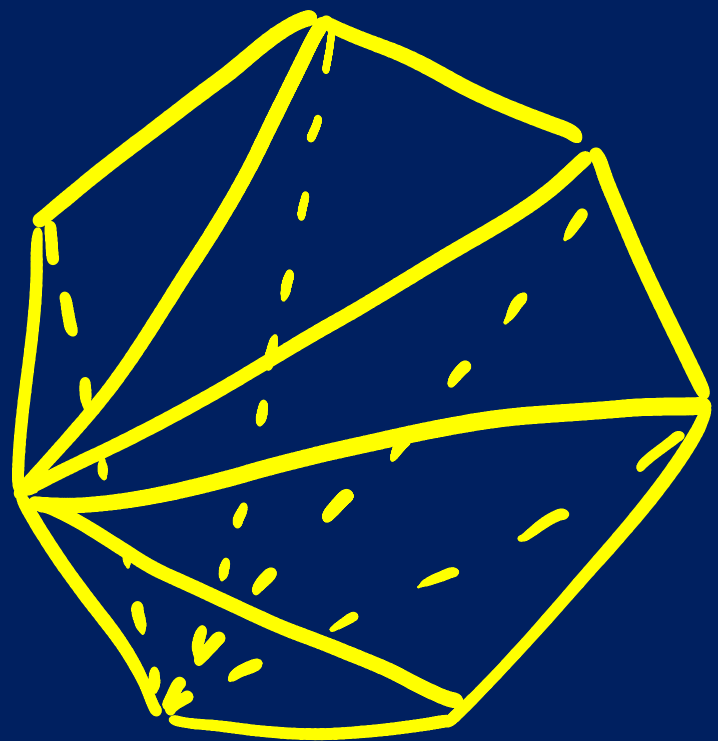
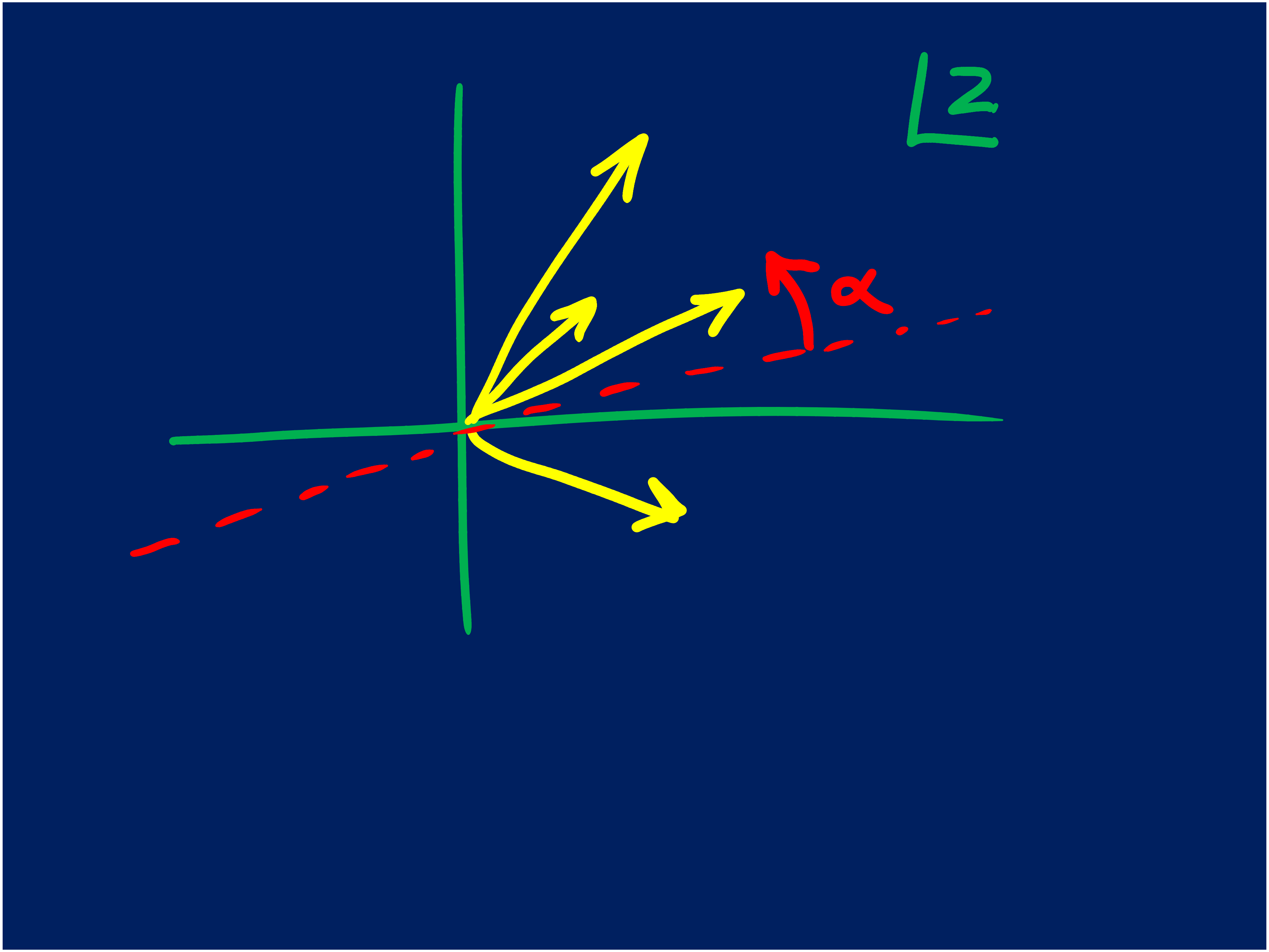
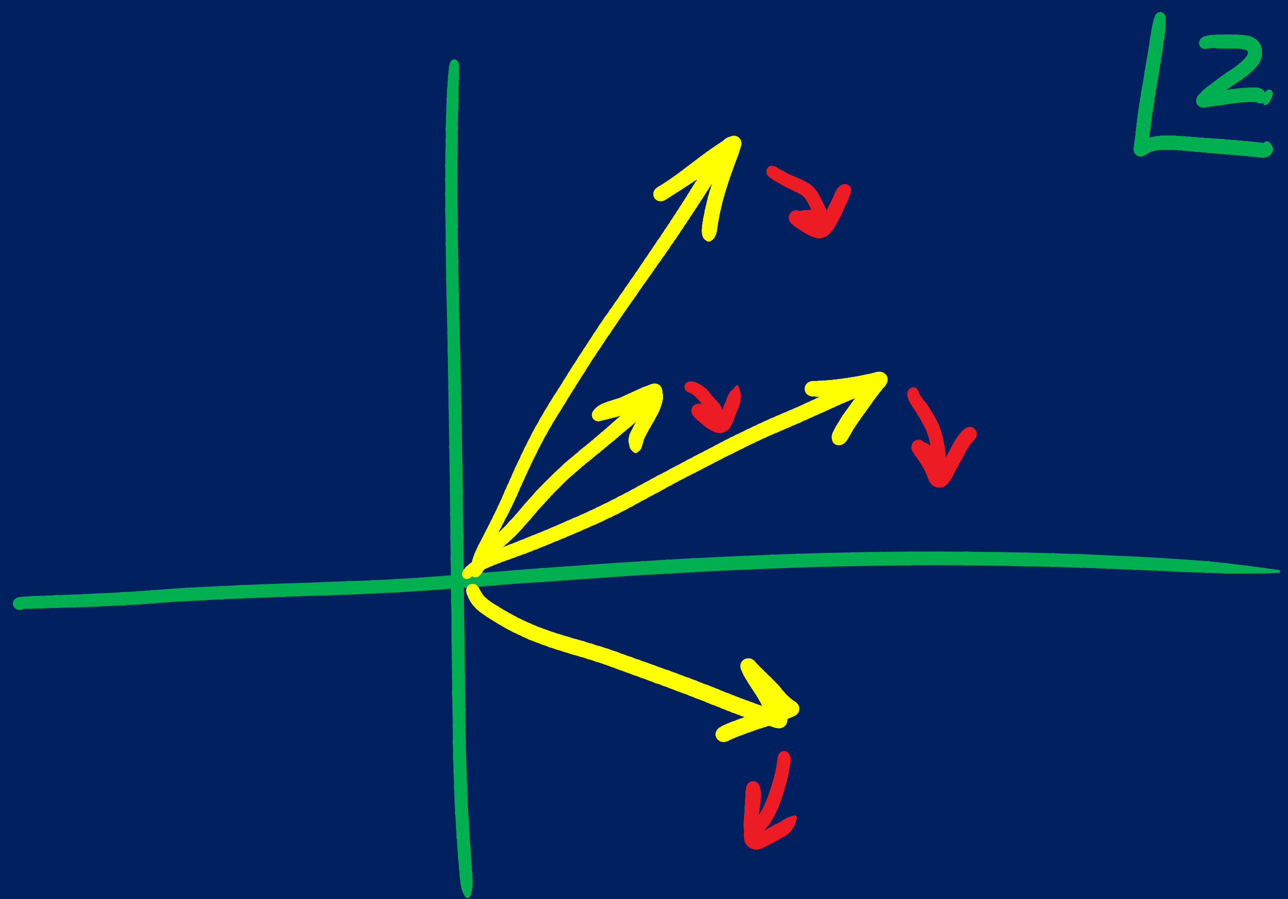


Can we view this as a 3d M5 brane?

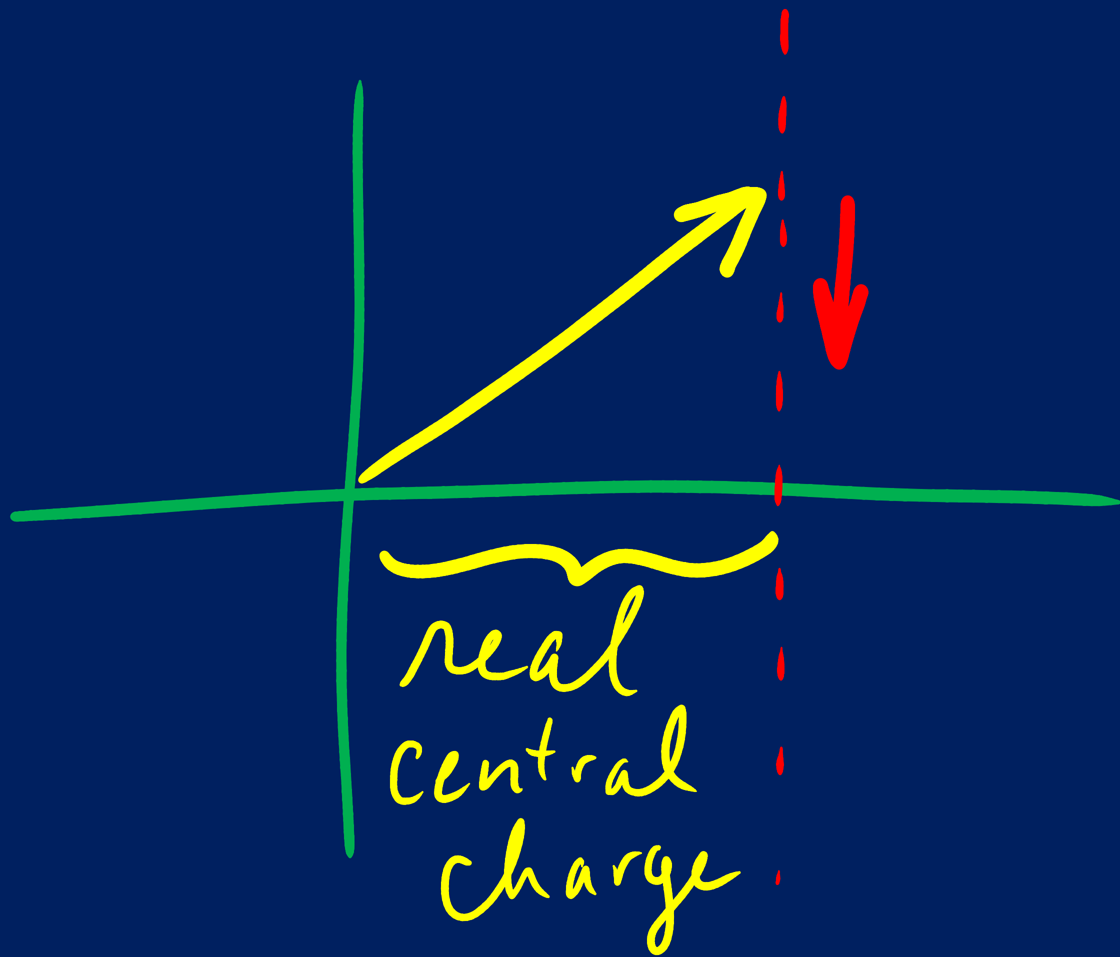
$$\Sigma \times \mathbb{R}$$

$\mathbb{R}$: change α?


