Brane Tilings and Algae

（ブレーンタイリングと藻類）

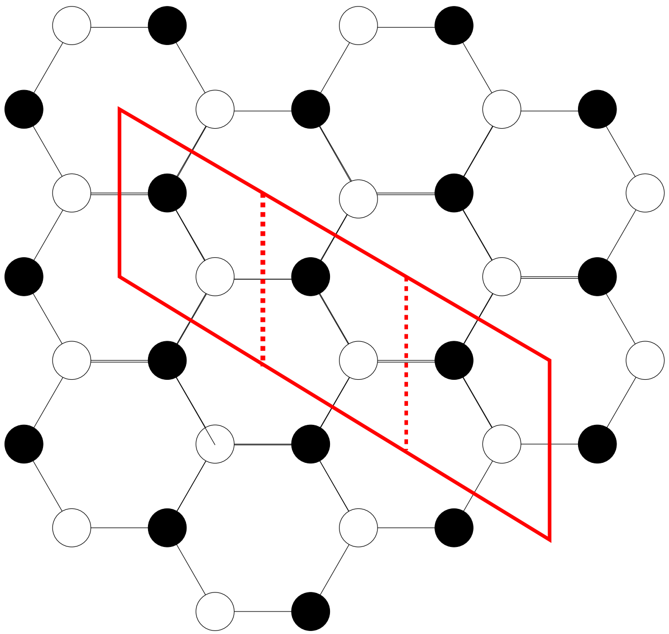
山崎雅人（やまざきまさひと）
yamazaki@hep-th.phys.s.u-tokyo.ac.jp

Univ. of Tokyo (Hongo)

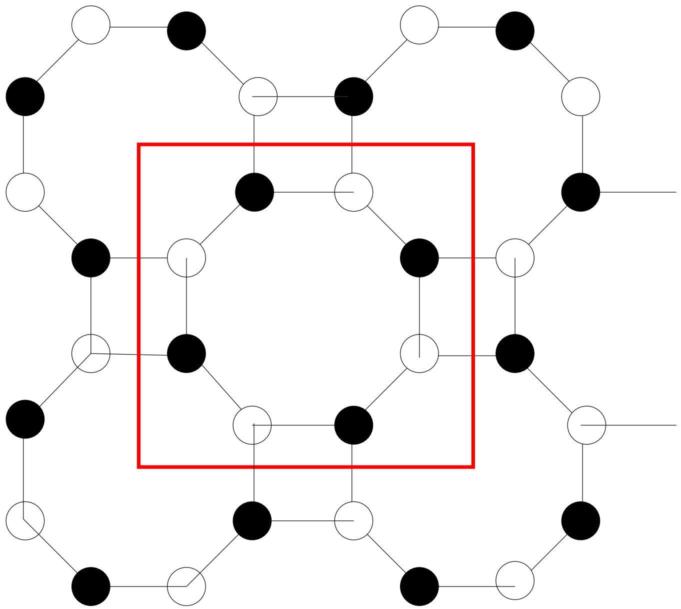
Based on math.AG/0605780, math.AG/0606548
(In collaboration with Kazushi Ueda)

Brane Tiling

Today's topic: "Brane Tiling" (Introduced in Mar/Apr, 2005),
a bipartite graph on torus



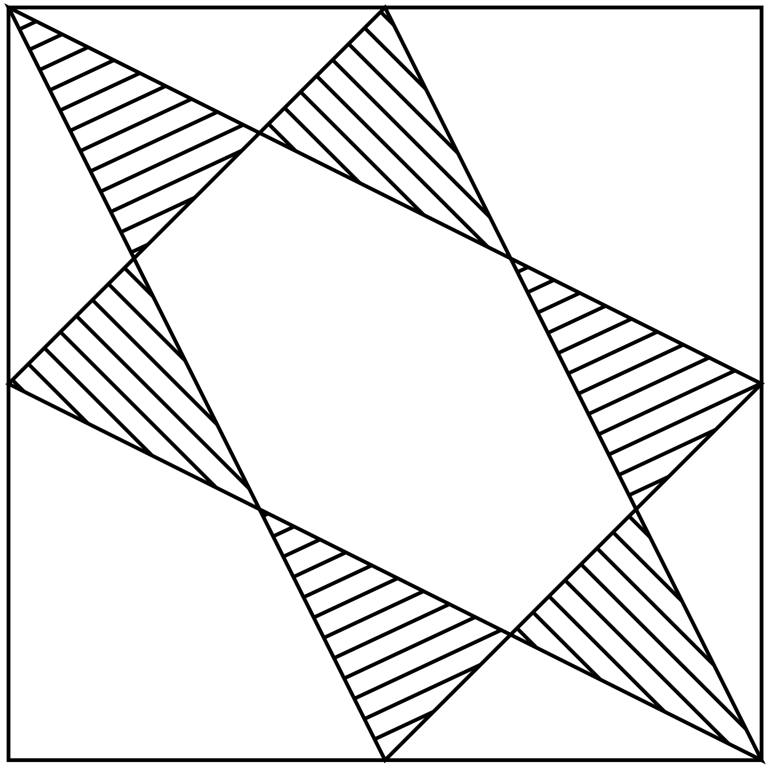
$\mathbb{C}^3/\mathbb{Z}_3$



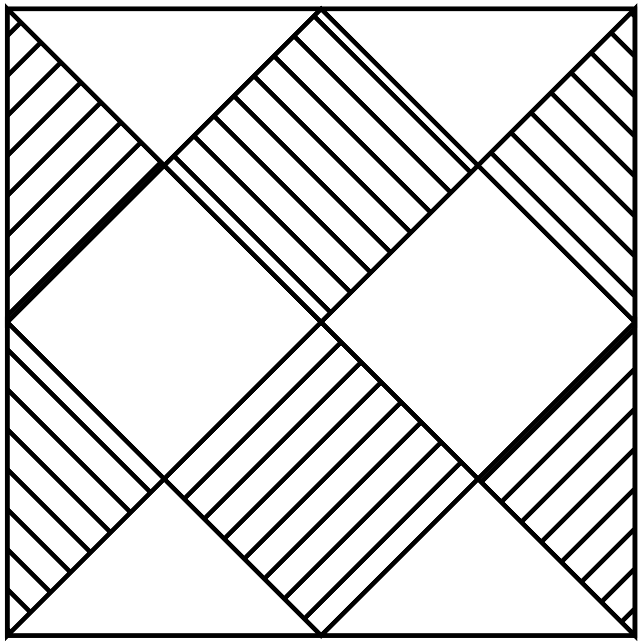
$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

Alga

... and Alga (Introduced in Nov. 2005)



$\mathbb{C}^3/\mathbb{Z}_3$



$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

Plan of This Talk

1. Introduction
2. Origin of Brane Tilings
3. Brane Tiling Algorithm
   - Brane Tiling $\longrightarrow$ Quiver and Superpotential
   - Brane Tiling $\longrightarrow$ Toric Diagram
   - Toric Diagram $\longrightarrow$ Brane Tiling
4. Homological Mirror Symmetry from Brane Tilings and Algae
5. Summary, Outlook and Speculations

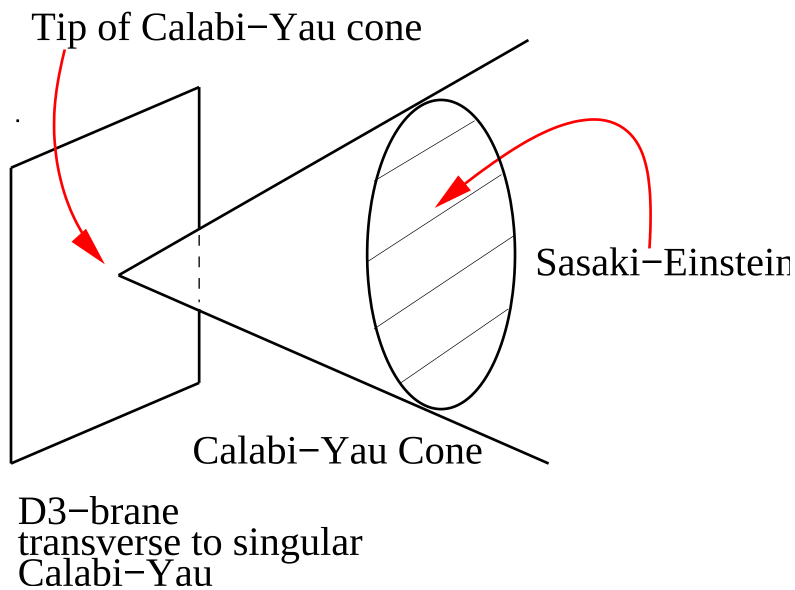
D-brane probe

D-branes: provide connection between
Geometry and Gauge Theory

- D-brane probes geometry (Geometry)
- We have Gauge Theory on D-brane world volume (Gauge Theory)

D3-brane Probing Singular Calabi-Yau

We probe cone-type non-compact singular Calabi-Yau by D3-branes.



X_5 : 5-dim. Sasaki-Einstein mfd $\Updownarrow$

$C(X_5)$: singular CY 3-fold

$$(ds^2_{C(X_5)} = dr^2 + r^2 ds^2_{X_5})$$

D3-brane is transverse to CY and placed at the apex of CY

cone: $\underbrace{\mathbb{R}^4}_{\text{D3-brane}} \times \underbrace{C(X_5)}_{\text{CY}}$

Motivation: AdS/CFT

Most studied case of AdS/CFT:

IIB on $AdS_5 \times S^5 \Leftrightarrow \mathcal{N} = 4$ SYM

We want to reduce SUSY to $\mathcal{N} = 1$! Consider IIB on $AdS_5 \times X_5$. X_5 must be a Sasaki-Einstein mfd in order for gauge dual to be $\mathcal{N} = 1$ SCFT. [Morrison-Plesser, Figueroa-O'Farrill]

Locally

$$AdS_5 \times X_5 \sim \mathbb{R}^4 \text{''} \times \text{''}\, C(X_5)$$

this is the geometry we probe.

$$\underbrace{\overbrace{\left(\frac{du^2}{u^2} + u^2 ds_4^2\right)}^{AdS_5} + ds_{X_5}^2}_{\text{direct product}} = \underbrace{u^2 ds_4^2 + \frac{1}{u^2}\left(du^2 + u^2 ds_{X_5}^2\right)}_{\text{warped product}}$$

Problem

We assume $C(X_5)$ (CY) is (local) toric

AdS/CFT (belief)

IIB on $AdS_5 \times X_5$ (X_5: Sasaki-Einstein, $C(X)$: toric CY) is dual to $\mathcal{N} = 1$ quiver gauge theory

- Geometry: encoded in toric diagram
- Gauge theory: encoded in quiver

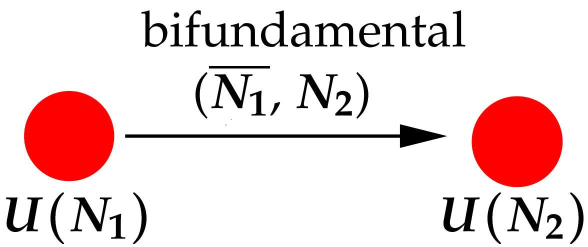
Problem

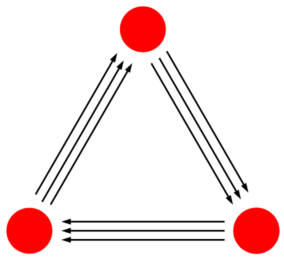
Which toric diagram gives which quiver gauge theory?

What is Quiver?