L2
α


L2

This rotation does not preserve conventional* supersymmetry, instead:



*There is an unconventional supersymmetry it can preserve (Cecotti,V)

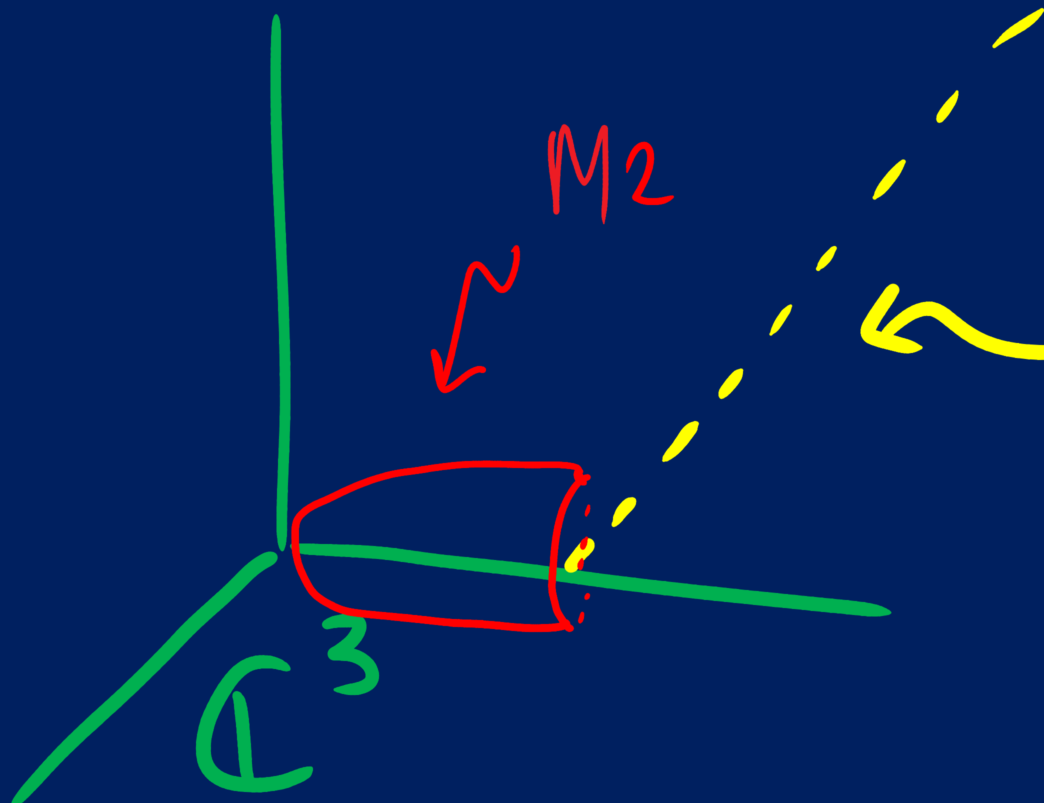
This leads to N=2, d=3 theory:

$$M5: \quad R^3 \times \underbrace{(\Sigma \times R)}_{R\text{-flow}}$$

$$K \subset CY^3$$

Special Lagrangian

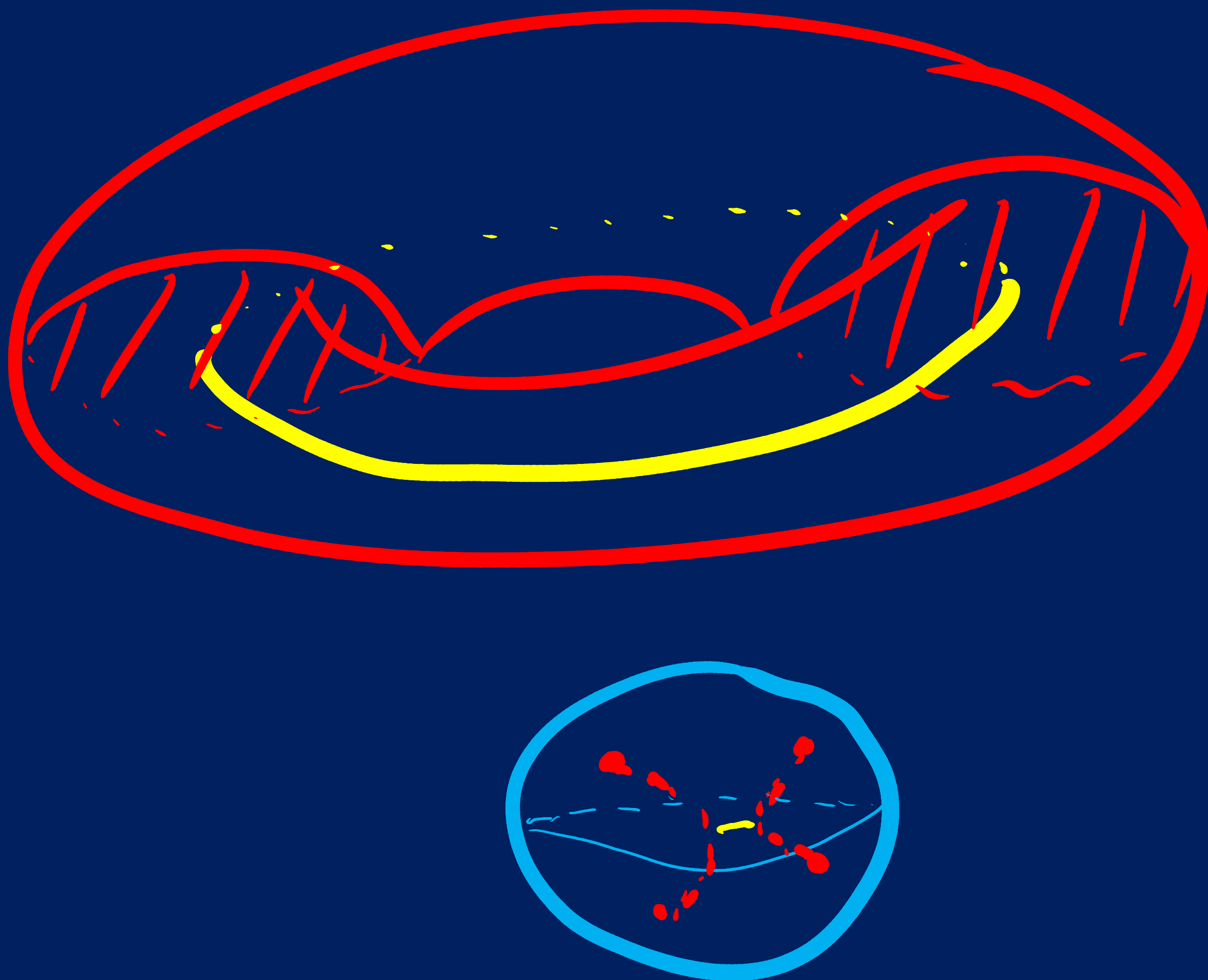
$$y^2 = x^2 - \mu \rightsquigarrow$$

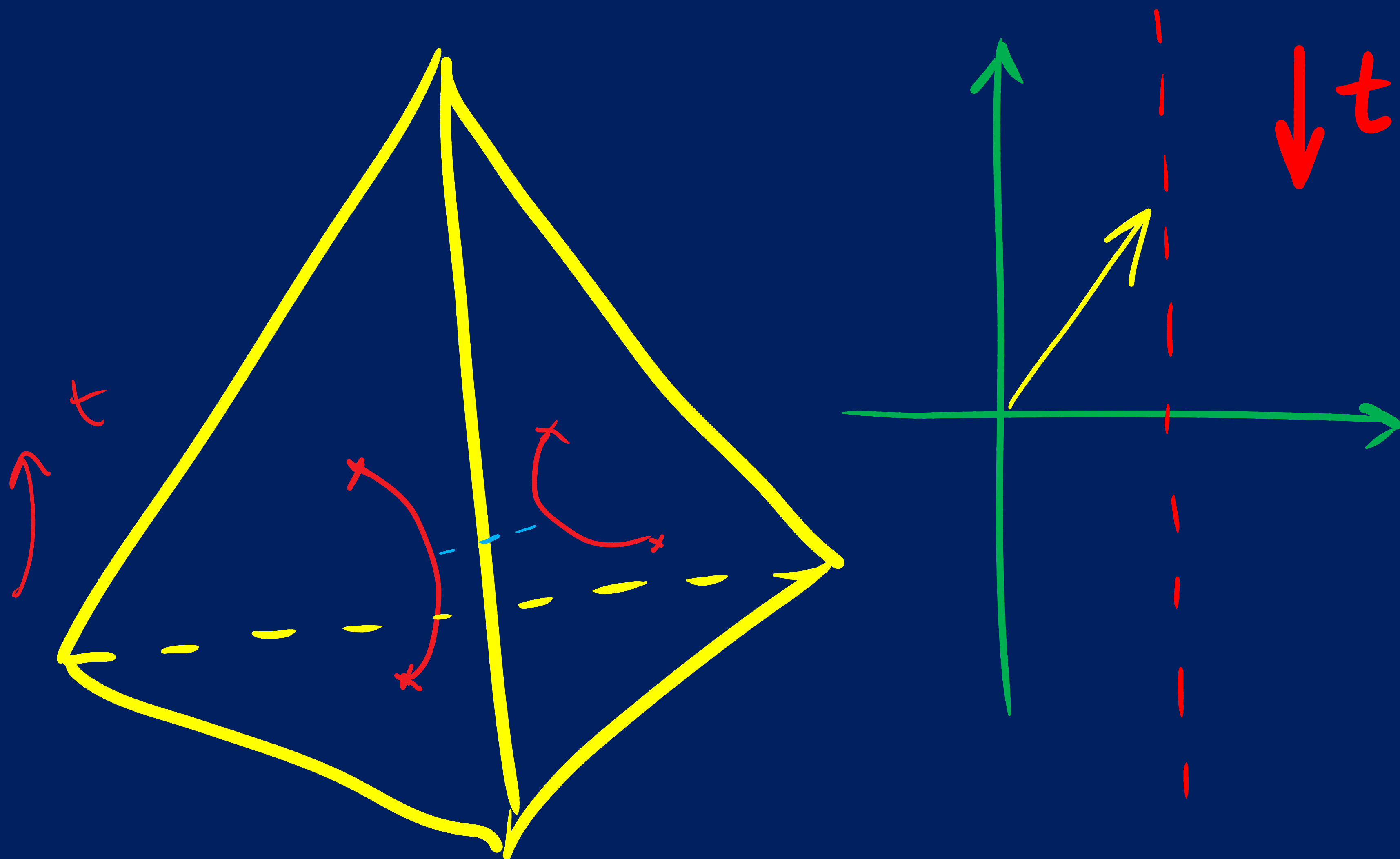


special Lagrangian

$S^1 \times R \times R$

Topology of the Special Lagragian is that of solid torus, or equivalently double cover of the solid ball:

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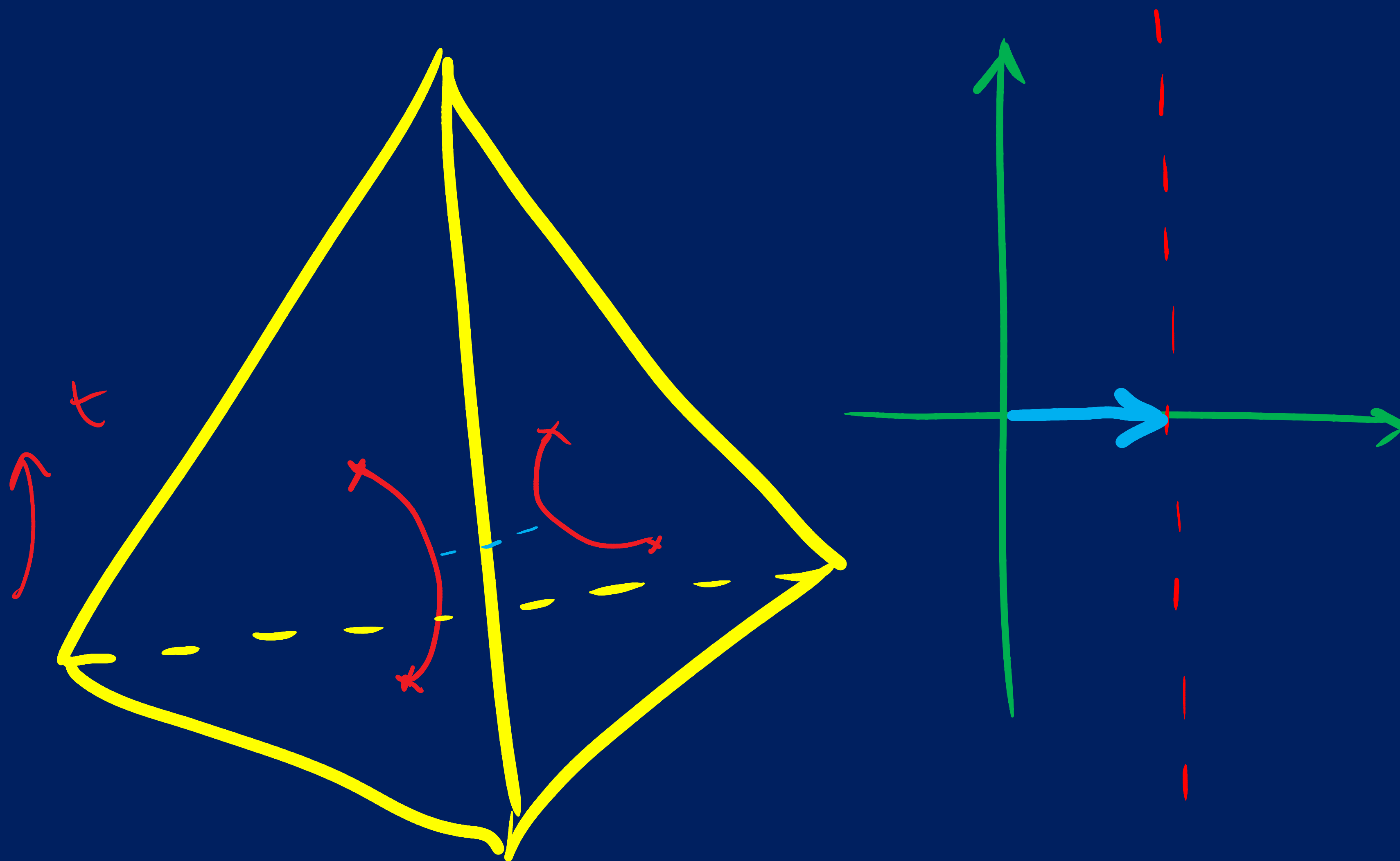

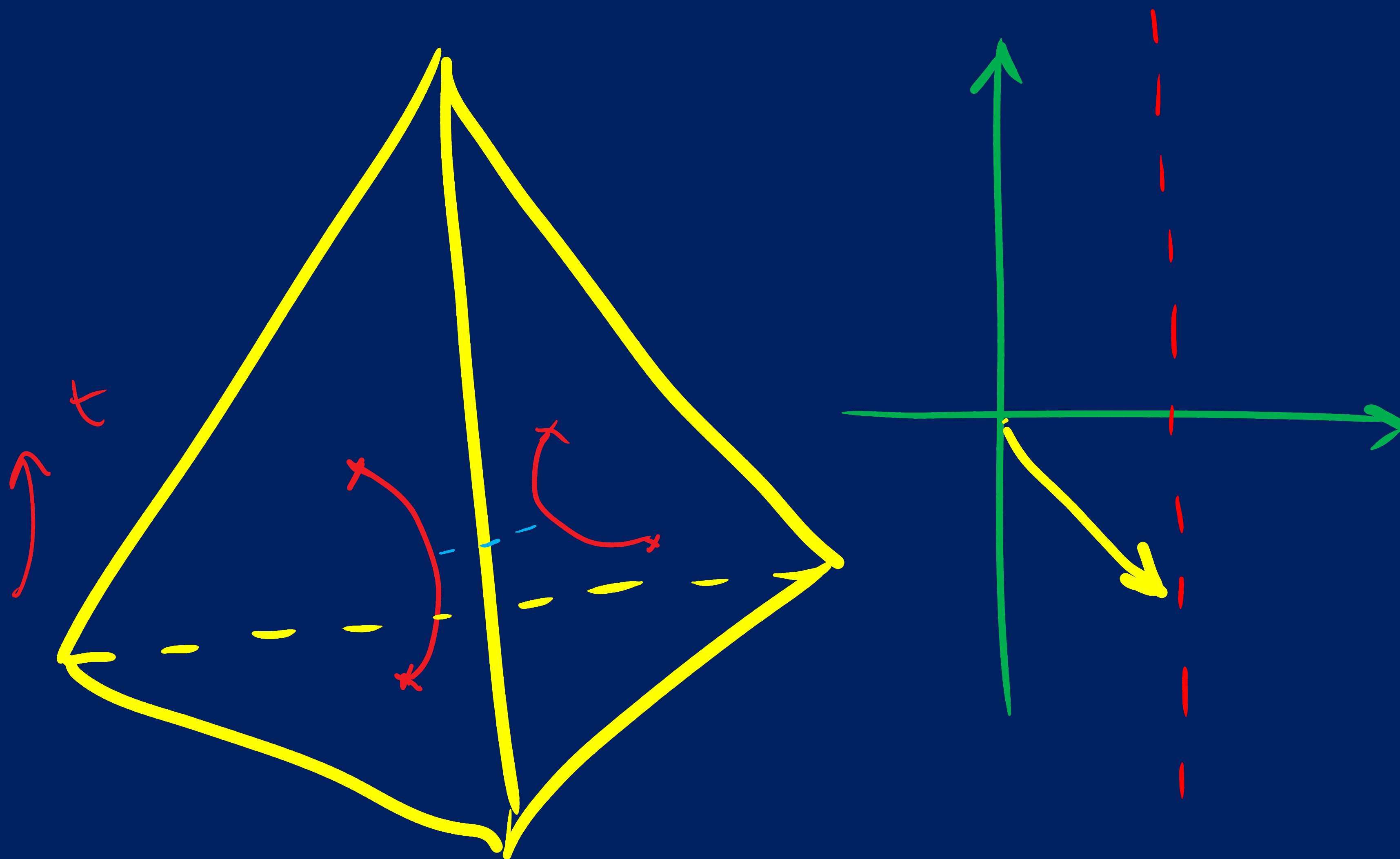
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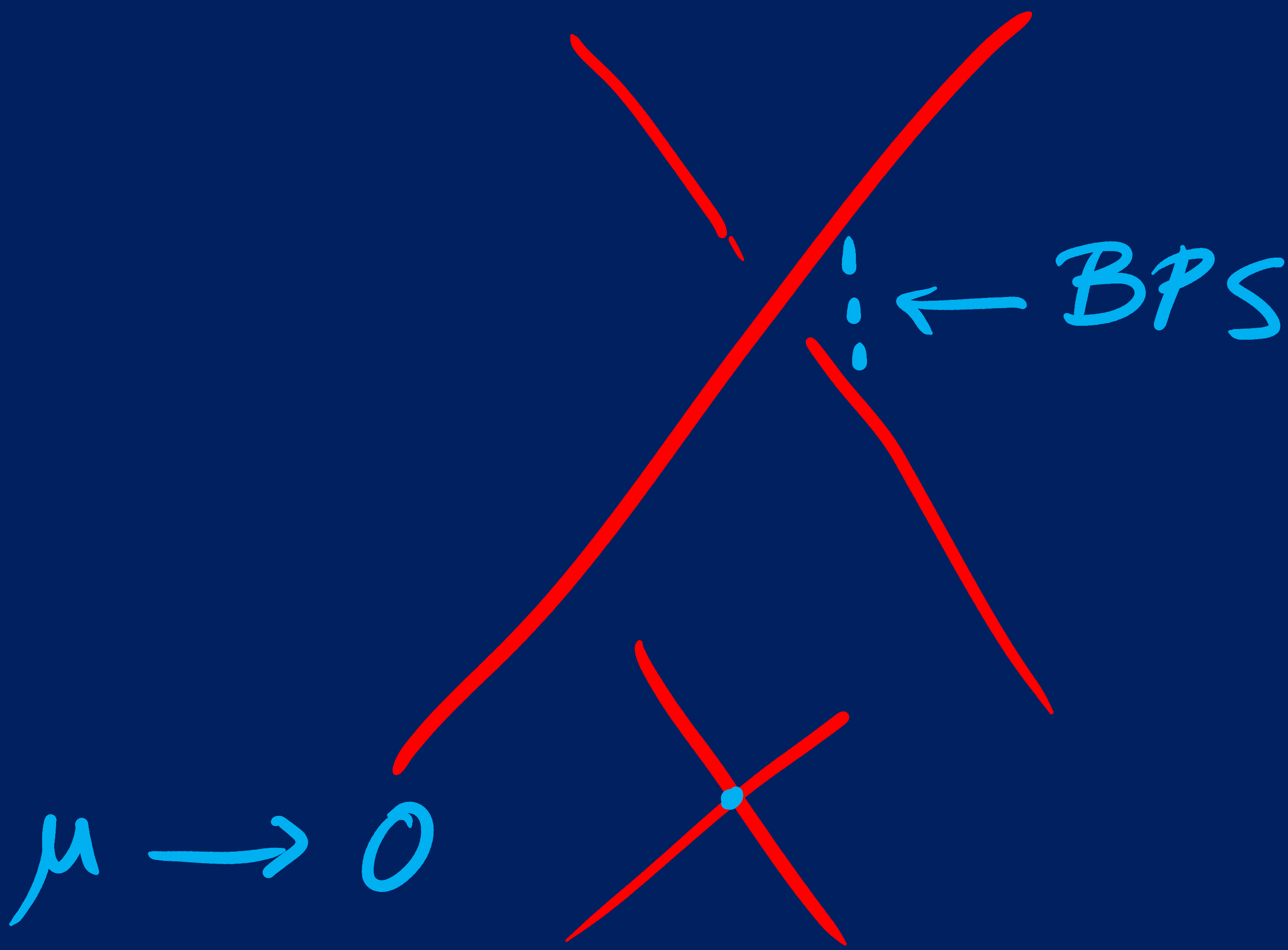
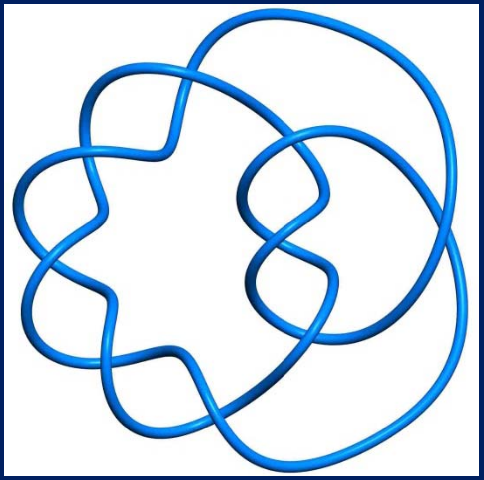
Branched loci form a braid:



How to describe the N=2, d=3, gauge theory on M5 branes wrapping a branched cover of three sphere?



Branch sheet (Seifert surface) is important for figuring out homology of the cover.

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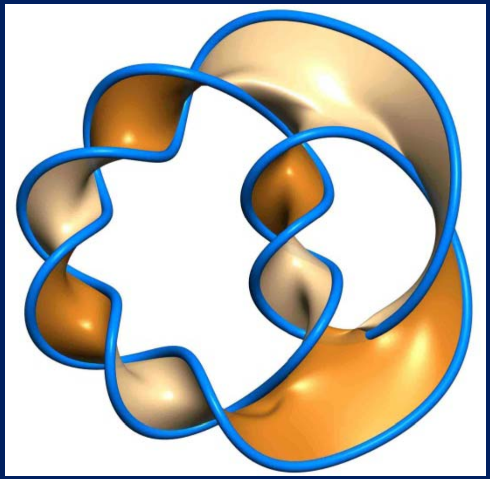

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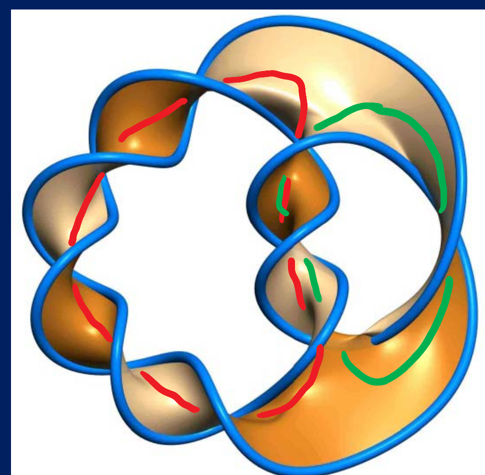
γ_1



γ_2

$$T = \alpha_i \wedge F_i + *\alpha_i \wedge *F_i$$

$$dT = 0 \rightarrow d\alpha_i \wedge F_i + *\alpha_i \wedge d * F_i = 0$$

Wedge the above with α_j and integrate over the 3-manifold:

$$\Big(\int_{S^3} \alpha_j \wedge *\alpha_i\Big) d * F_i + k_{ji} F_i = 0$$

where

$$k_{ji} = \int_{S^3} \alpha_j \wedge d\alpha_i$$

For each cycle γ_i, it is bounded by some disc D_i in S^3. Letting α_i be the Poincare dual 1-form one finds that the matrix k_{ji} is the symmetrized linking number of the cycles $< \gamma_i, \gamma_j >$.