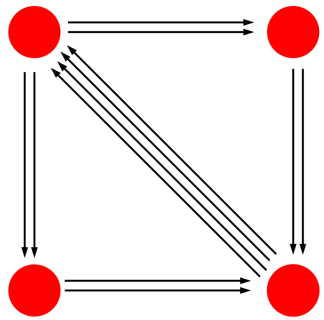
Quiver（箙）：矢を入れる筒
$\longrightarrow$ 有向グラフ

- node= gauge group : D-branes
- oriented arrow= bifundamental : open string streched between D-branes
- loop= gauge-invariant operator (mesonic operator)



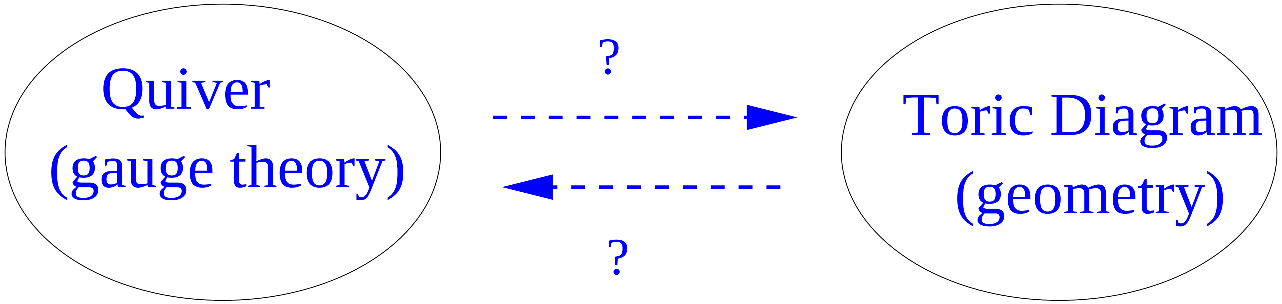


$\mathbb{C}^3/\mathbb{Z}_3$

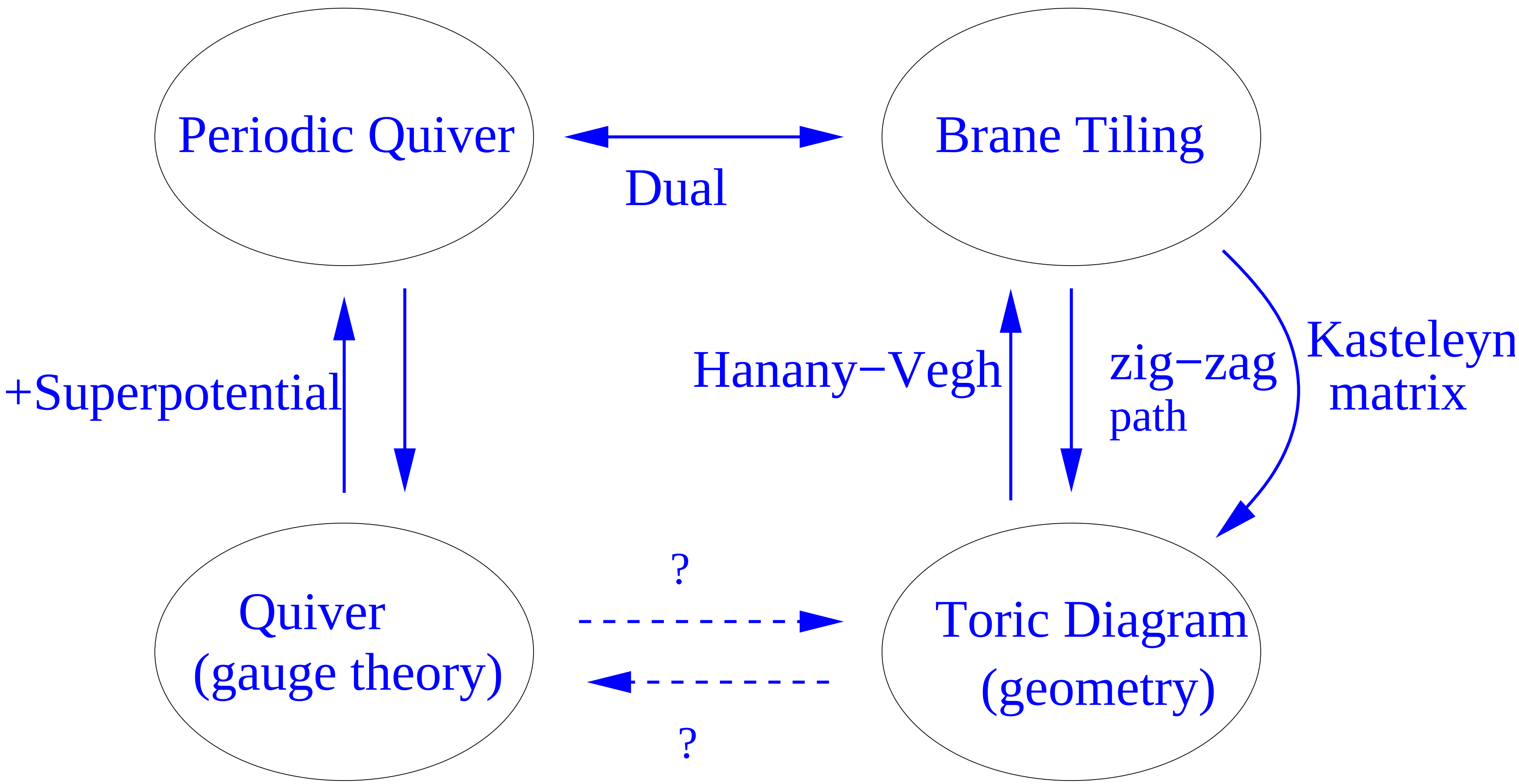


$\mathbb{F}_0 = \mathbb{P}_1 \times \mathbb{P}_1$

Strategy of Brane Tiling Algorithm



Strategy of Brane Tiling Algorithm



Comparison with Previous Algorithms

- Orbifold case: quiver already known (including non-abelian case) [Douglas-Moore, Lawrence-Nekrasov-Vafa]
- More general type: algorithm already exists but impractical (requires much computational time) [Douglas-Greene-Morrison, Feng-Hanany-He]

Brane tiling:

- Fast computation for general toric singularity
- Gives superpotential
- Graphical understanding of some operations

Further Motivations for Brane Tiling

- Gauge theory has several realization in string theory:
  - D-brane probe (D3)
  - Elliptic model (D4)
  - Hanany-Witten type setup (D5/NS5)
  - Geometric engineering (D6)
- Relation with Statistical Models: crystal, topological string, random plane partition, Nekrasov's formula, free fermion, Brownian motion
- Mathematical Aspects: (Homological) Mirror symmetry, alga, amoeba, derived category, exceptional collection...
- Phenomenological Applications (e.g. Dynamical SUSY breaking, Gauge Mediation)

Plan of This Talk

1. Introduction
2. Origin of Brane Tilings
3. Brane Tiling Algorithm
   - Brane Tiling $\longrightarrow$ Quiver and Superpotential
   - Brane Tiling $\longrightarrow$ Toric Diagram
   - Toric Diagram $\longrightarrow$ Brane Tiling
4. Homological Mirror Symmetry from Brane Tilings and Algae
5. Summary, Outlook and Speculations

Brane Tilings as NS5-D5 system

Suppose we have D3-branes probing singular Calabi-Yau. If we T-dualize along 2-cycle (which is the fiber direction of toric Calabi-Yau), then we have configuration of D5-branes and NS5-branes:

	0	1	2	3	4	5	6	7	8	9
D5	○	○	○	○		○		○		
NS5	○	○	○	○	Σ (2-dim surface)					

- D5-brane worldvolume: $\mathbf{R}^4 \times T^2$
- NS5-brane worldvolume: $\mathbf{R}^4 \times \Sigma$

Real shape of branes: difficult to determine in general (we need to solve EOM)

Weak Coupling Analysis

Consider the weak coupling limit $g_s \to 0$. Then

$$T_{NS5} \gg T_{D5}$$

Then NS5-brane worldvolume Σ is a holomolophic curve $W(x,y) = 0$ in $\left(\mathbb{C}^\times\right)^2$, where

- $x = \exp(x_4 + ix_5), y = \exp(x_6 + ix_7)$
- $W(x,y)$ is a Newton polynomial of the toric diagram

$W(x,y) = 0$ is a 2-dim. surface in 4-dim. space (x_4, x_5, x_6, x_7) and difficult to analyze visually. We consider projection to

- x_4, x_6-direction (perpendicular to D5): Amoeba
- x_5, x_7-direction (parallel to D5): Alga

Amoeba

The amoeba [Gelfand-Kapranov-Zelevinsky] of a Newton polynomial $W(x,y) \in \mathbb{C}[x^{\pm}, y^{\pm}]$ of toric diagram Δ is the image of its zero locus $\{(x,y) \in (\mathbb{C}^{x})^2 \mid W(x,y) = 0\}$ by the map Log:

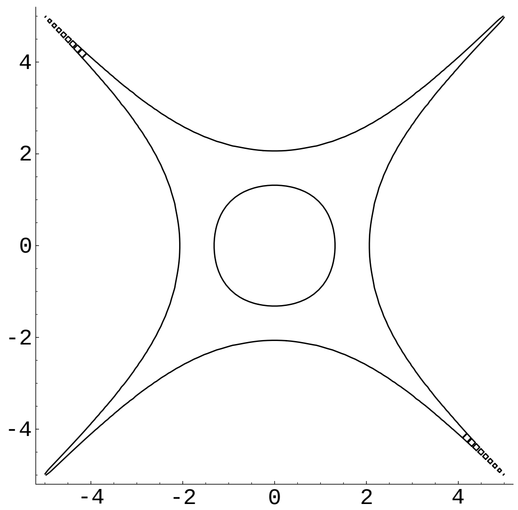
$$\begin{array}{ccc} \mathrm{Log} : (\mathbb{C}^{\times})^2 & \to & T = (\mathbb{R}/\mathbb{Z})^2 \\ \Cup & & \Cup \\ (x,y) & \mapsto & (\log|x|, \log|y|). \end{array}$$

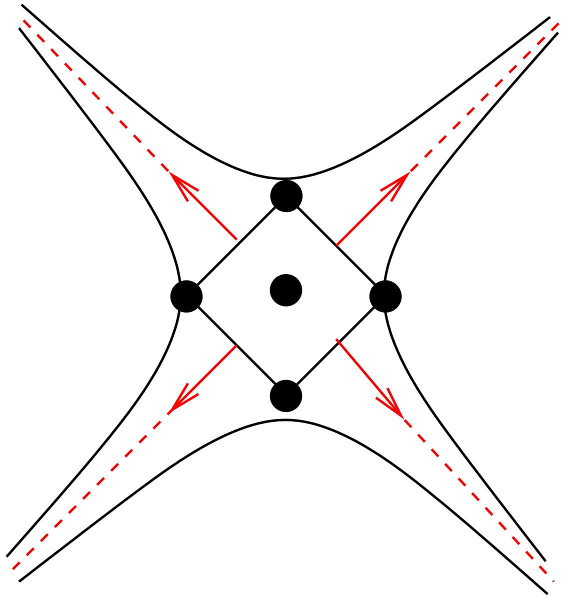
Relation with

- monodromy of hypergeometric function
- real algebraic geometry (Hilbert's 16th problem)
- instanton counting [Maeda-Nakatsu]

Amoeba: Example

example: $\mathbb{F}_0 = \mathbb{P}_1 \times \mathbb{P}_1$





Tropicalization(ultradiscretization) of amoeba gives (p,q)-web, which represents NS5-brane

Alga

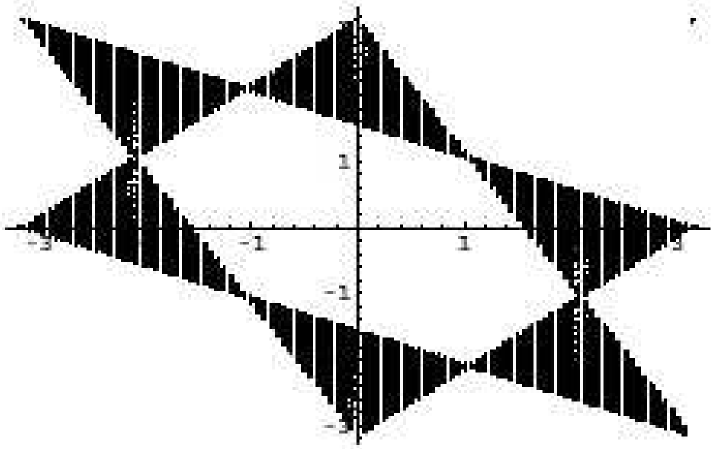
A new notion of algae (藻、藻類) was introduced in [Feng-He-Kennaway-Vafa (hep-th/0511287)]

The alga of a Newton polynomial $W(x,y) \in \mathbb{C}[x^{\pm}, y^{\pm}]$ of Δ is the image of its zero locus $\{(x,y) \in (\mathbb{C}^{\times})^2 \mid W(x,y) = 0\}$ by the argument map

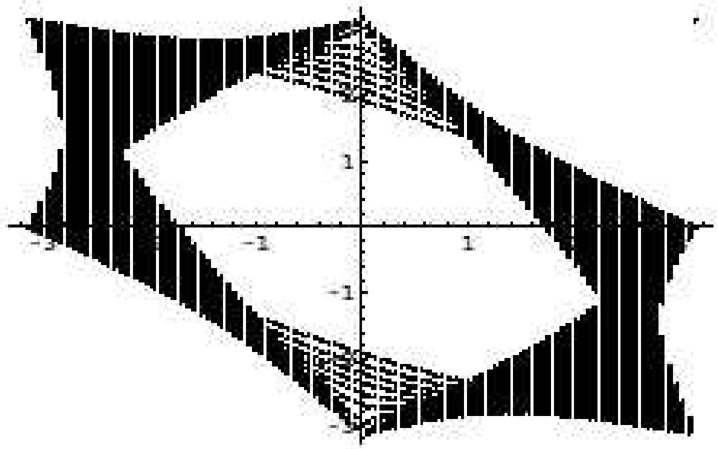
$$\begin{array}{ccc} (\mathbb{C}^{\times})^2 & \to & T = (\mathbb{R}/\mathbb{Z})^2 \\ \Cup & & \Cup \\ (x,y) & \mapsto & \dfrac{1}{2\pi}(\arg(x), \arg(y)). \end{array}$$

Actually, Tsikh et. al. have considered the same thing before and called them coamoeba (コアメーバ？余アメーバ？虚アメーバ？)

Alga: Examples



$$x + y + \frac{1}{xy} = 0$$

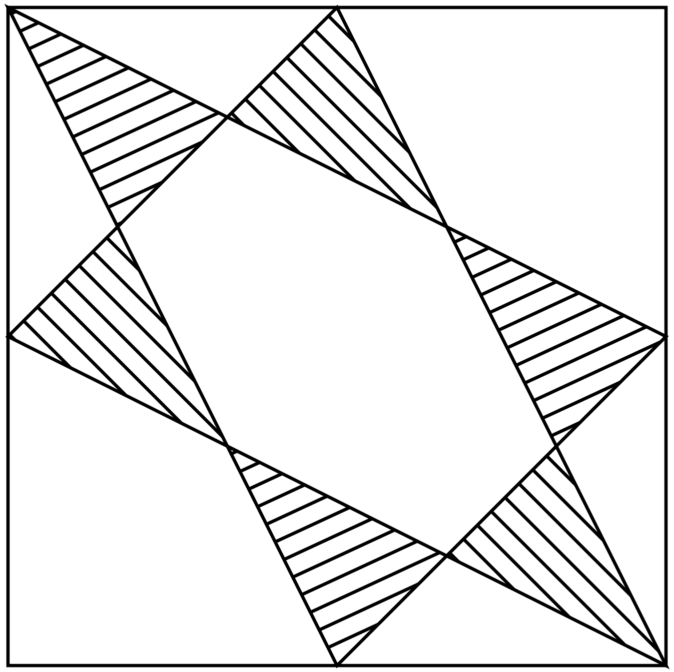


$$x + y + 3 + \frac{1}{xy} = 0$$

Alga takes very beautiful form when we choose suitable coefficients, but not in general.