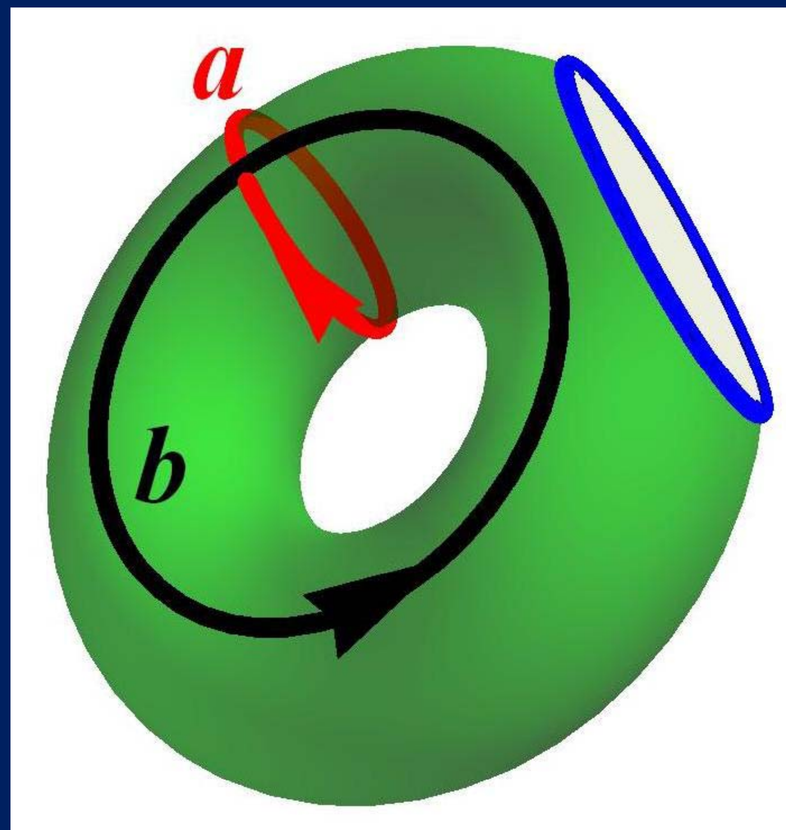
We thus see that the geometry of the branched cover gauge group can be interepreted in the IR as the $U(1)^n$ gauge group where n is the number of generators for the Seifert surface, and with the Chern-Simons level k_{ji} coming from the linking number.

H_1 (cover) :

generated by Γ_i

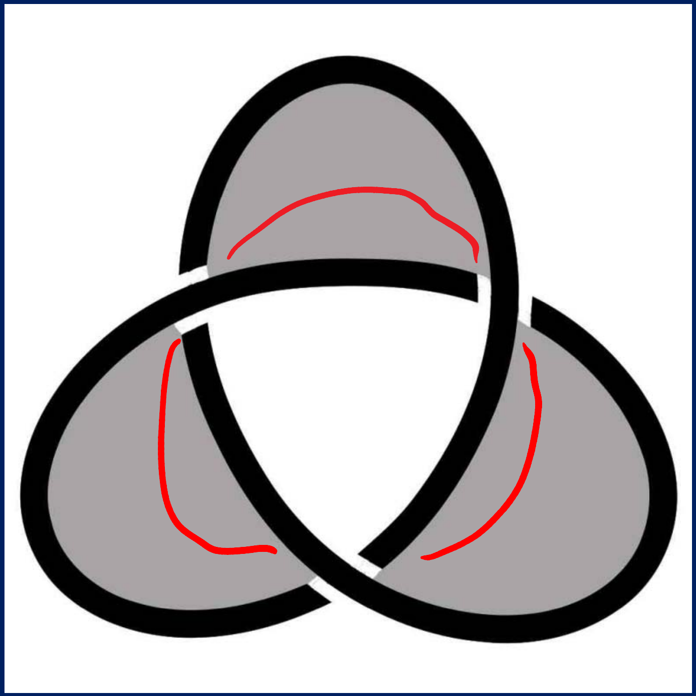
Relations $k_{ij}\Gamma_j = 0$

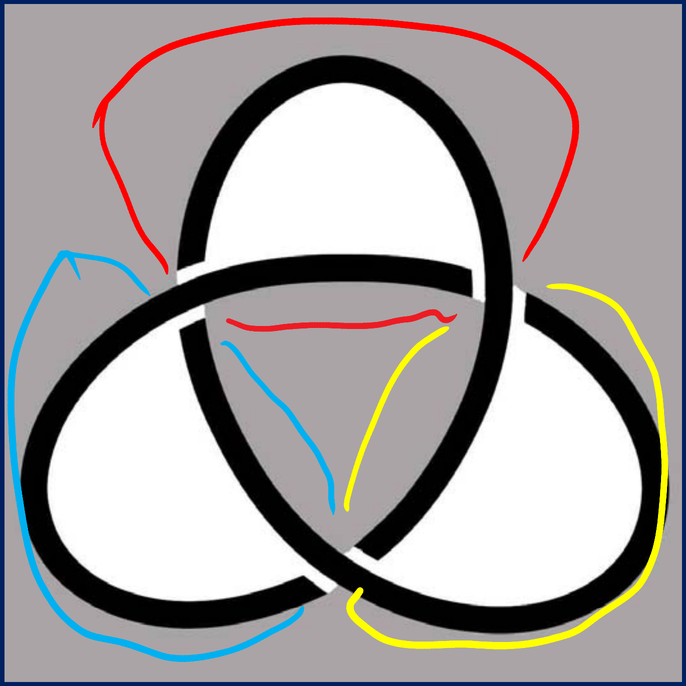
Example:
If the branching braid is unknot the cover has no first homology

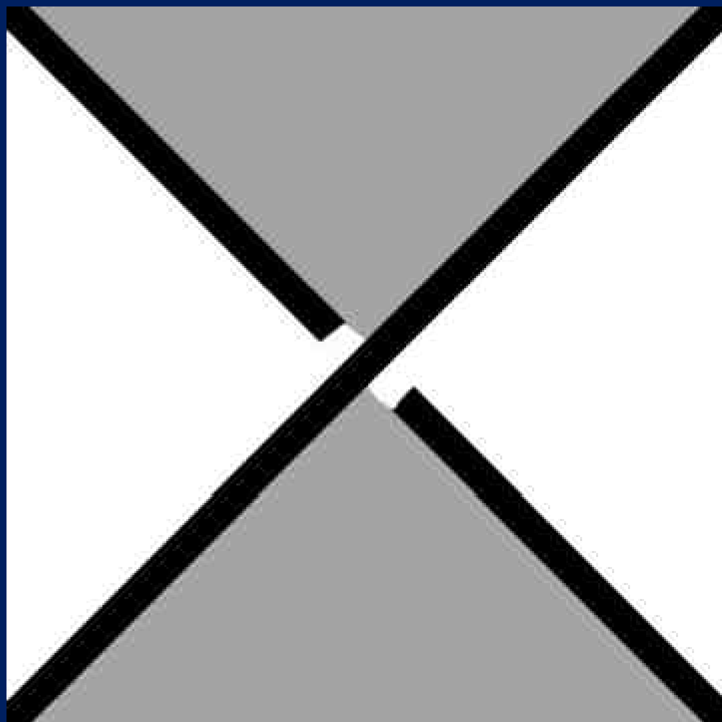


$U(1)^2$

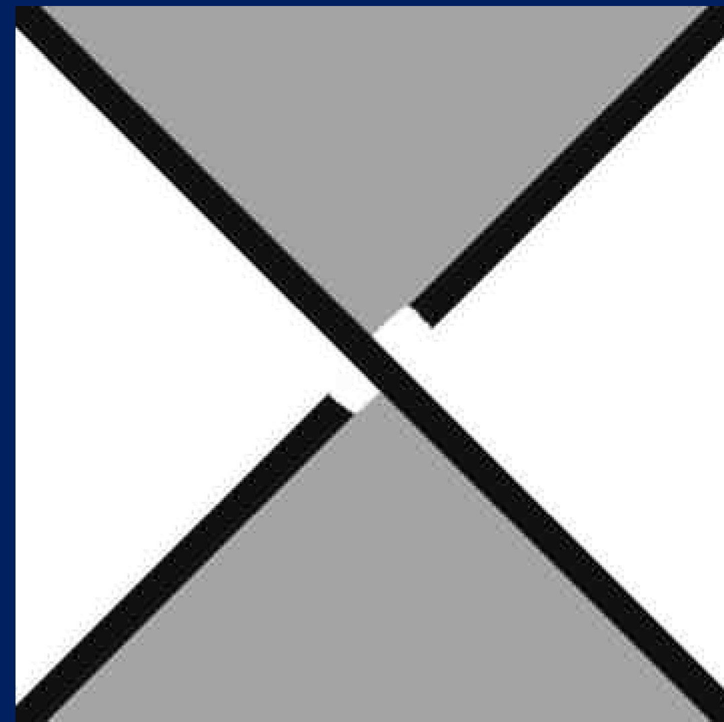
$\rightarrow k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow$ trivial (witten)







$\varepsilon = -1$

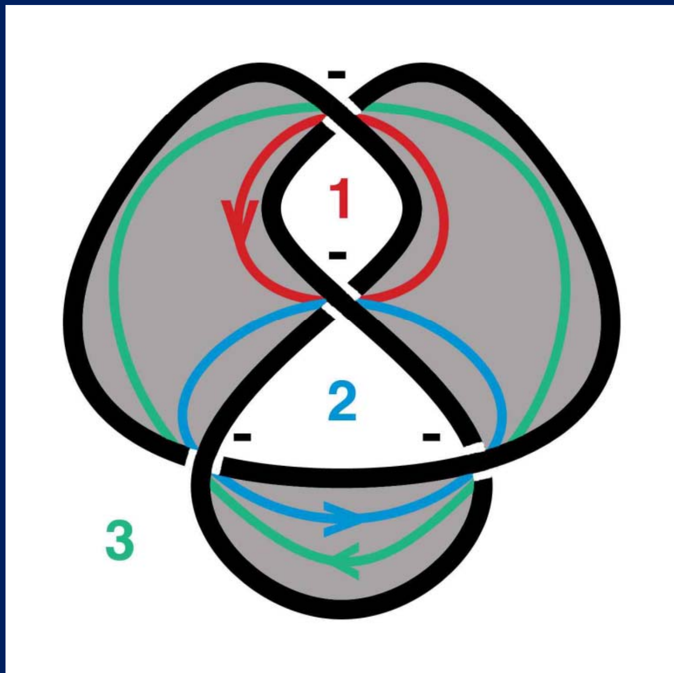


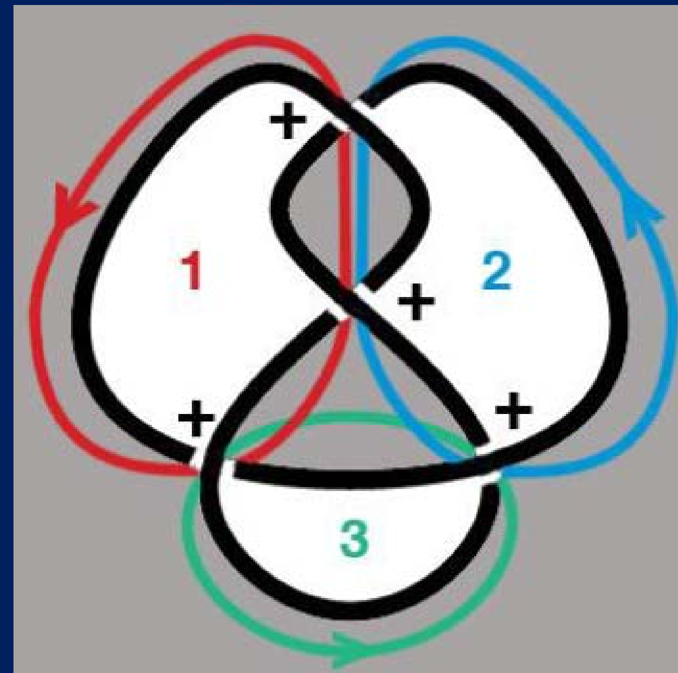
$\varepsilon = +1$

$$k_{ij} = \sum_{ij \text{ crossing}} \varepsilon \qquad i \neq j$$

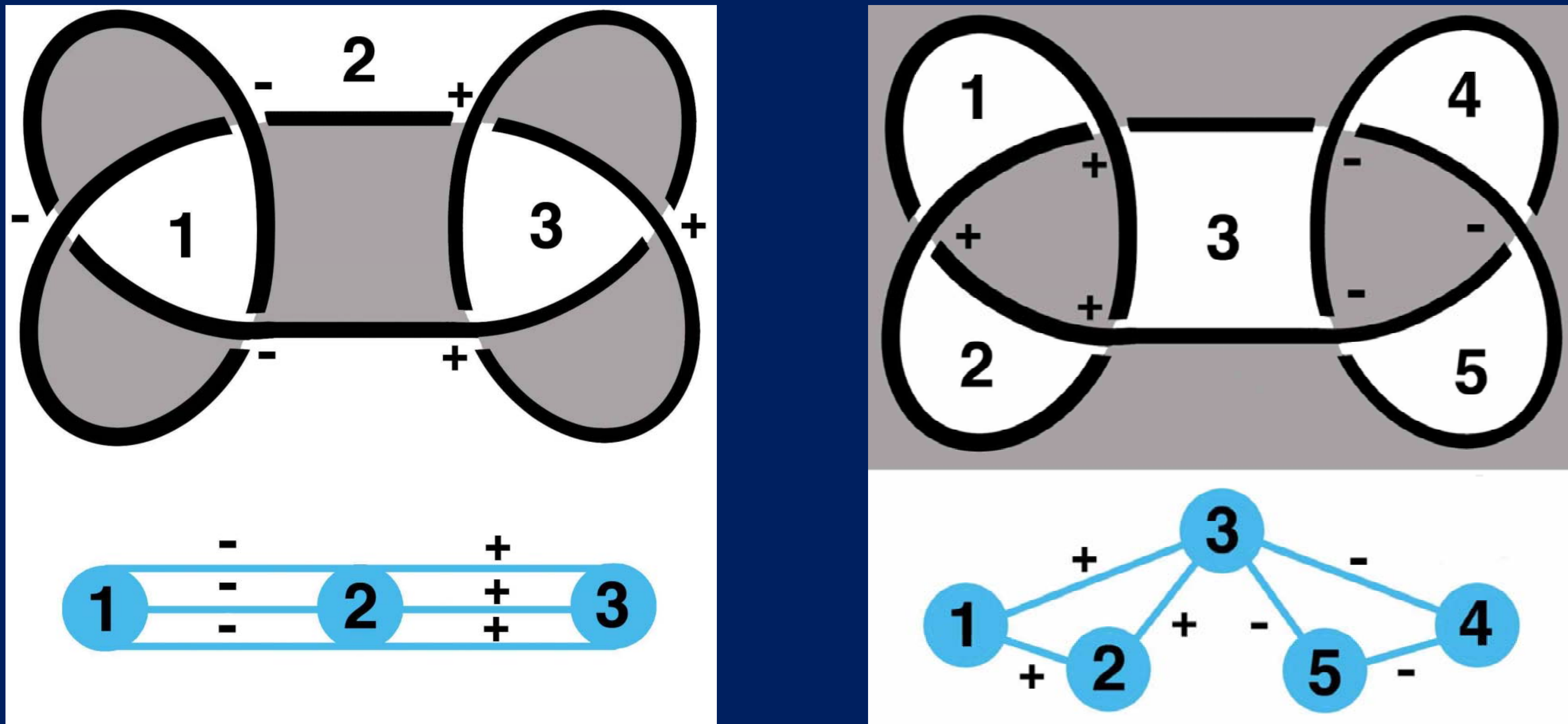
Checkerboard coloring gives a way to count the number of U(1)'s. We get one U(1) for each white region except for one (which we have a choice).

The diagonal component of k is determined by the condition that the sum along each row, if we were to keep the extra U(1) would be zero.





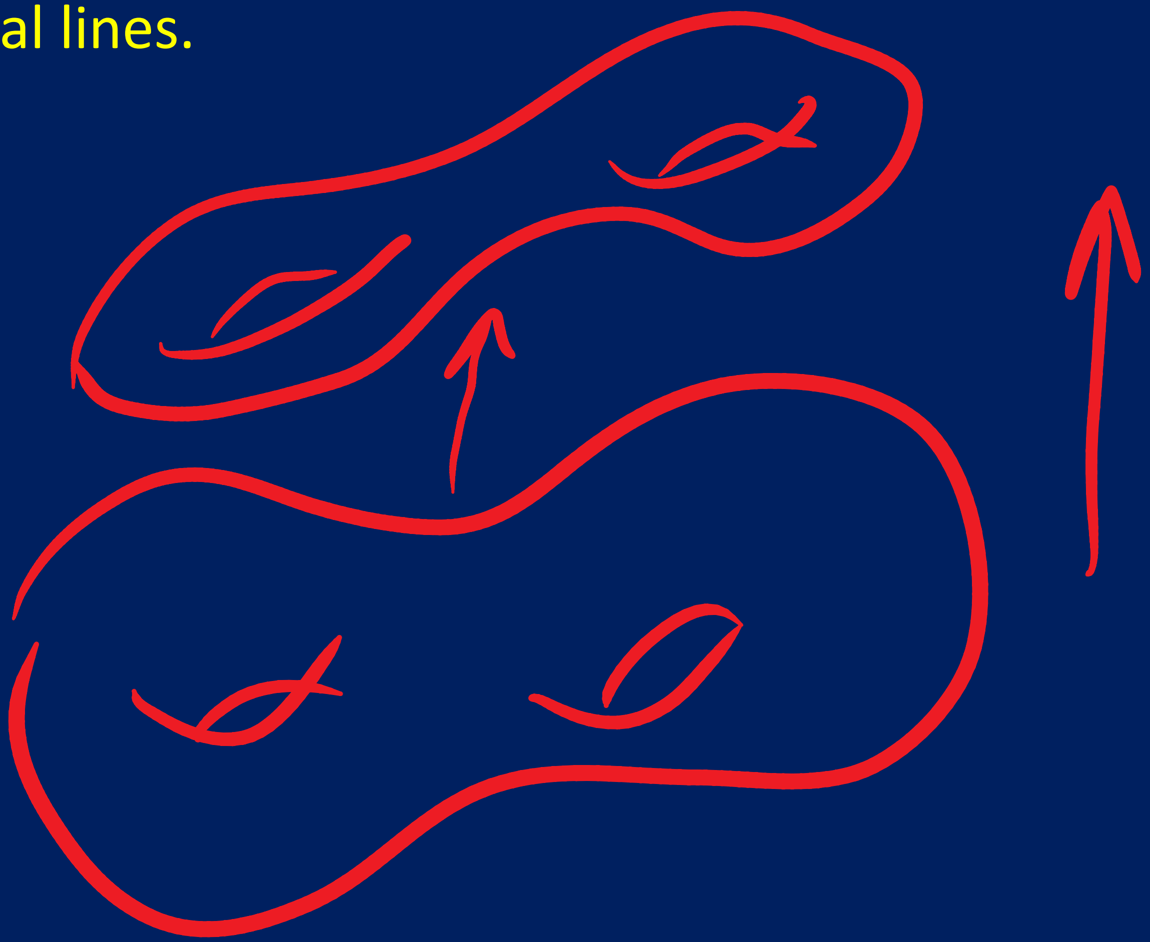
$$k = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \qquad k = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$$



Tait graphs

Starting from an N=2 in 4d, the 1-parameter family which leads to N=2 in 3d, necessitates the central charges to move along real lines.

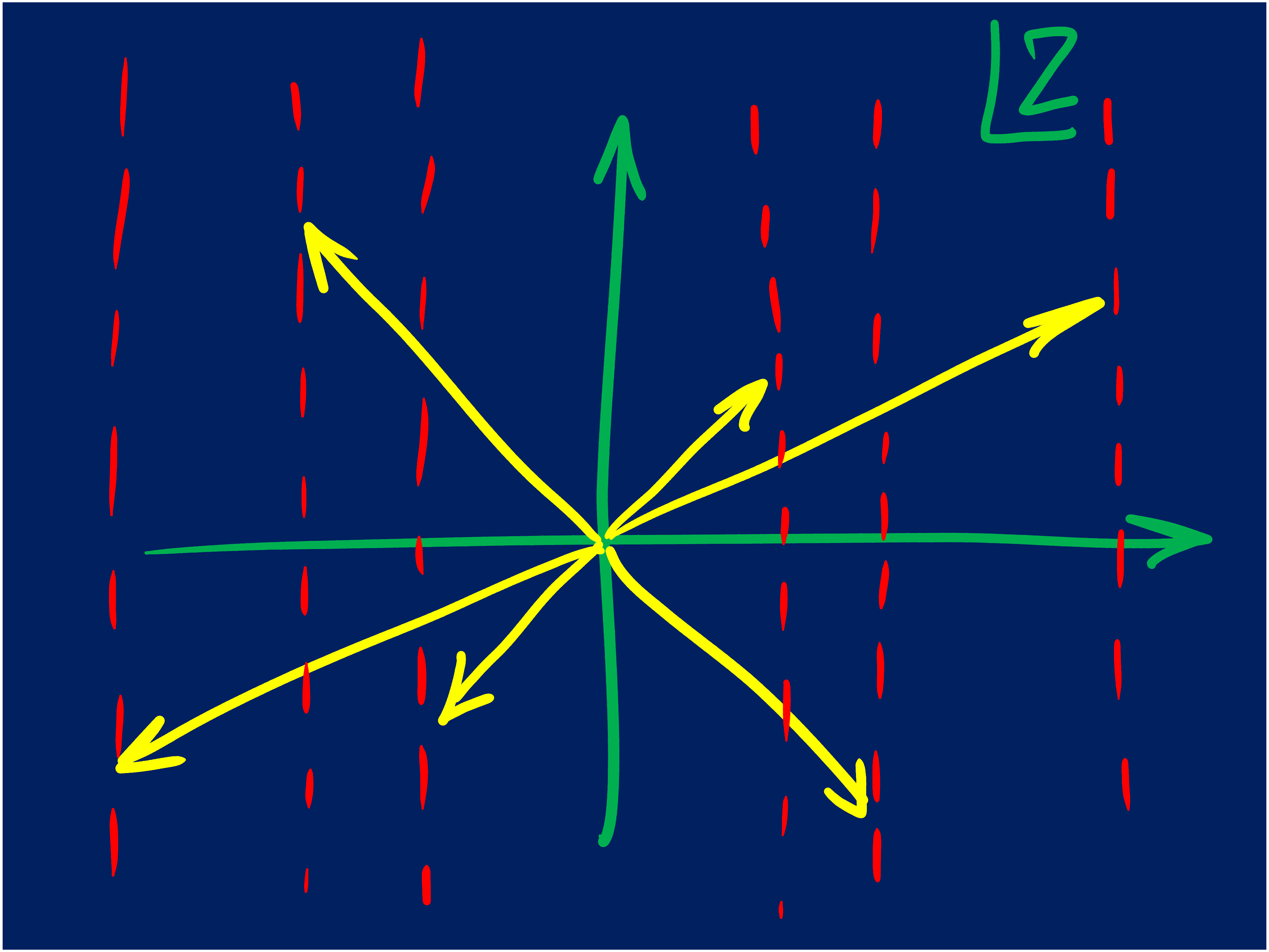
R-flow

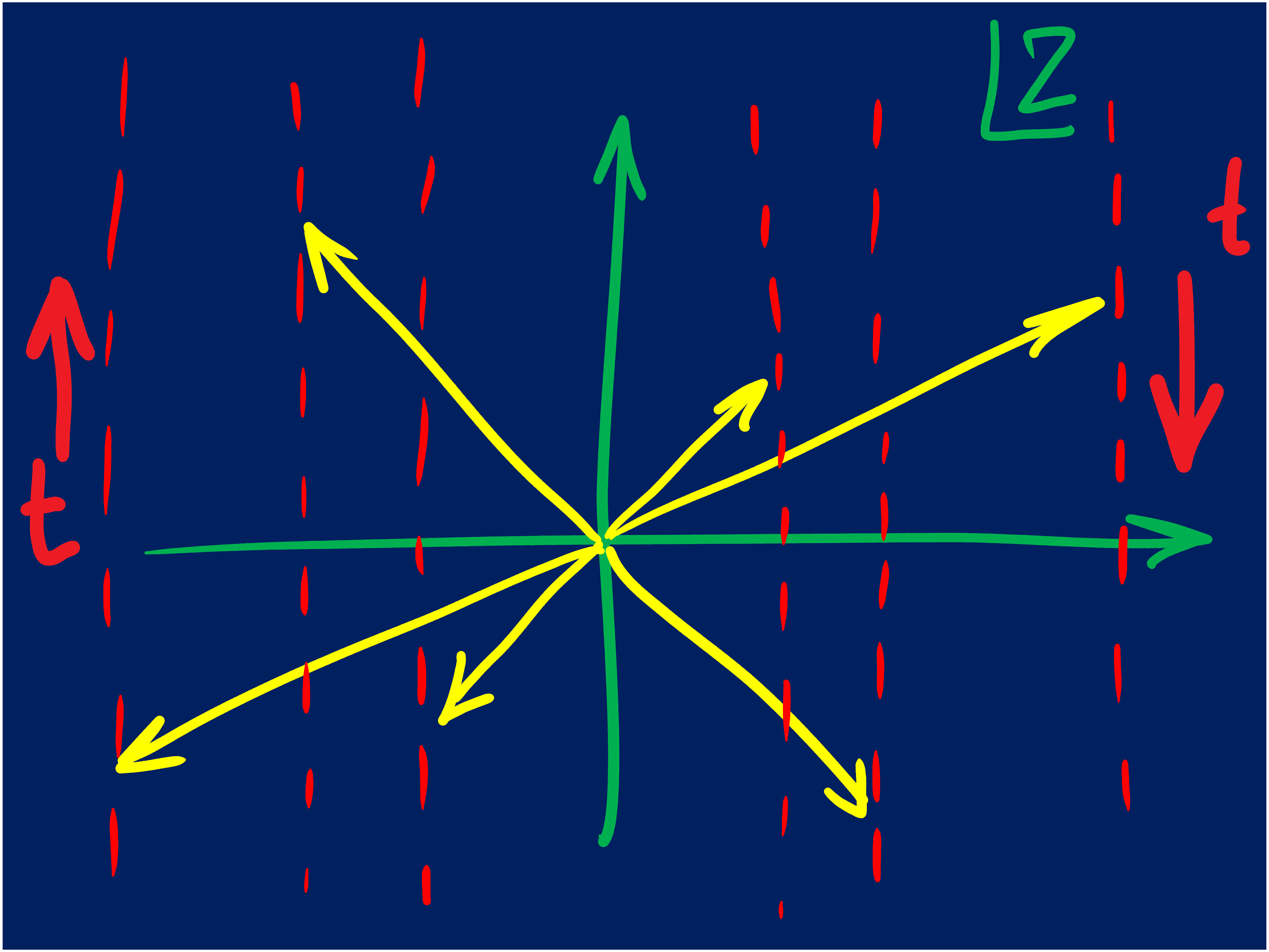


Flow has been studied by Joyce and can be reformulated as finding BPS states in the SQM with 4-supercharges with superpotential