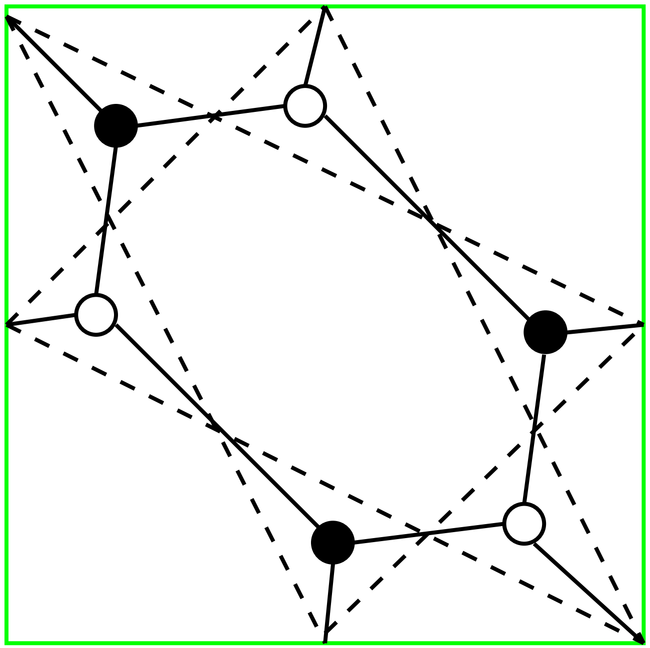
From Algae to Brane Tilings

You can obtain brane tiling from alga!



$\mathbb{C}^3/\mathbb{Z}_3$

$$W = x^3 + y^3 + 1$$



From Alga to Brane Tiling: $\mathbb{C}^3/\mathbb{Z}_n$ case

Suppose W(x,y) is given by sum of three monomials ($\mathbb{C}^3/\mathbb{Z}_n$ case). then

- $W^{-1}(0)$ is disjoint union of triangles (without boundary) and their vertices.
- Each triangle is diffeomorphic to a disk (without boundary)
- The argument map from $W^{-1}(0)$ to its alga is diffeomorphis when restricted to each disk.

From Alga to Brane Tiling: $\mathbb{C}^3/\mathbb{Z}_n$ case

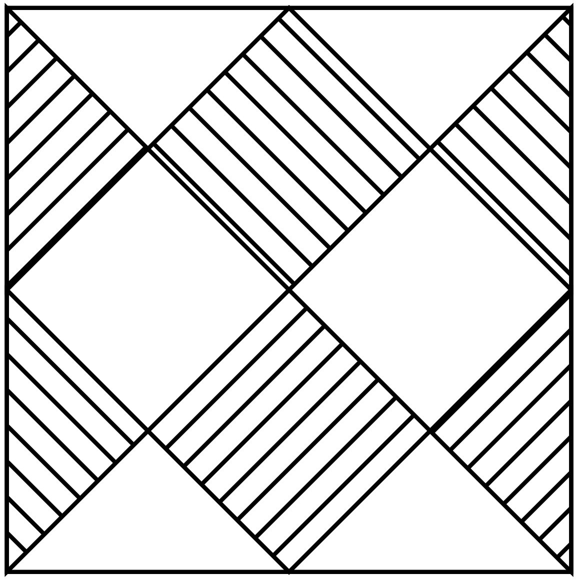
Suppose W(x,y) is given by sum of three monomials ($\mathbb{C}^3/\mathbb{Z}_n$ case). then

- When this diffeomophism is orientation preserving(reversing), place white(black) vertex at the center of gravity and
- Connect two vertices for each intersection point of triangles

Then the bipartite graph given by this method is the brane tiling!

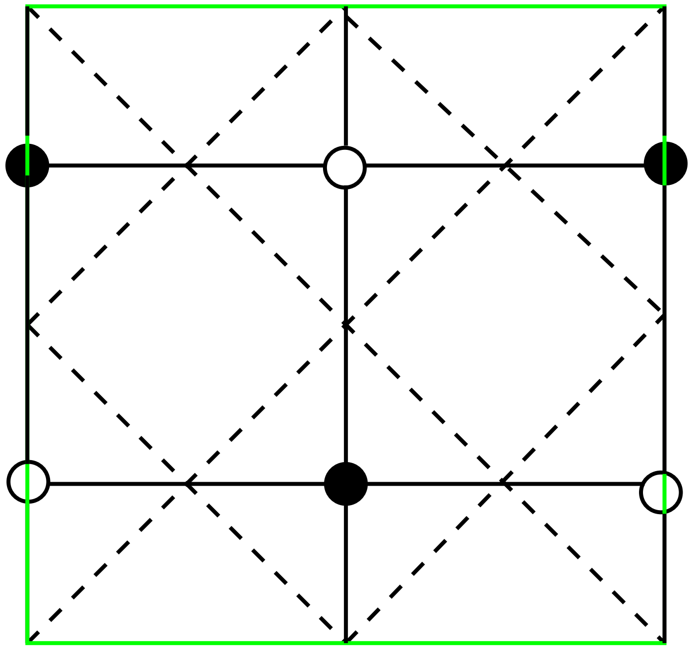
From Alga to Brane Tiling: $\mathbb{P}^1 \times \mathbb{P}^1$ case

The case with $\mathbb{P}^1 \times \mathbb{P}^1$ is similar.



$$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$$

$$W = x - \frac{1}{x} + y + \frac{1}{y}$$



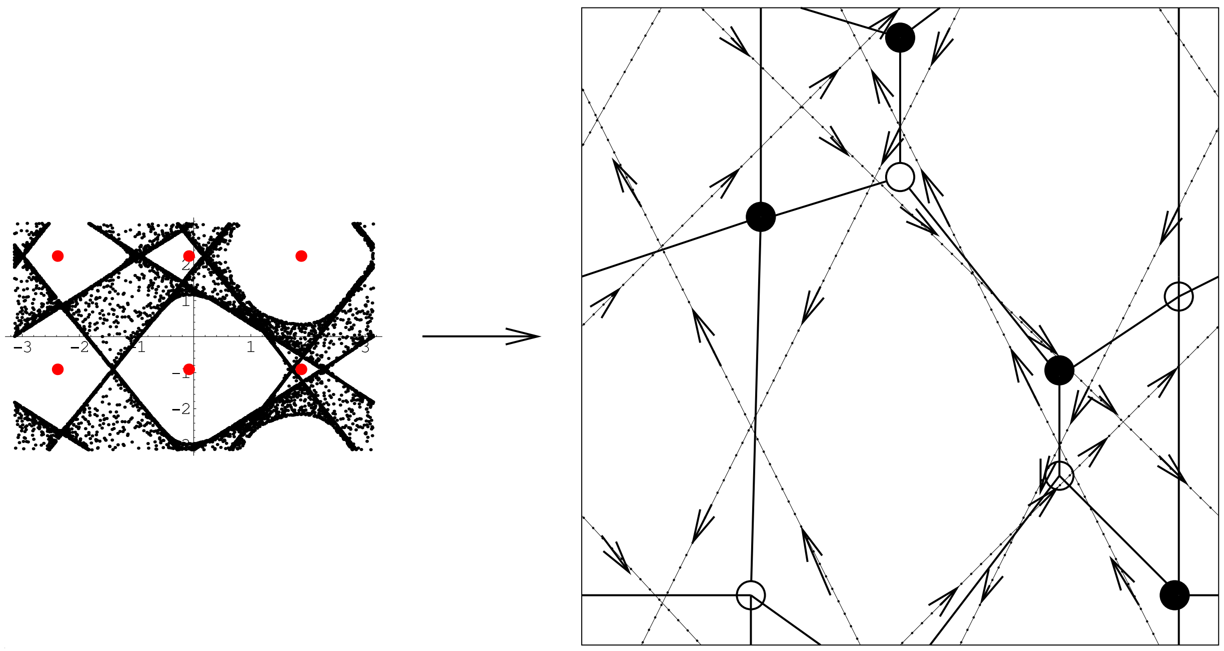
Subtleties in Alga

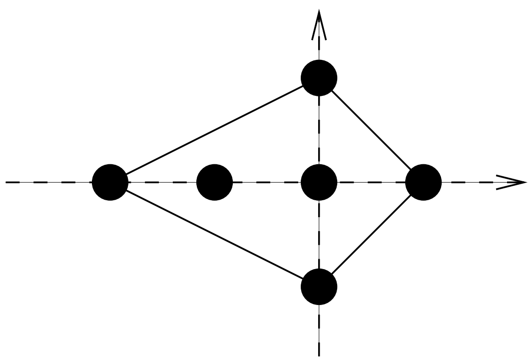
Actually, there are some subtleties when we obtain brane tiling from alga:

- Even in simple cases like $\mathbb{P}^1 \times \mathbb{P}^1$, we have to choose carefully the coefficients to obtain "good" alga.
- In general it is not known whether we can always choose coefficients to obtain good alga. (cf. Harnack curve for amoeba)
- Even if we cannot obtain "good" alga, we may still be able to read off brane tiling. We hope to report on this point in the near future ("asymptotic boundary")

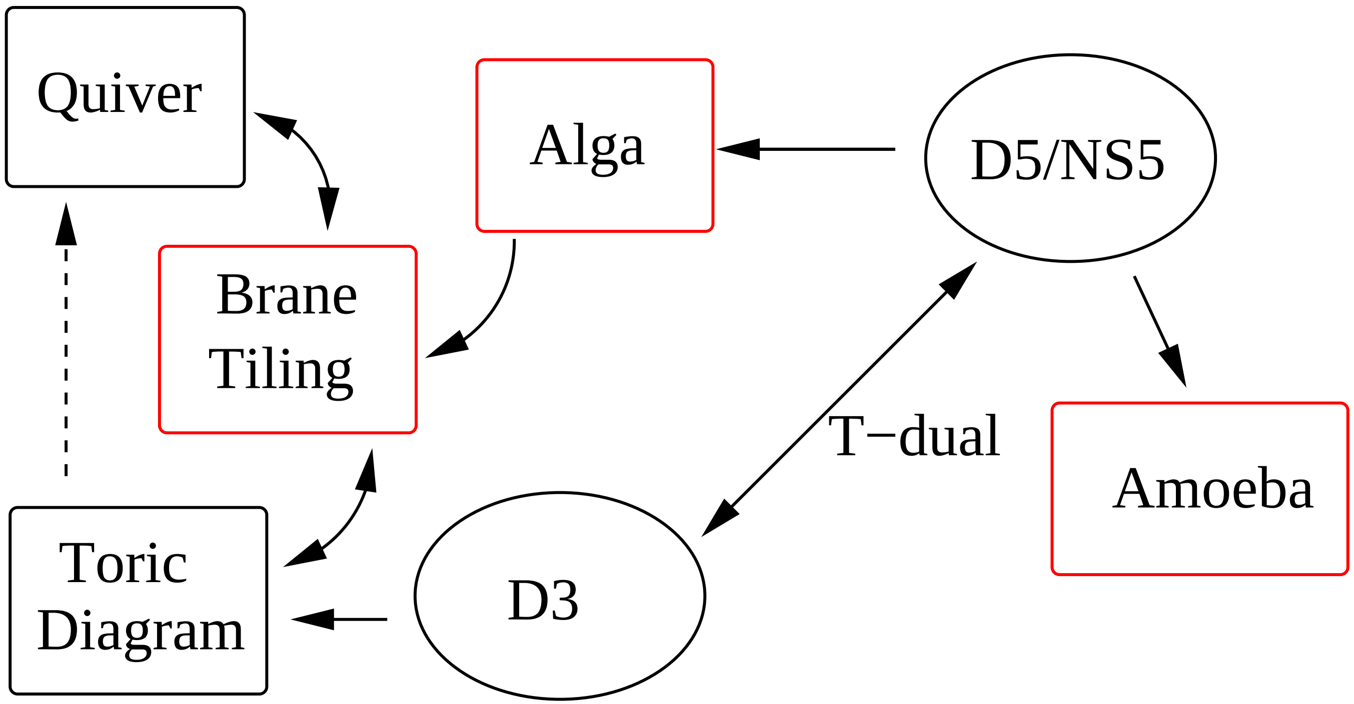
Subtleties in Alga

$Y^{3,1}$ case





Summary: Origin of Brane Tiling



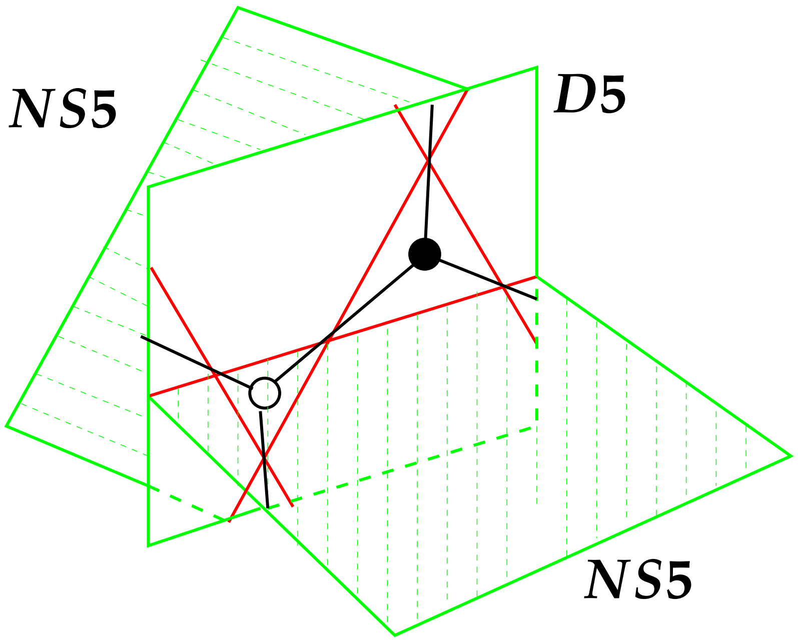
- Alga is projection of NS5-brane onto D5-brane direction
- slogan: Brane Tiling is "Tropicalization" of Alga

Strong Coupling Analysis

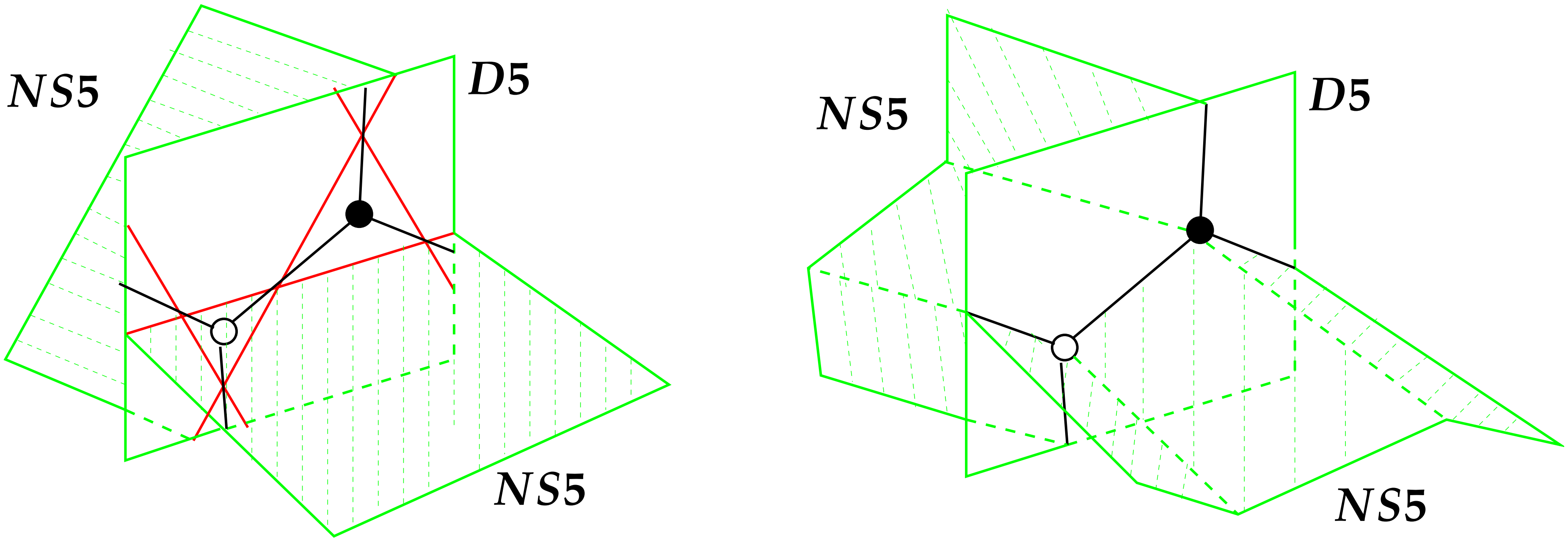
Consider the strong coupling limit $g_s \to \infty$. Then

$$T_{D5} \gg T_{NS5}$$

Then D5-branes become flat and NS5-branes are orthogonal to D5.



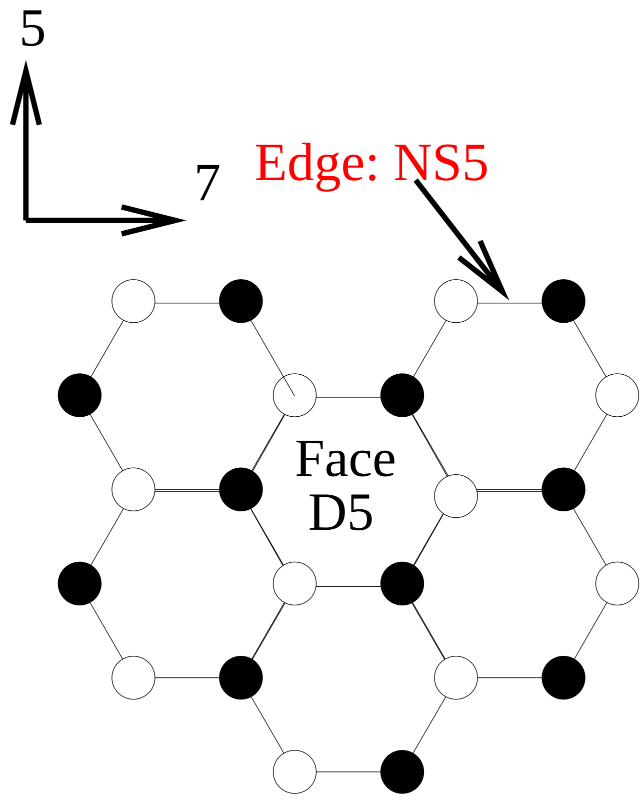
Strong Coupling Analysis



Real shape of branes is different from brane tiling!!

What is Brane Tiling?

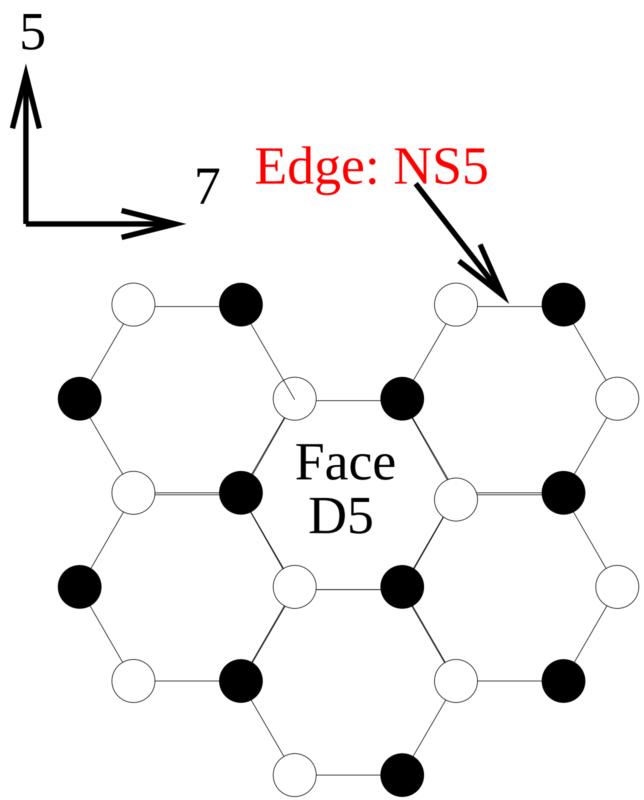
- "tropicalization" of alga
- a bipartite graph written on a torus
- sophisticated form of "brane box" and "brane diamond"
- face: D5-brane
- edge: open strings streched between D5-branes through NS5-brane
- vertex: regions where strings interact
- color of vertex: orientation of NS5-branes



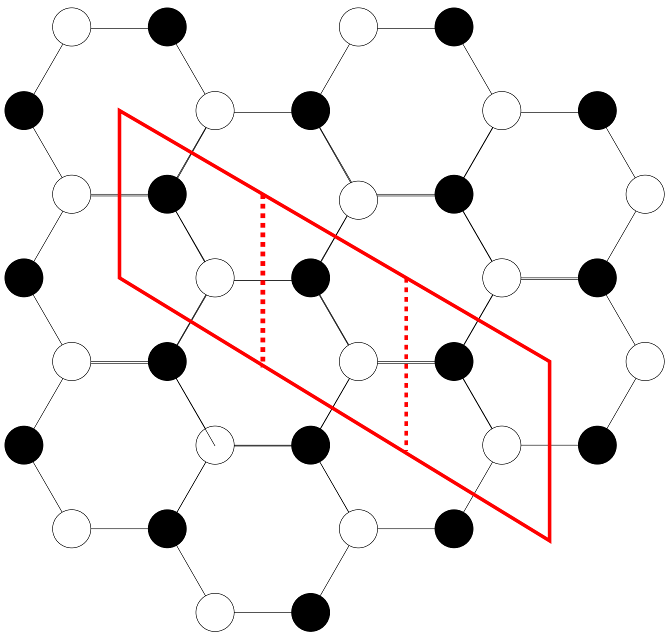
What is Brane Tiling?

- face: D5-brane (gauge group)
- edge: open strings streched between D5-branes through NS5-brane (bifundamentals)
- vertex: regions where strings interact (gauge invariant operator)
- color of vertex: orientation of NS5-branes (sign in superpotential)

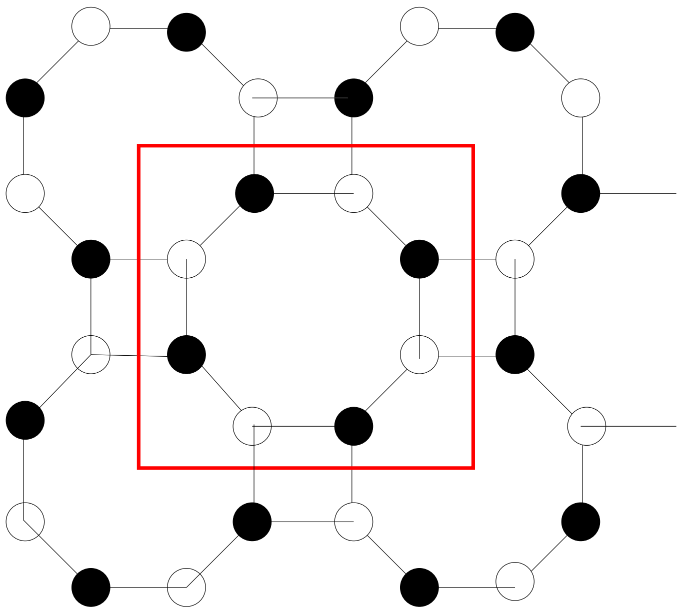
This is basically the dual of quiver!