$$\phi_t : \Sigma \to \mathbb{C}^3$$

$$W = \int_{\Sigma} \Omega_{ijk} \phi^i d\phi^j d\phi^k$$


L2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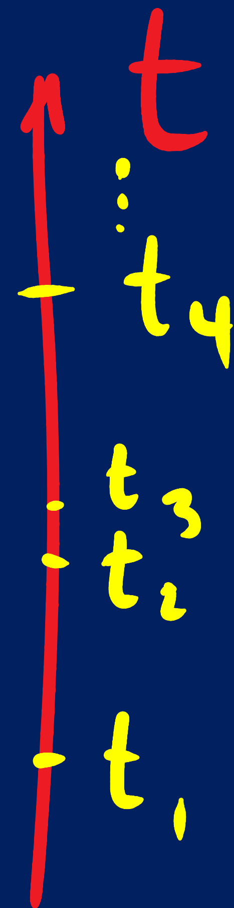

Somewhat analogous to the Domain walls constructed in Drukker, Gaiotto, Gomis; Terashima, Yamazaki

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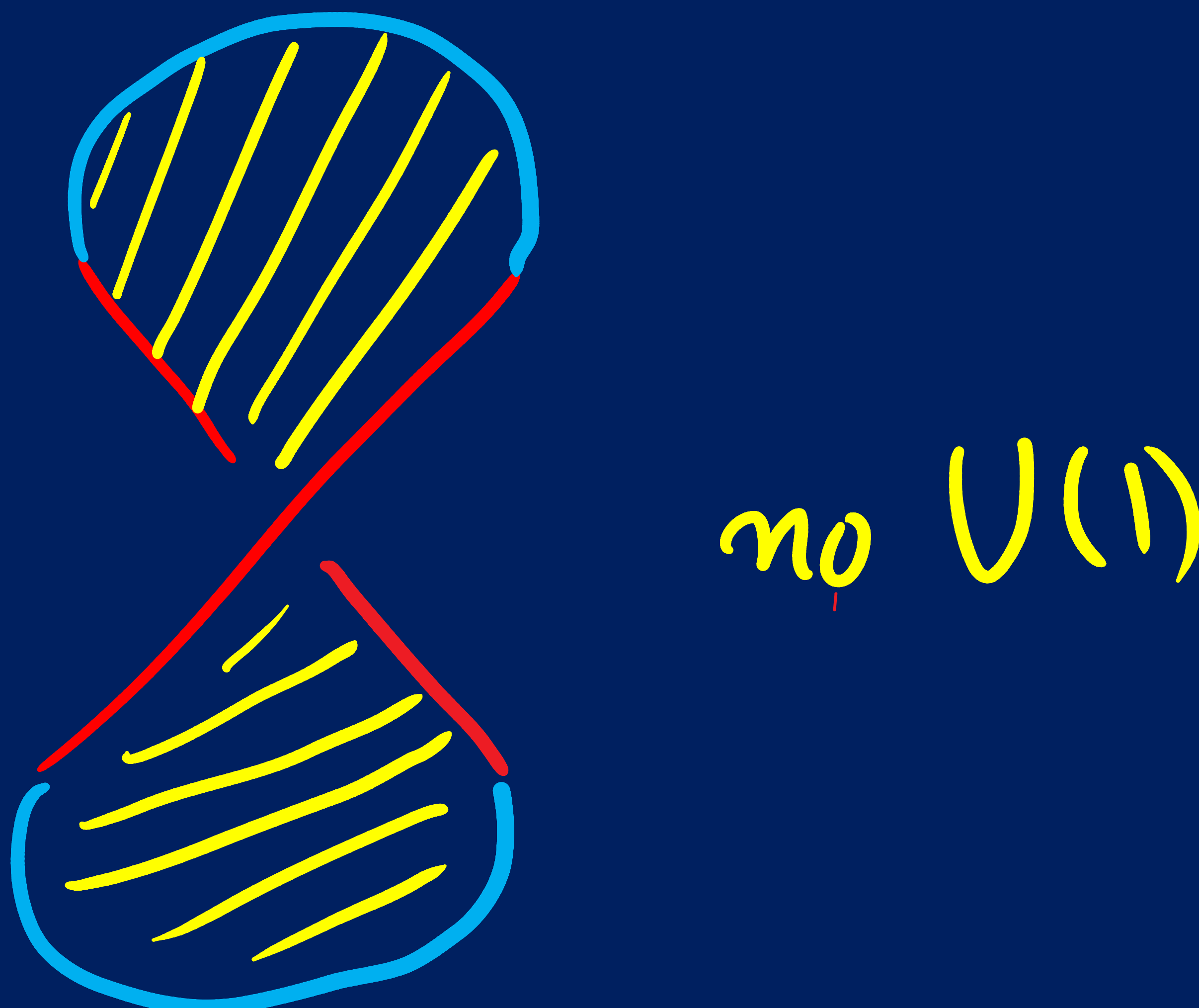

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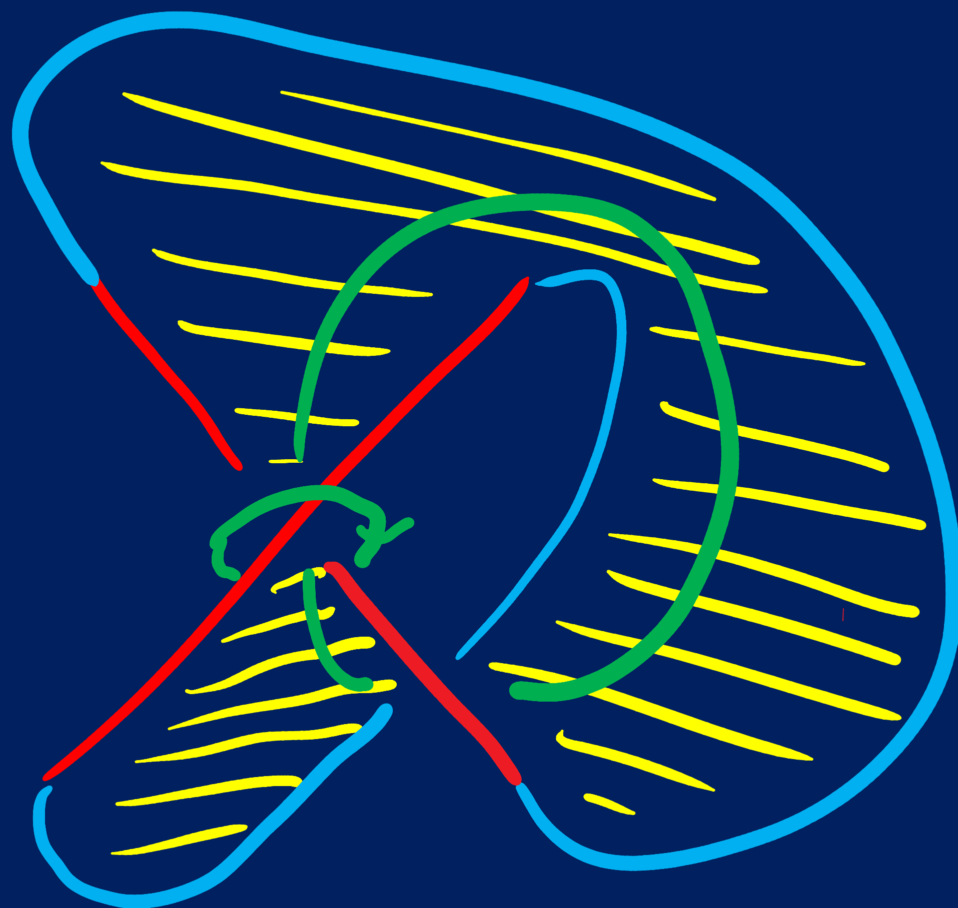
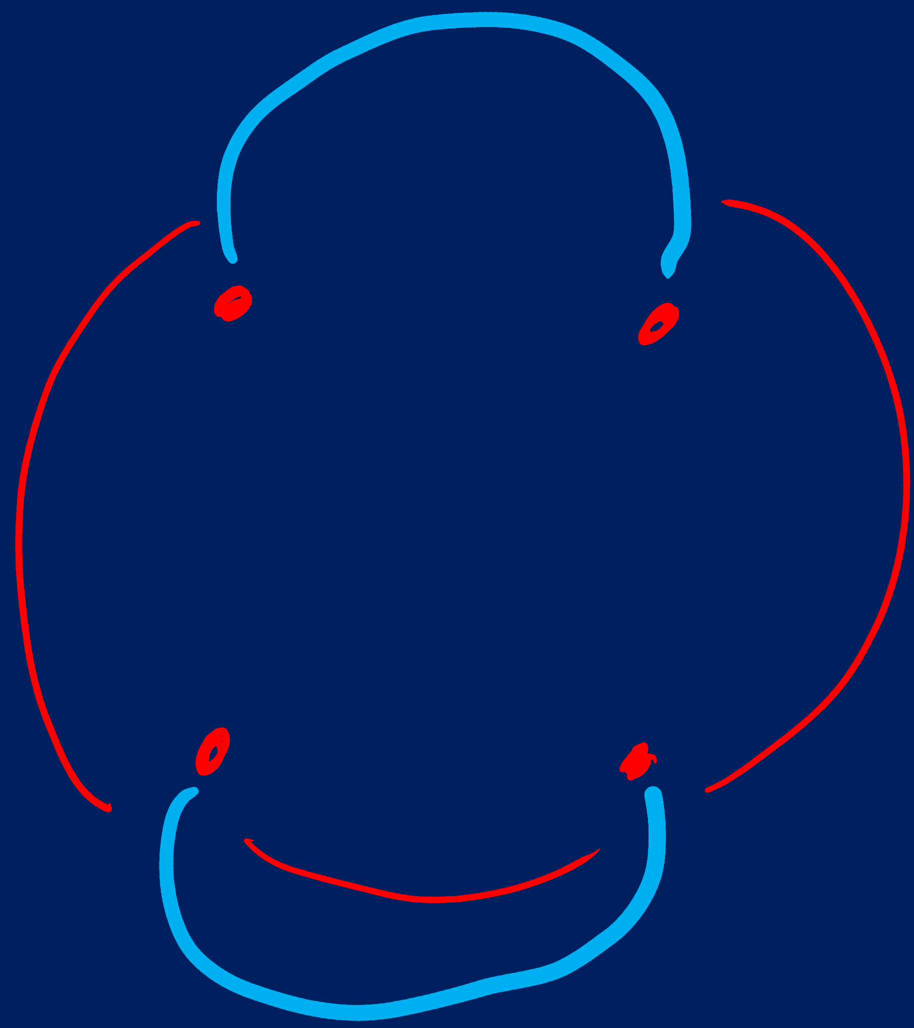
U(1)