Example of Brane Tiling



$\mathbb{C}^3/\mathbb{Z}_3$

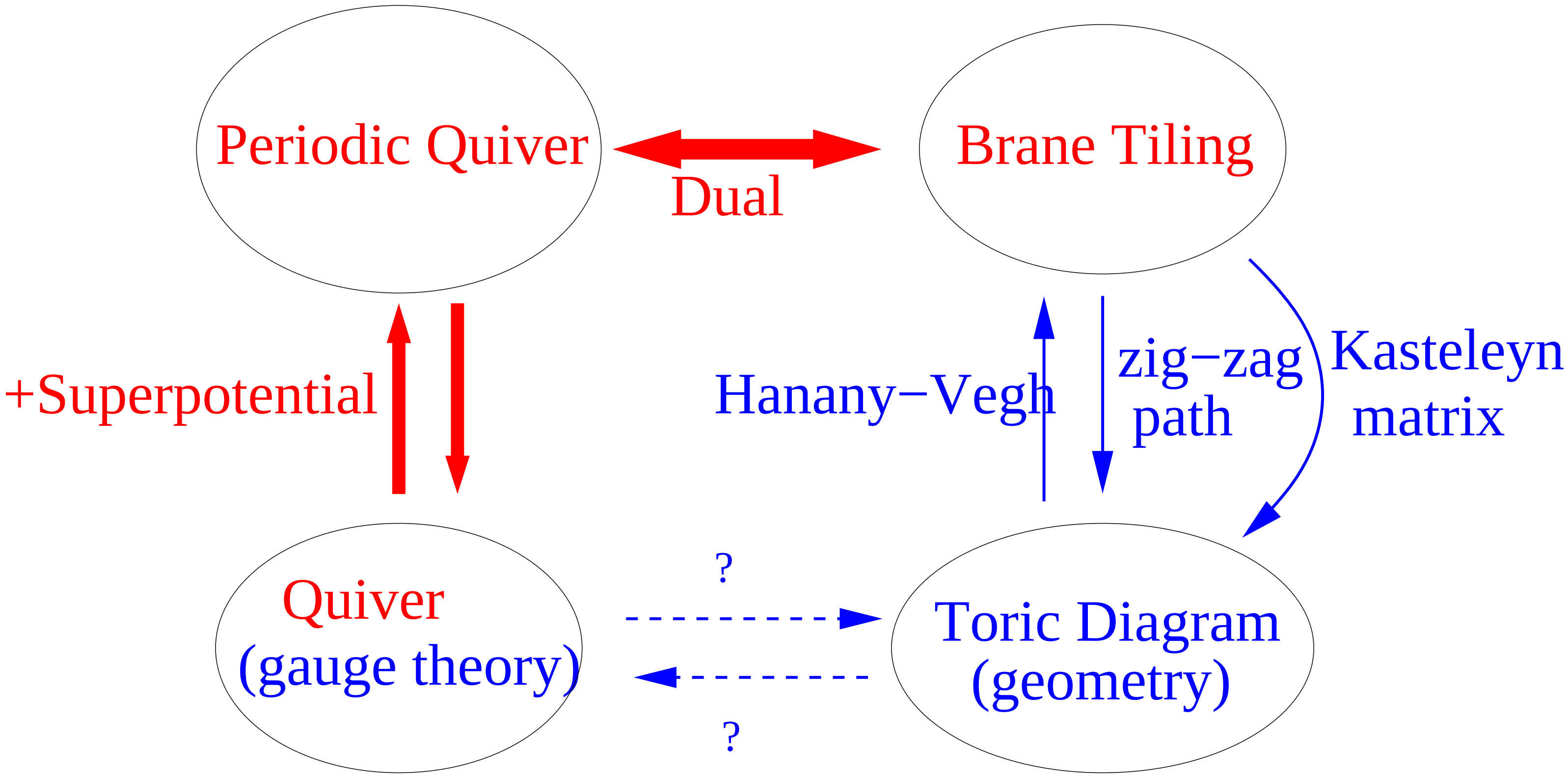


$\mathbb{F}_0$

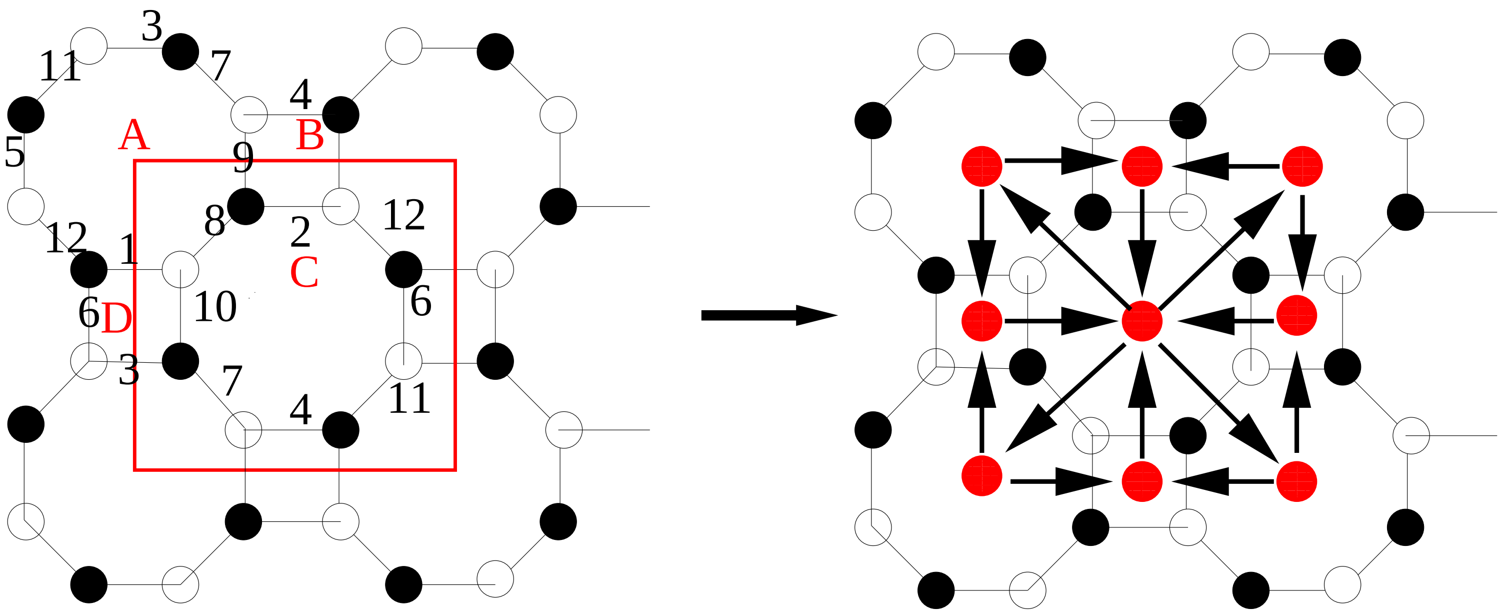
Plan of This Talk

1. Introduction
2. Origin of Brane Tilings
3. Brane Tiling Algorithm
   - Brane Tiling $\longrightarrow$ Quiver and Superpotential
   - Brane Tiling $\longrightarrow$ Toric Diagram
   - Toric Diagram $\longrightarrow$ Brane Tiling
4. Homological Mirror Symmetry from Brane Tilings and Algae
5. Summary, Outlook and Speculations

Brane Tiling $\rightarrow$ Quiver & Superpotential



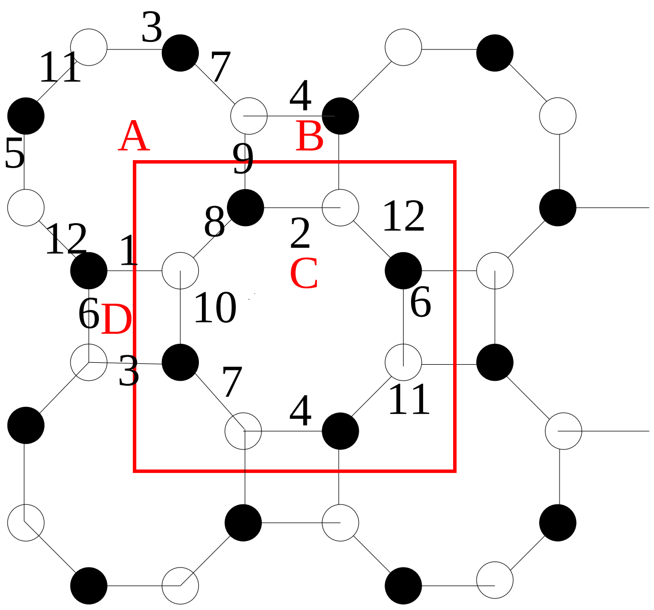
Example: $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$



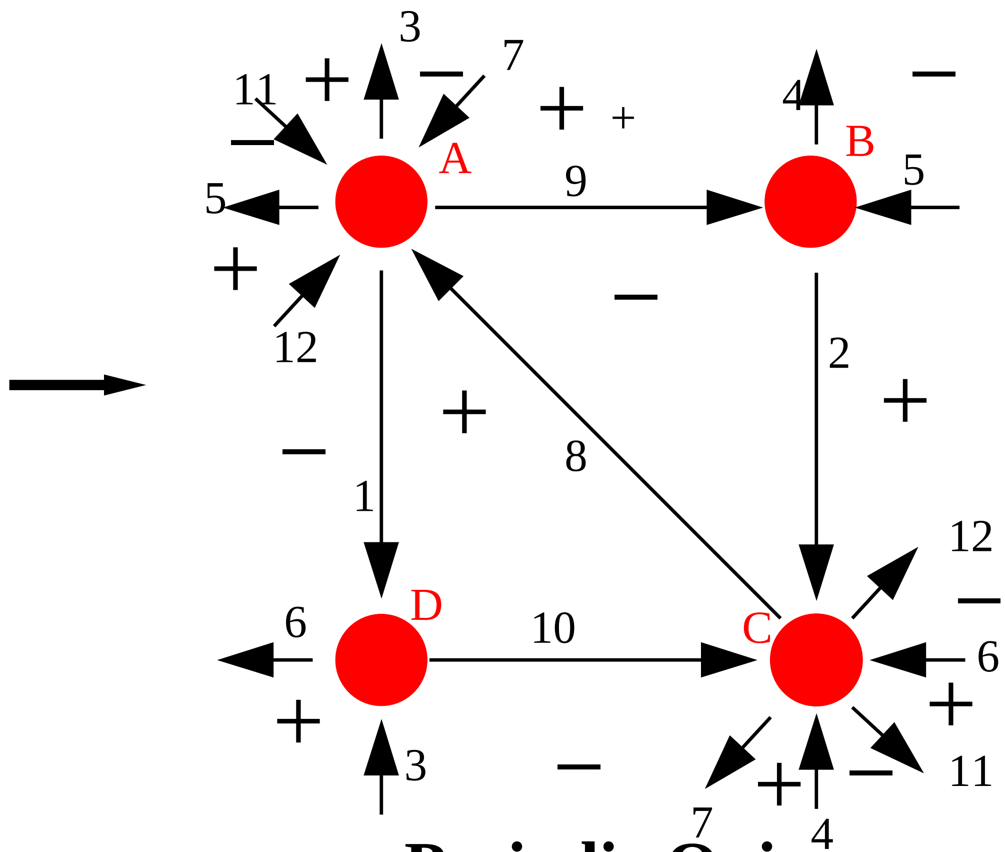
Brane Tiling

Dual of Brane Tiling

Example: $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

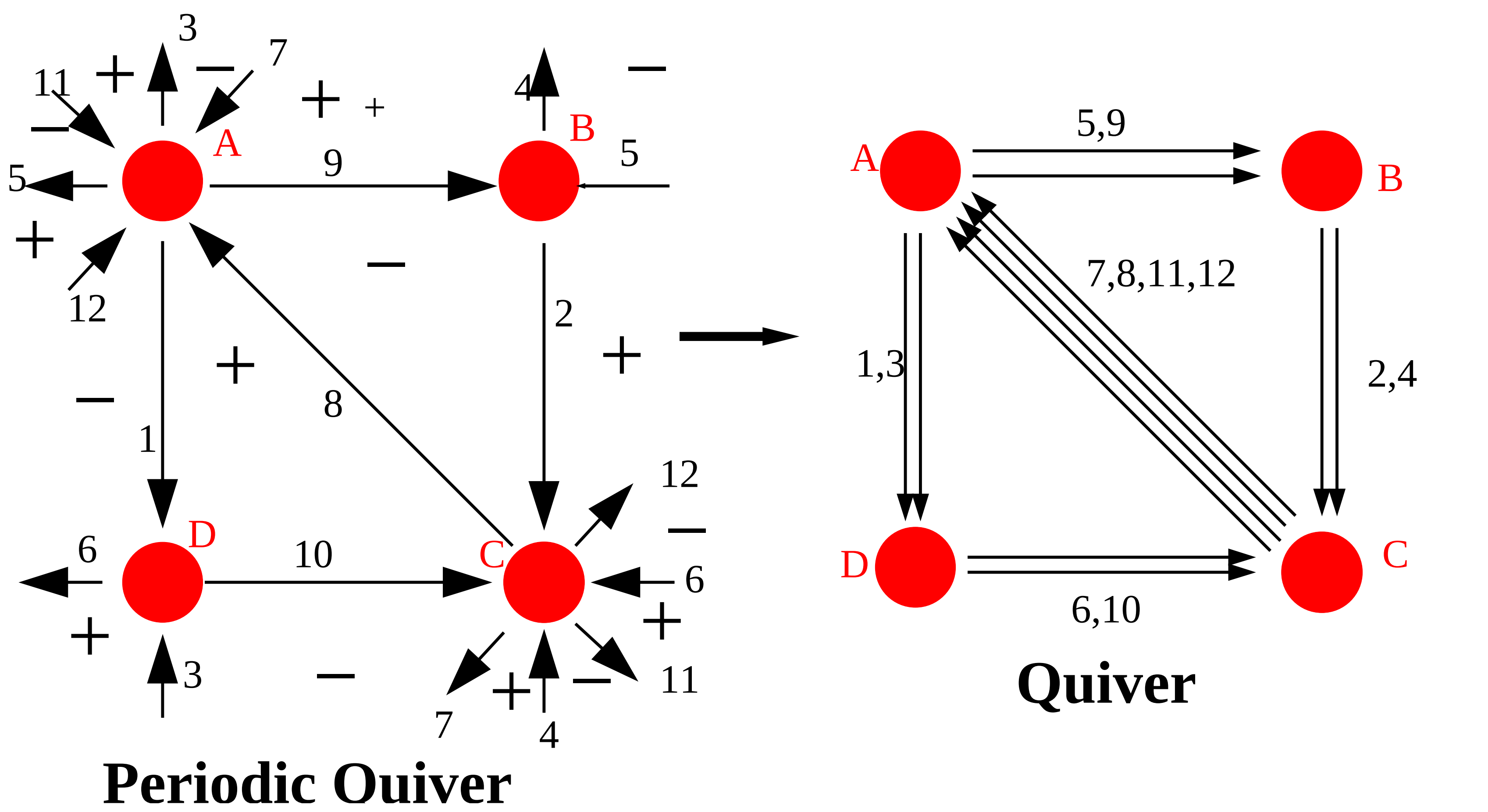


Brane Tiling

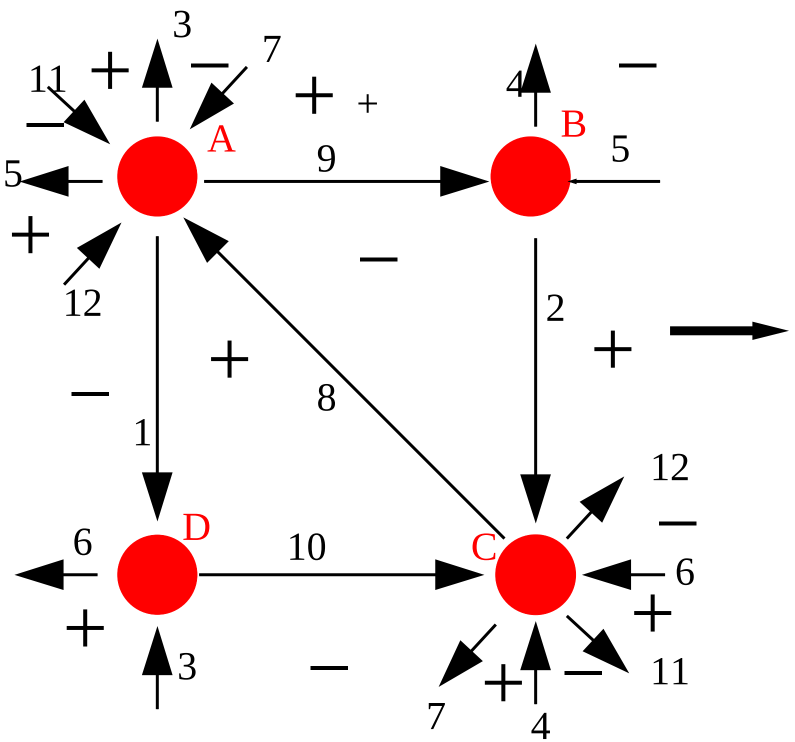


Periodic Quiver

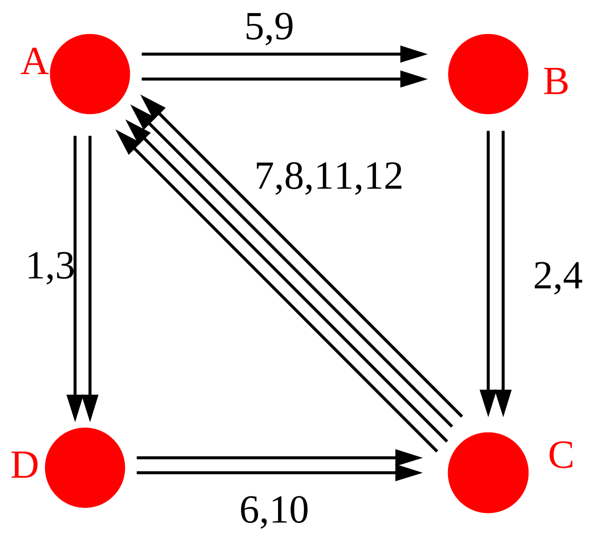
Example: $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$



Example: $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$



Periodic Quiver

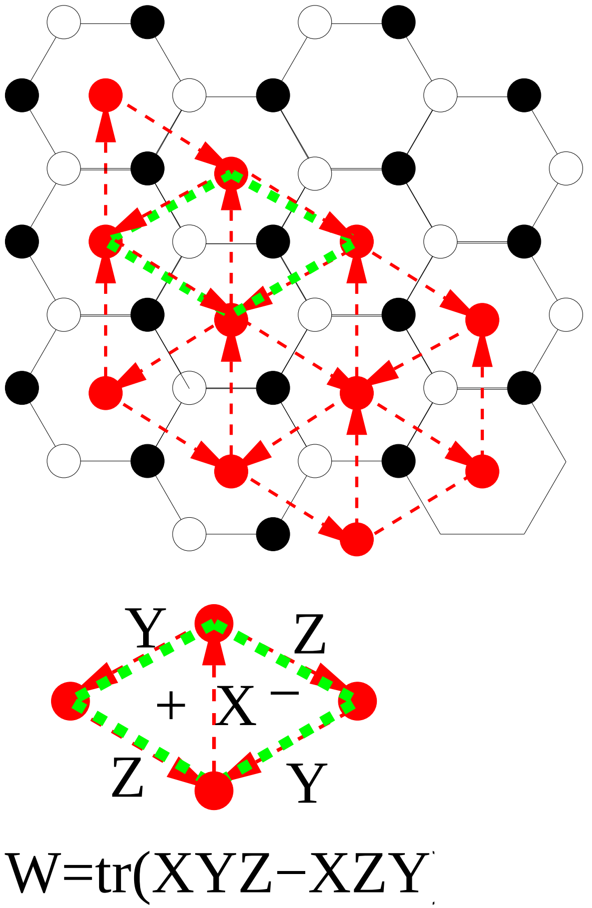


Quiver

$$W = \mathrm{tr}\,(X_1X_{10}X_8 - X_8X_9X_2 + X_7X_9X_4 - X_7X_3X_{10} + X_{11}X_3X_6 - X_{11}X_5X_4 + X_2X_{12}X_5 - X_1X_6X_{12})$$

Example of Brane Tiling: $\mathbb{C}^3$

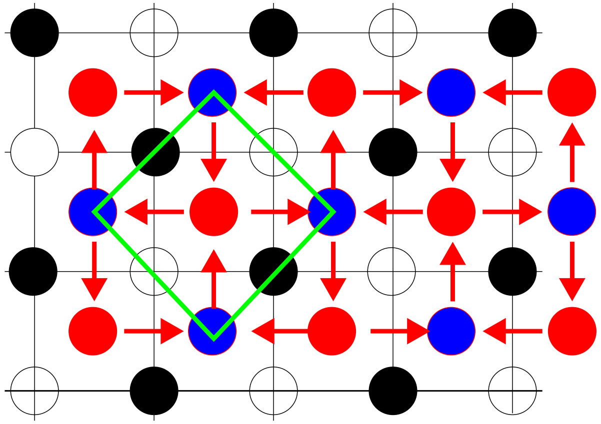
The simplest example is $\mathbb{C}^3$, which corresponds to $\mathcal{N} = 4$ Super Yang-Mills.

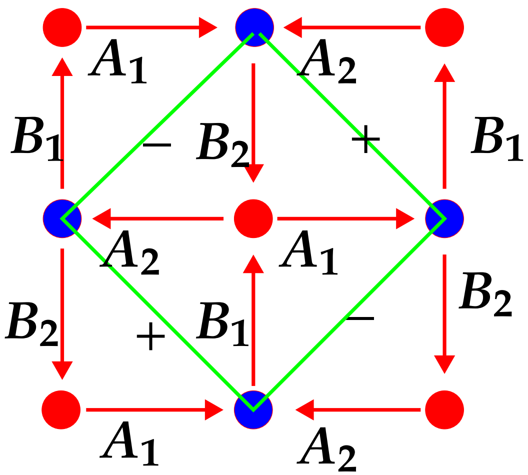


We have the familiar superpotential of $\mathcal{N} = 4$ SYM:

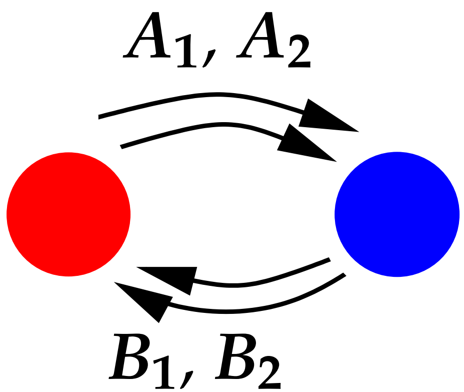
$$W = \mathrm{tr}\left(X\left[Y, Z\right]\right)$$

Example of Brane Tiling: $T^{1,1}$ (Conifold)





We have the familiar quiver
[Klebanov-Witten]



and the superpotential

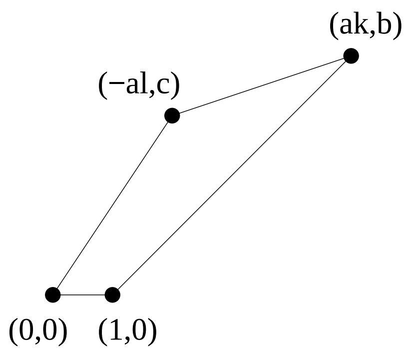
$$W = \mathrm{tr}\left(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1\right)$$

Explicit Metric of 5d Sasaki-Einstein mfds

For a long time, explicit form of metric for 5d Sasaki-Einstein mfd is known only for S^5 and $T^{1,1}$ (conifold).

Recently a new infinite familiy of metric for SE mfds was constructed, by taking certain scaling limit from 5d Kerr BH solution [Gauntlett-Martelli-Sparks-Waldram, Cvetic-Lu-Page-Pope].

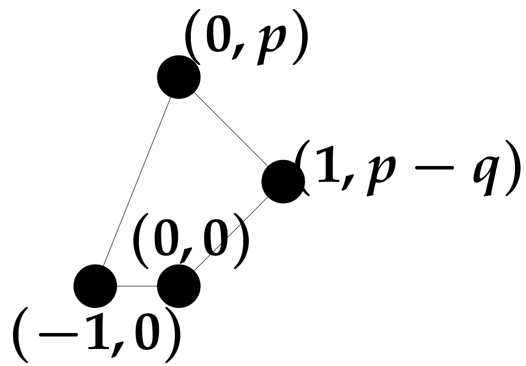
These manifolds (named $L^{a,b,c}$) are later shown to be toric:



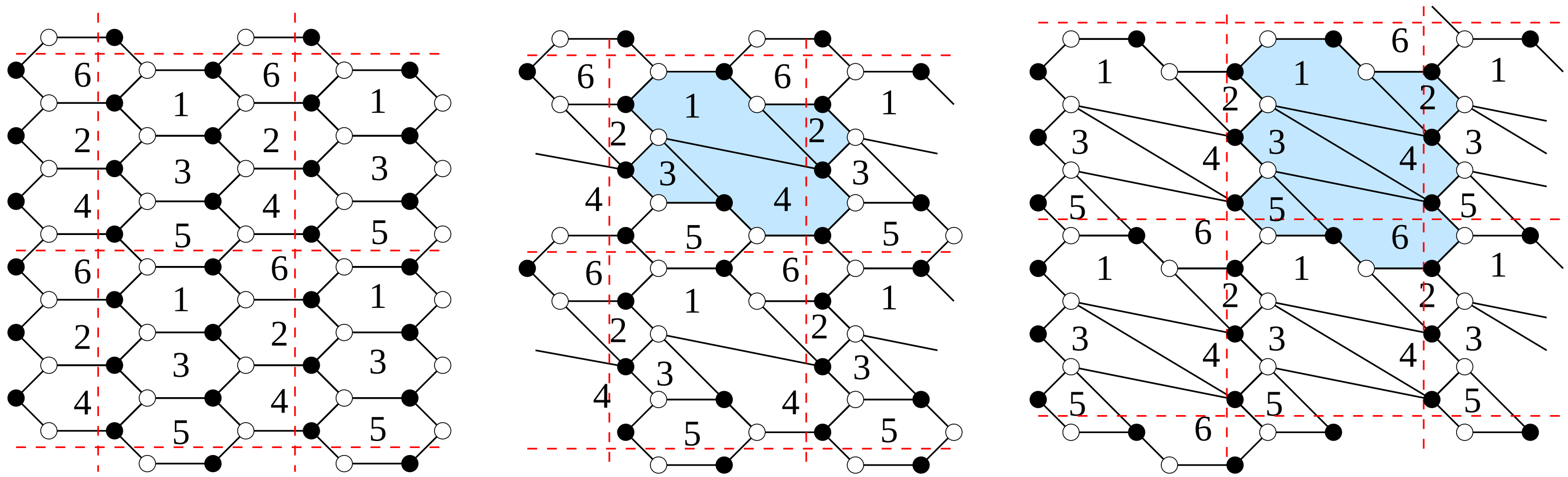
where k and l are two integers satisfying $ck + dl = 1$.