SL(2,Z) worth of completion of the boundary data (related to which cycle of torus we shrink). Consistent with general arguments (Kapustin,Strassler; Witten)



$$y^2 = x^3 - \alpha x + \beta$$

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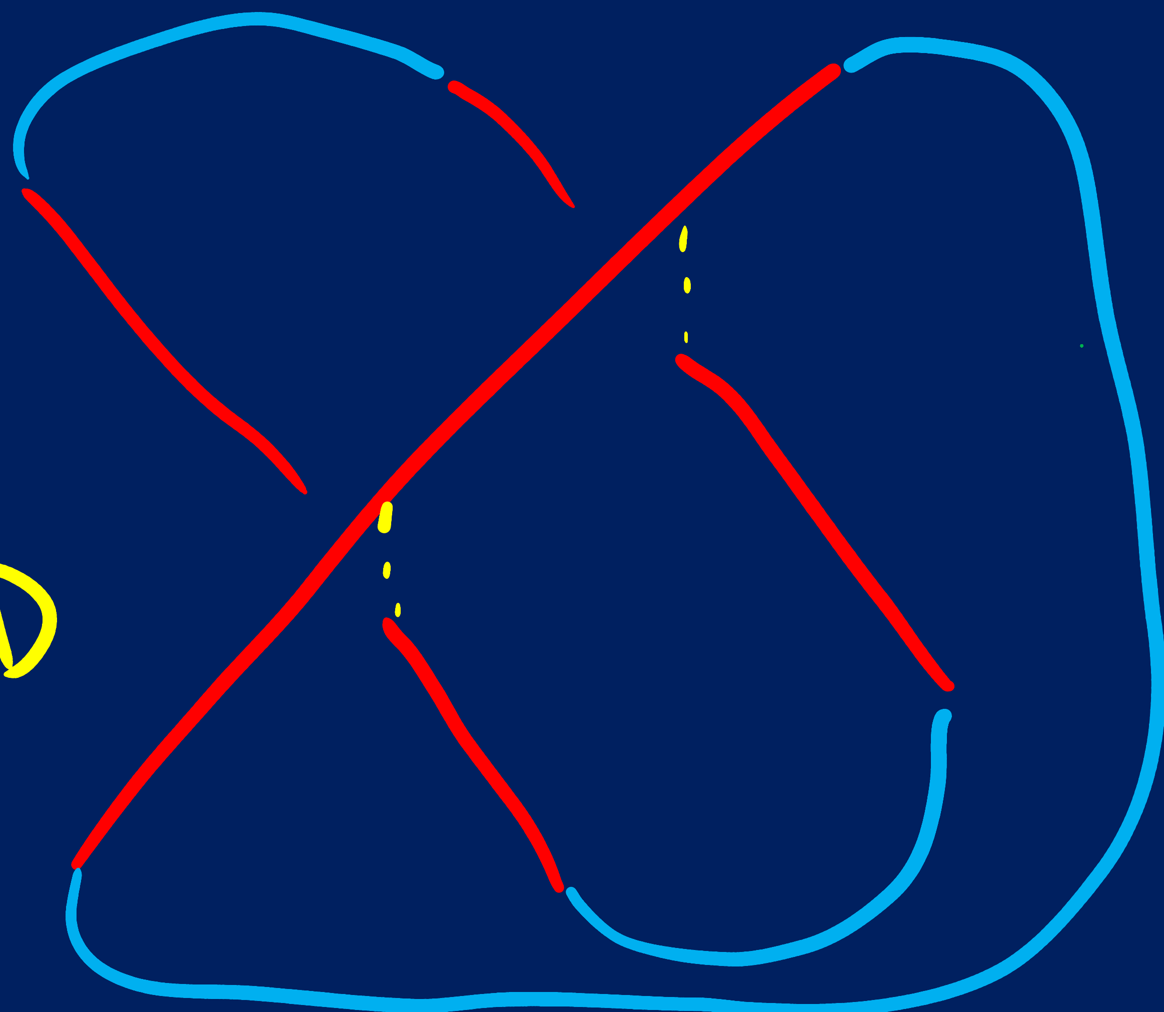
U(1)
SQED



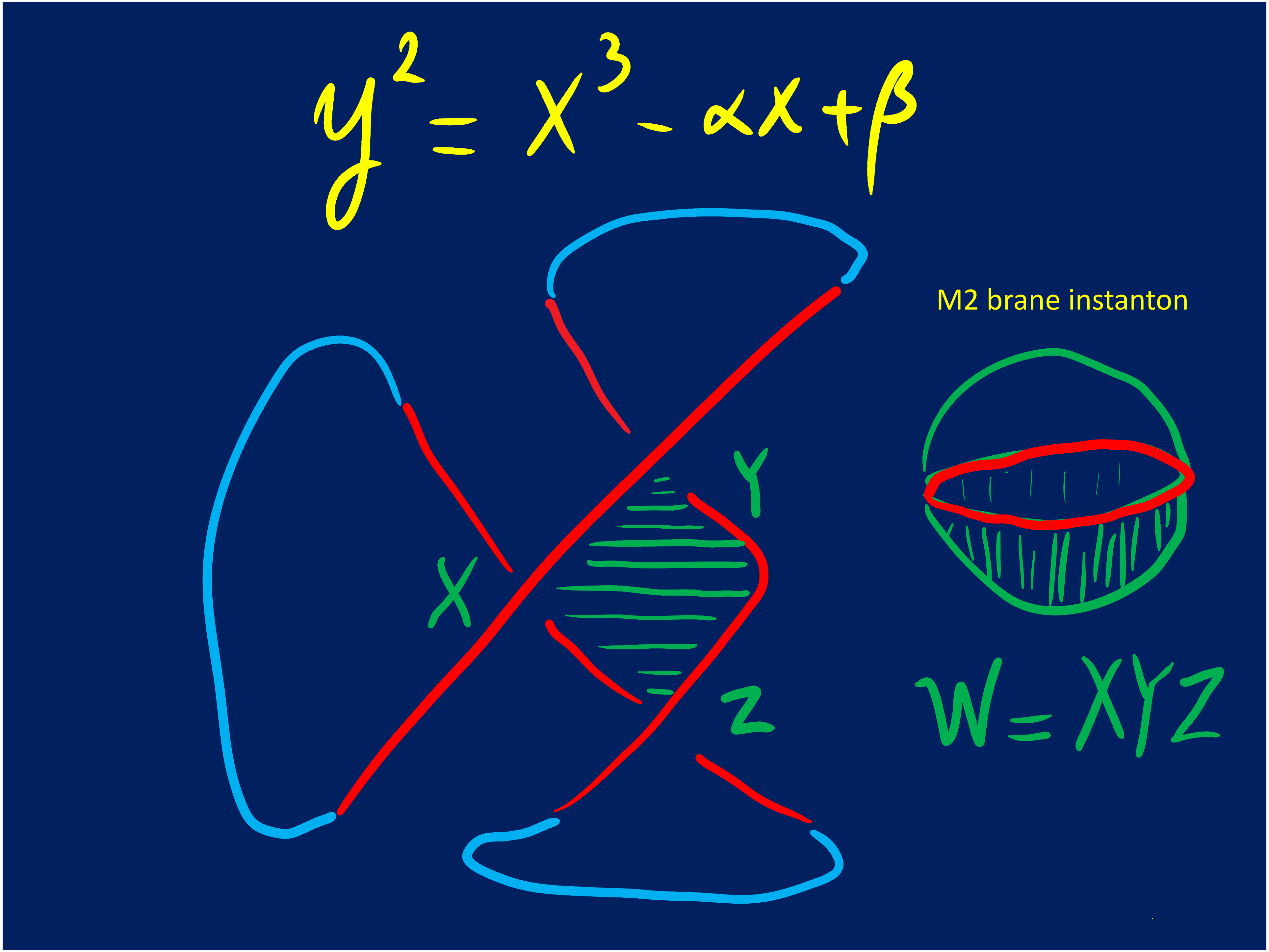
$$y^2 = x^3 - \alpha x + \beta$$

$$y^2 = x^3 - \alpha x + \beta$$

$$y^2 = x^3 - \alpha x + \beta$$

no $U(1)$

$$y^2 = x^3 - \alpha x + \beta$$


$y^2 = x^3 - \alpha x + \beta$
M2 brane instanton
X
Y
Z
$W = XYZ$

The 4d wall-crossing of Argyres-Douglas theory

+ R-flow

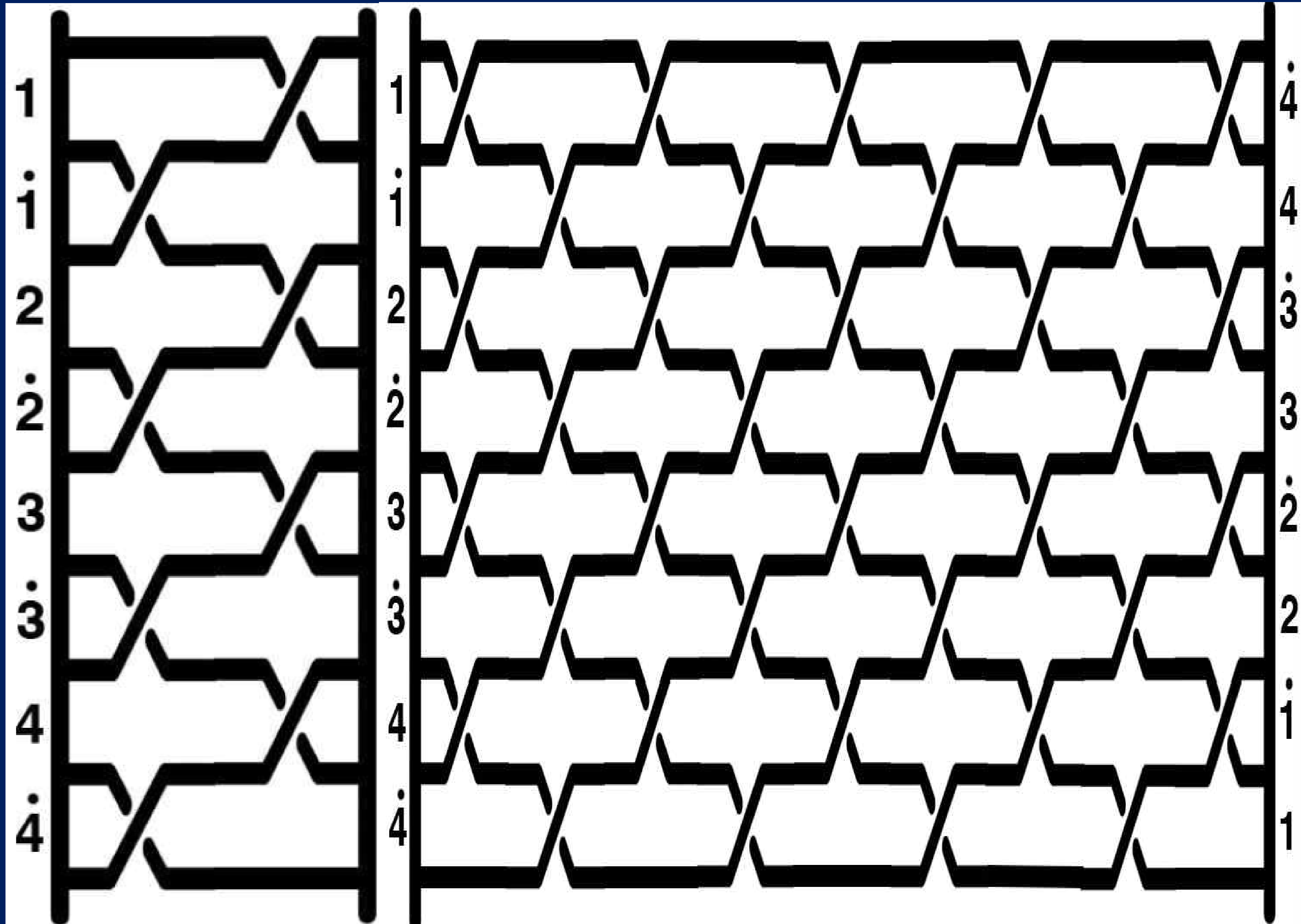
→ 3d theories which are mirrors

I.S.

A.H.I.S.S.