Example of Brane Tiling: $Y^{3,q}$

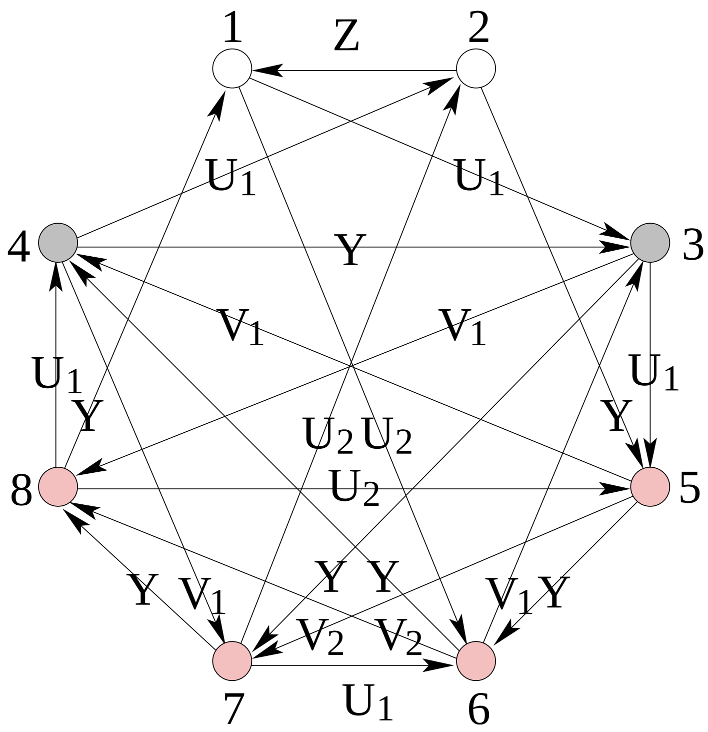
$Y^{p,q}$ is a special case of $L^{a,b,c}$ ($a = p + q$, $b = p - q$, $c = p$)

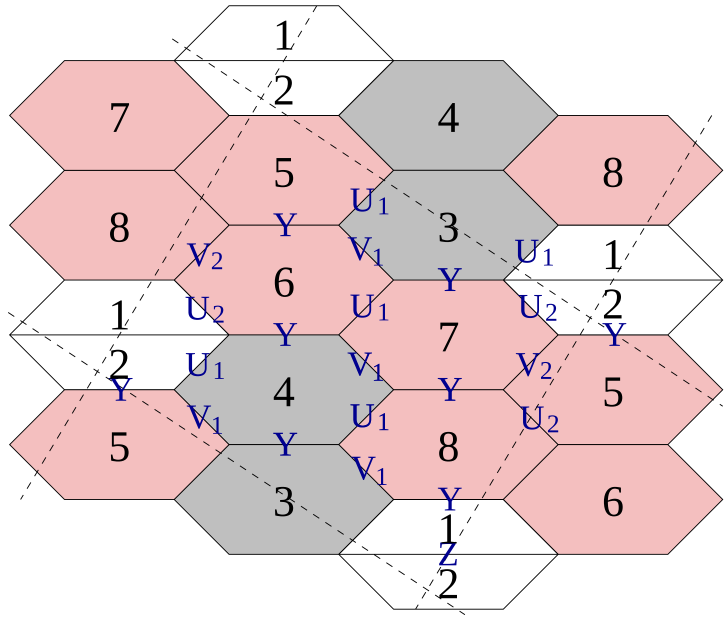


$Y^{3,3}$=$\mathbb{C}^3$/ $(\mathbb{Z}_2 \times \mathbb{Z}_3)$: enlarge the fundamental domain by 2×3. $Y^{3,2}$ and $Y^{3,1}$ is obtained by "adding impurity";



Example of Brane Tiling: $L^{1,7,3}$

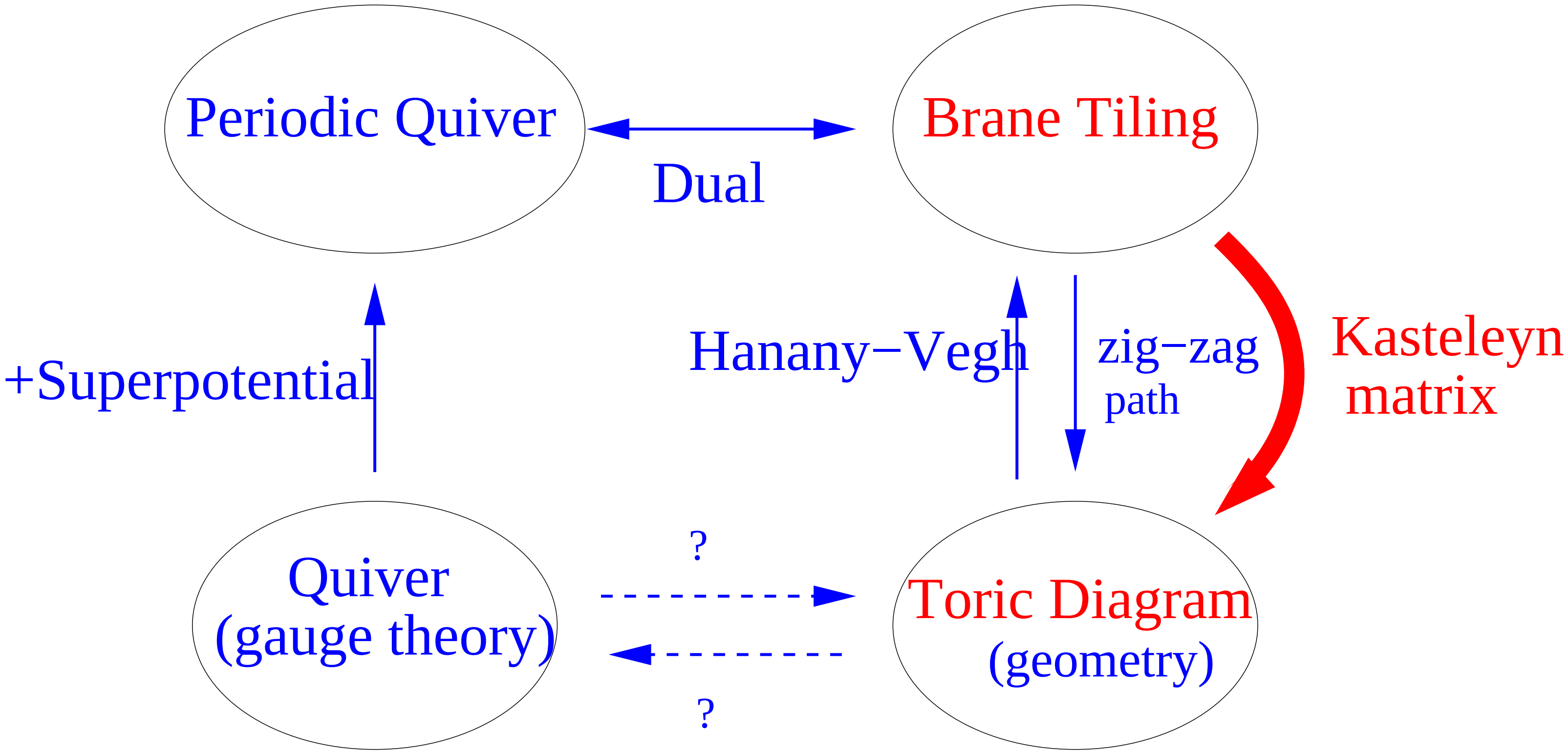




Plan of This Talk

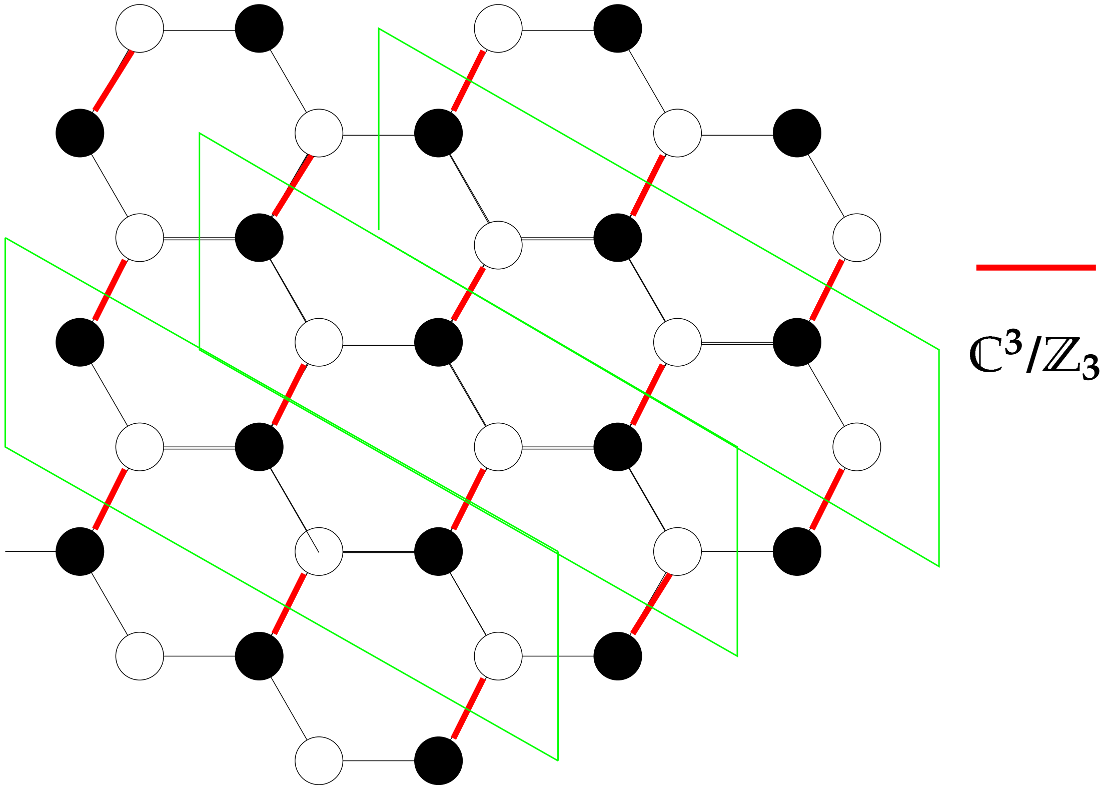
1. Introduction
2. Origin of Brane Tilings
3. Brane Tiling Algorithm
   - Brane Tiling $\longrightarrow$ Quiver and Superpotential
   - Brane Tiling $\longrightarrow$ Toric Diagram
   - Toric Diagram $\longrightarrow$ Brane Tiling
4. Homological Mirror Symmetry from Brane Tilings and Algae
5. Summary, Outlook and Speculations

Brane Tiling $\to$ Toric diagram



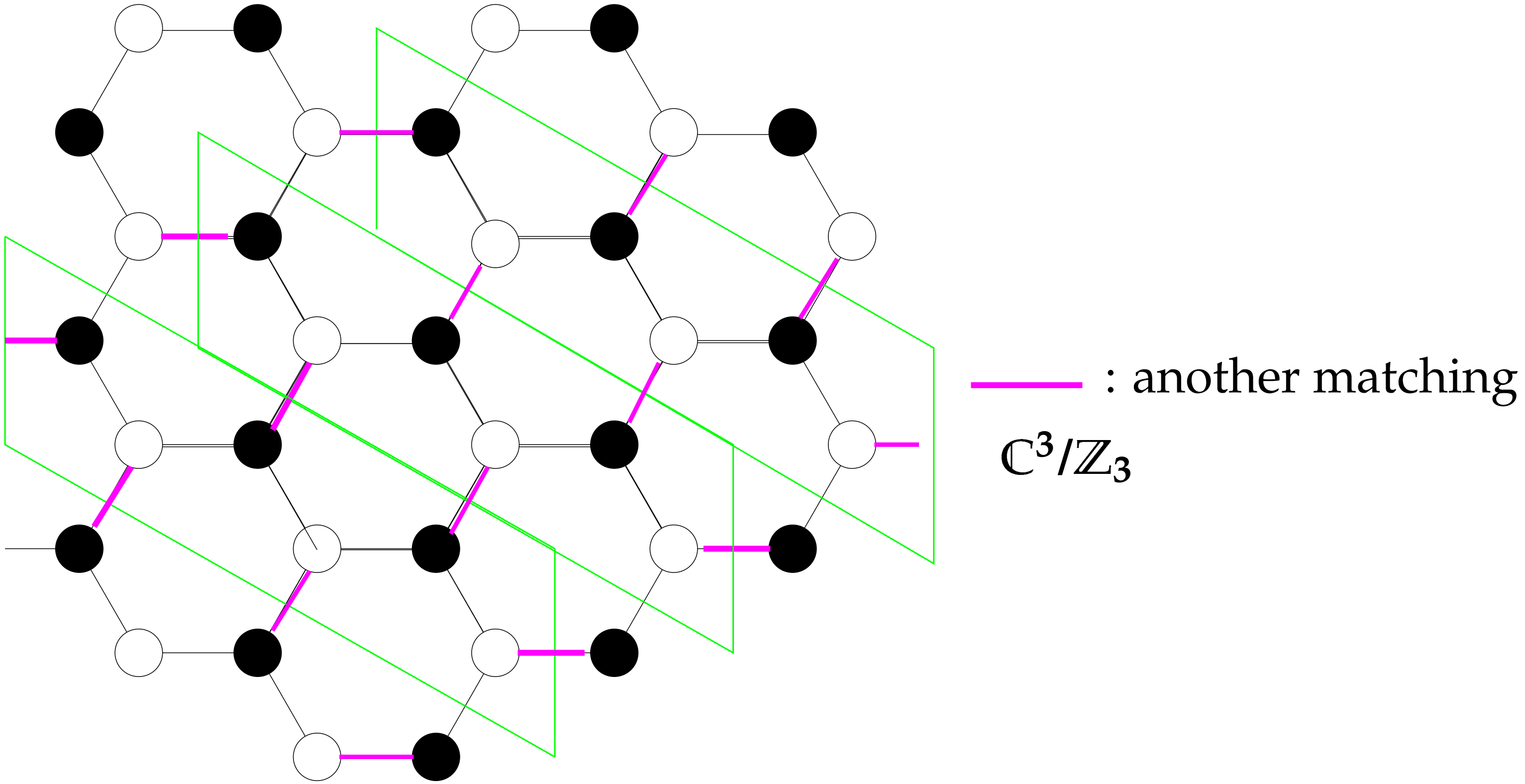
Height Function

Choose any reference matching, then we have height function.