This turns out to be the general story:

Wall-Crossing in 4d ←→ Mirror Symmetry in 3d


1
1̇
2
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3
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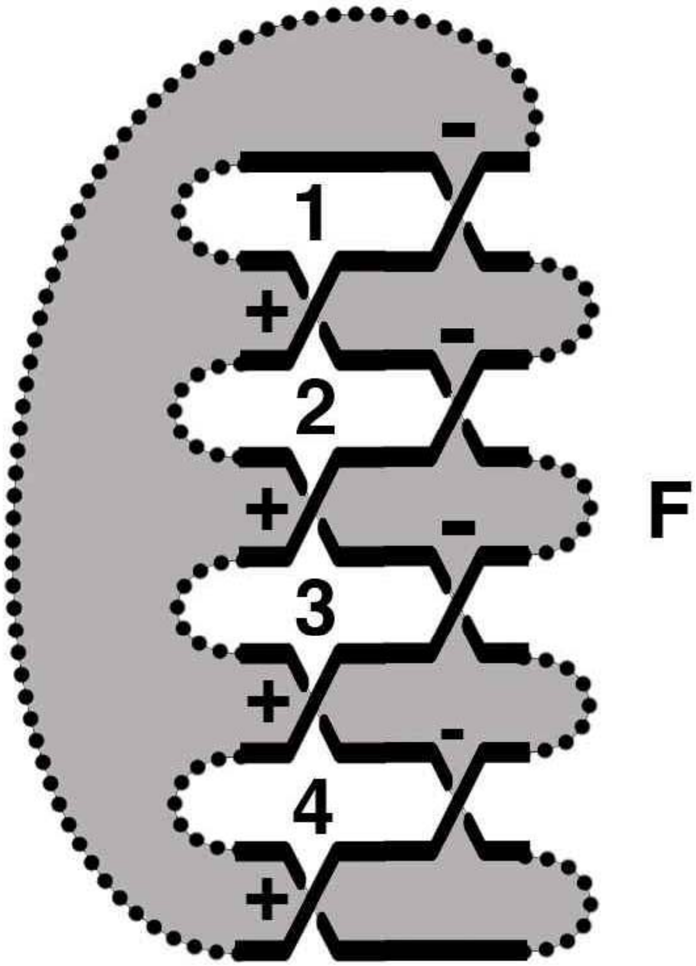
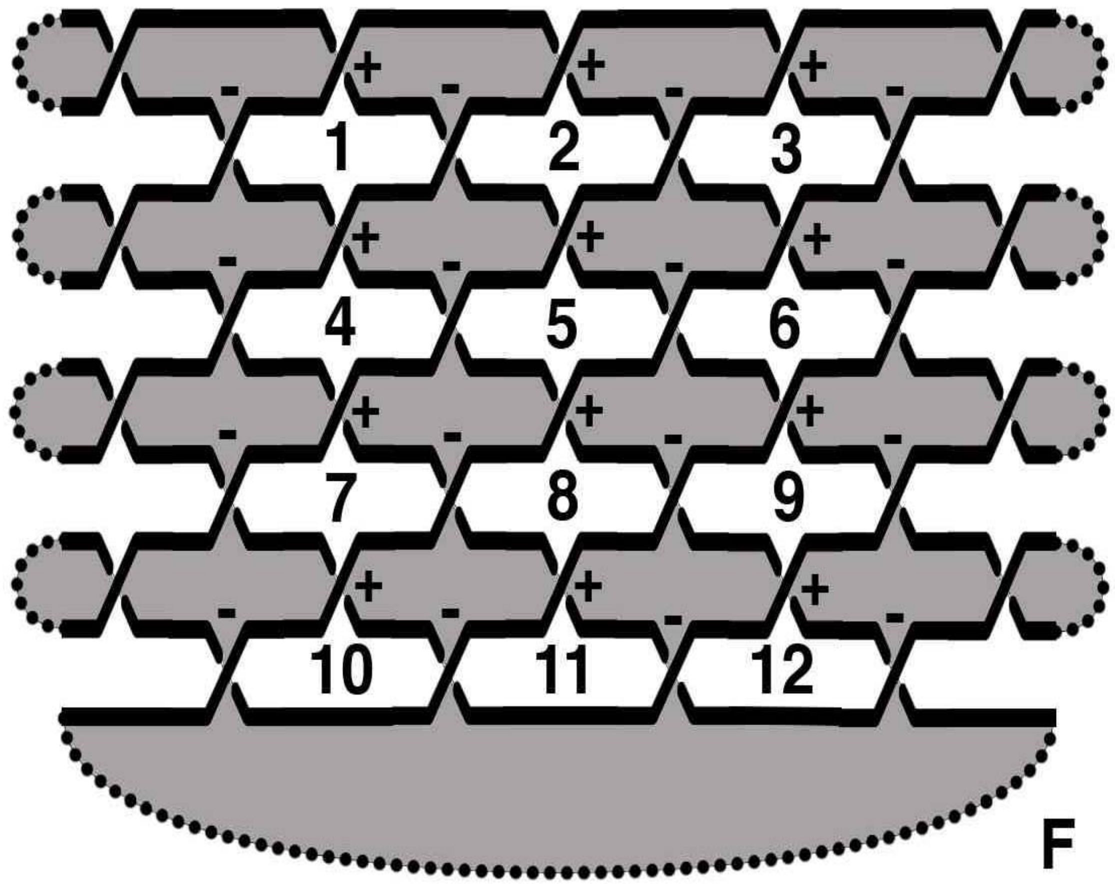
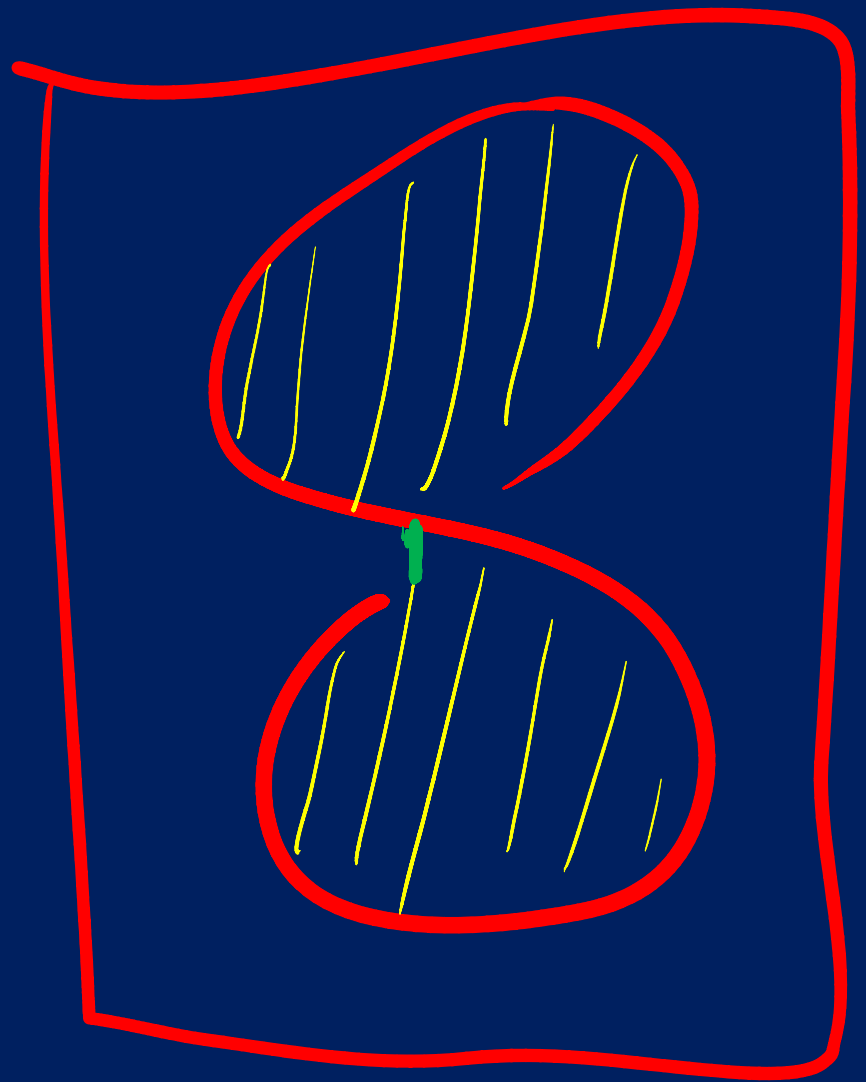




Each internal region gives a superpotential term. The black ones give ordinary ones, the white ones, include monopole operator.

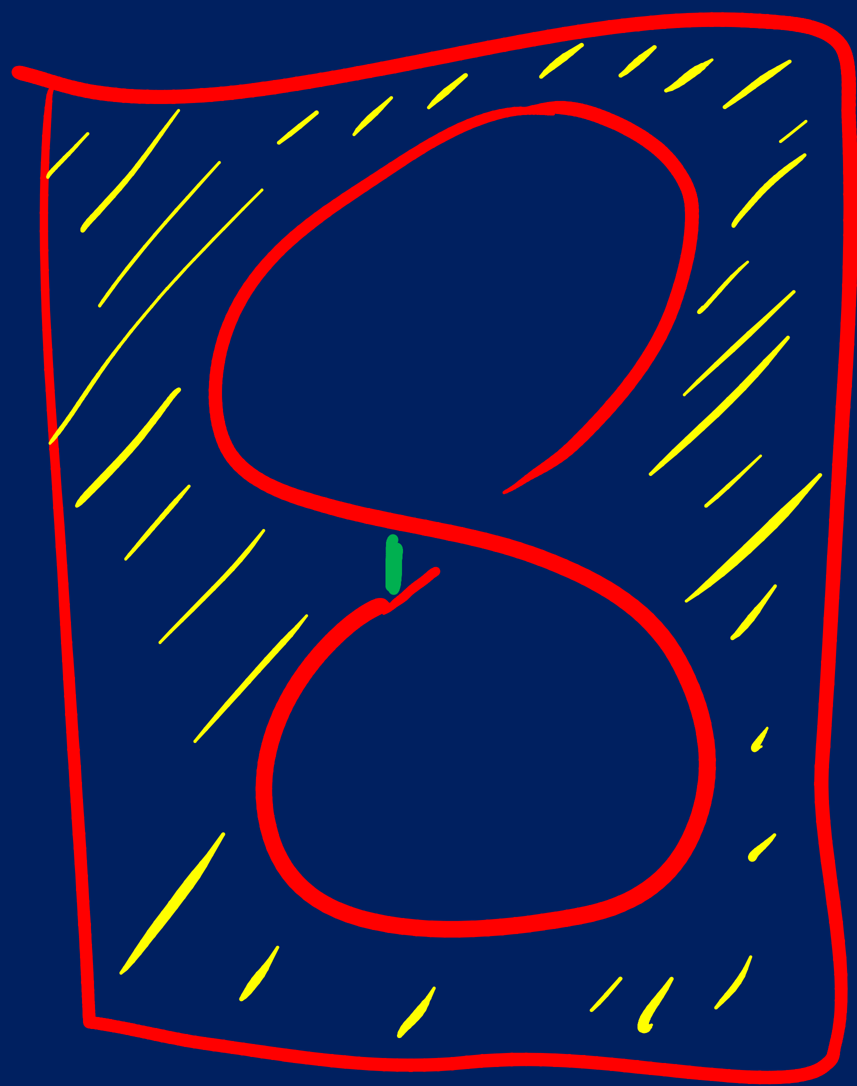
Black/White duality also maps to known 3d duality:



no U(1)

1 chiral

Black/White duality maps to known 3d duality:



U(1)
$k=1$,
+1 chiral

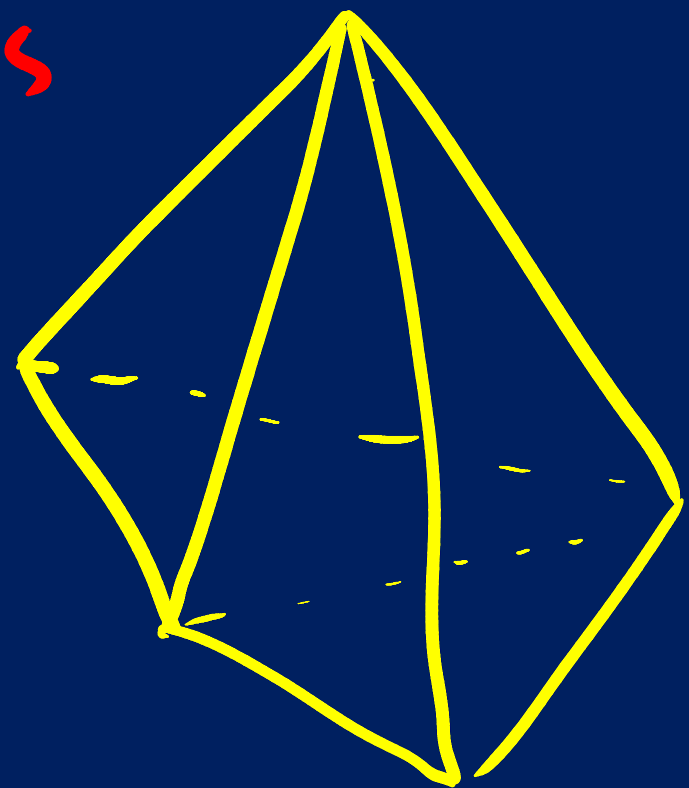
Connections with SL(2,R) Chern-Simons
(relation to work of Dimofte Gaiotto, Gukov)

$$y^2 = f(x)$$

2 M5 branes on X

$$\{\} \times \Sigma_x \times R$$

2 M5 branes on



Different 4d chambers $\leftarrow\rightarrow$ Different Tetrahedral decompositions

Partition function of M5 on Melvin cigar

$\leftarrow\rightarrow$

Partition function of SL(2,R) Chern-Simons

$$Z^{M5}\left(M^3 \times \underset{\frac{1}{2}S^3 \approx}{\mathrm{Melvin}_{\mathrm{Cigar}}}\right) = \underset{''}{Z^{CS}}(M^3)$$

$$Z = \langle \cdot \,|\, \mathrm{T}\left[\prod_{\alpha} \left(\mathcal{O}^{\alpha}_{BPS} \right) \right] |\, \cdot \rangle$$

Kontsevich-Soibelman Operators
(made from quantum dilogs acting on
U(1) Chern-Simons Hilbert Space)

Equivalence of partition function of 3d mirror pairs

Invariance of KS operator under wall crossing