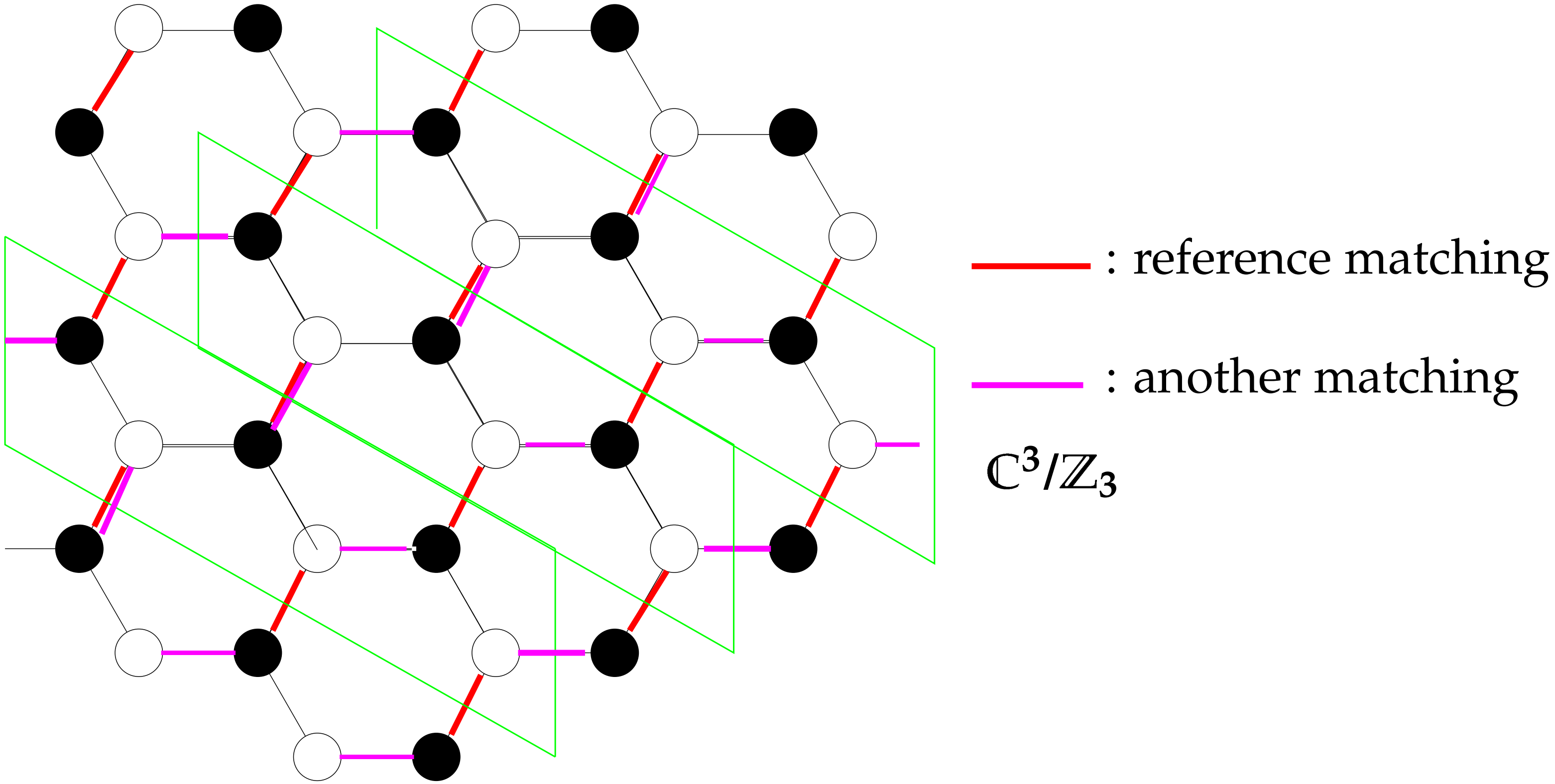
Height Function

Choose any reference matching, then we have height function.



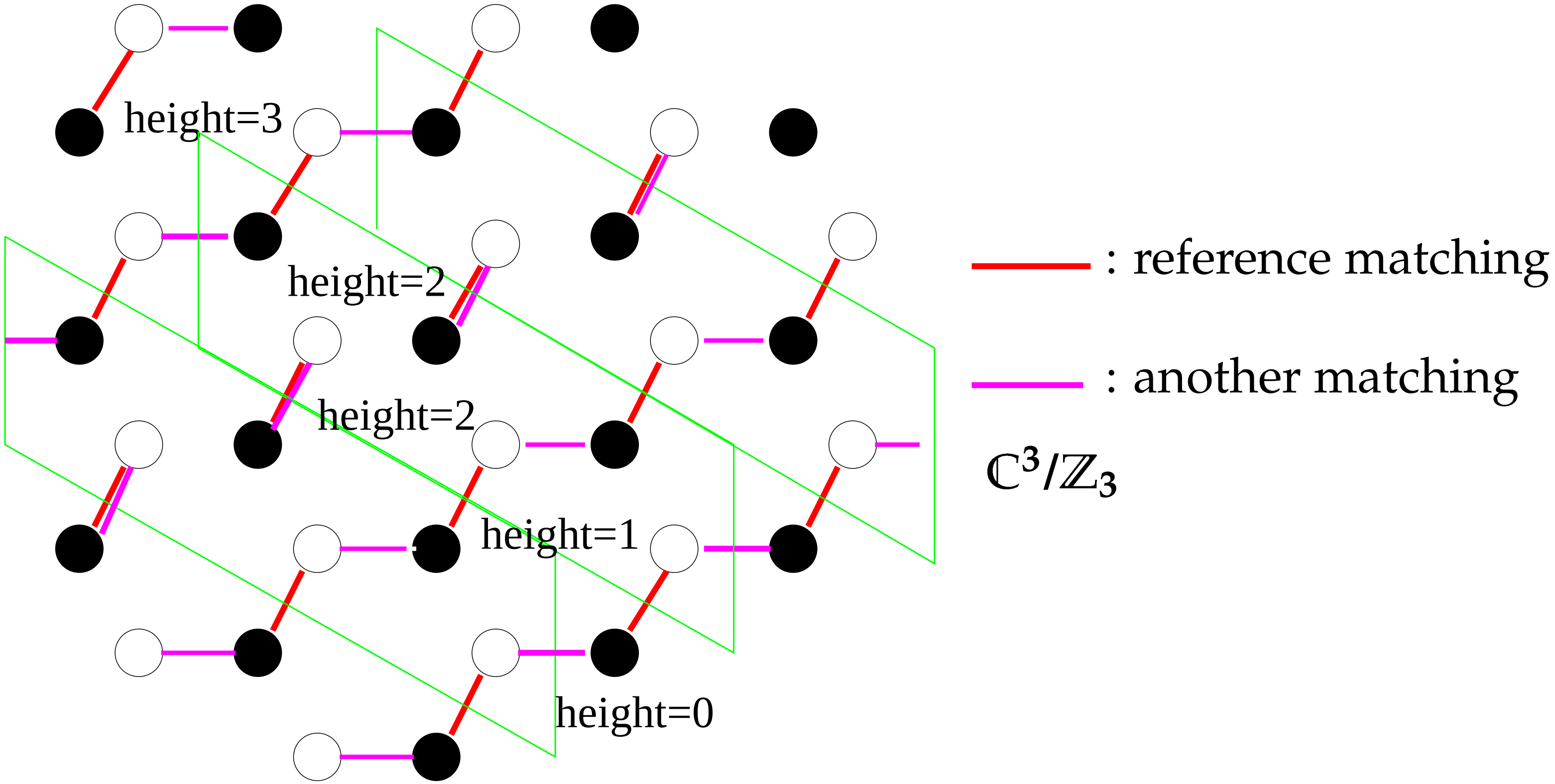
Height Function

Choose any reference matching, then we have height function.



Height Function

Choose any reference matching, then we have height function.



Characteristic Polynomial of Dimer

Define the characteristic polynomial

$$P(z,w) = z^{h_{x0}} w^{h_{y0}} \sum_{\text{P. M.}} c_{h_x,h_y} (-1)^{h_x+h_y+h_x h_y} z^{h_x} w^{h_y}$$

and (where $h_x(h_y)$ is the height change along the $\alpha(\beta)$-cycle) then the toric diagram is the Newton polygon of $P(z,w)$:
Note: Different choice of reference matching and α, β-cycle of the torus leads to same toric diagram up to $SL(2,\mathbb{Z})$ and translation.

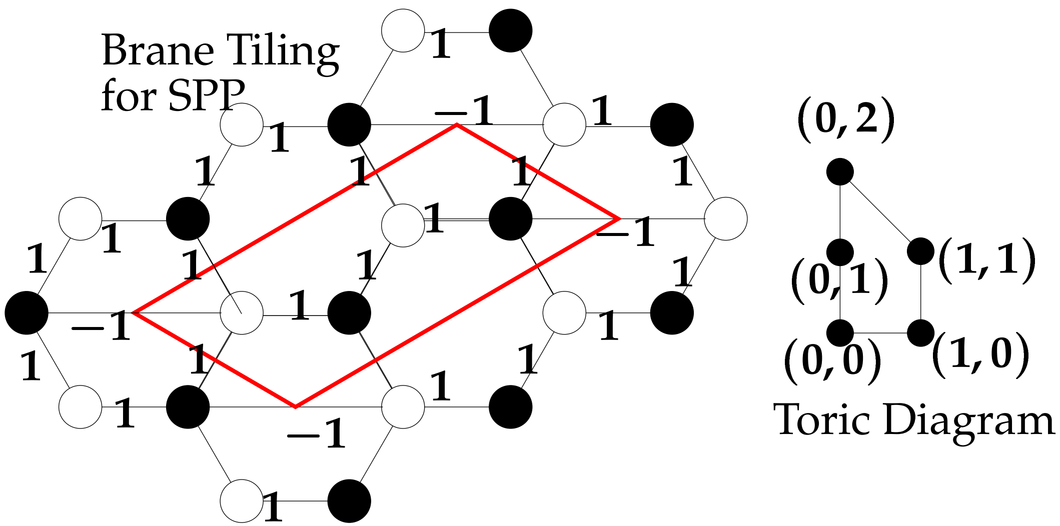
$P(z,w)$ can also be calculated as the determinant of the Kasteleyn matrix.

Kasteleyn matrix

Suppose we have N white and N black nodes on torus. Kasteleyn matrix K is then $N \times N$-matrix whose rows(columns) indexed by white(black) nodes.

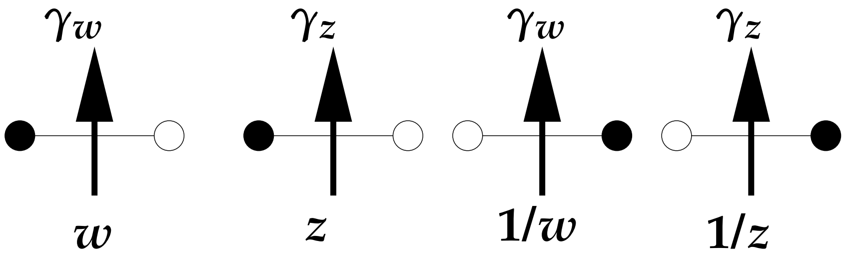
1. Assign weight ± 1 to each edge s.t. the products of weights around each face is

$$\begin{cases} +1 & \text{if } (\#\text{ of edges} \equiv 2 \bmod 4) \\ -1 & \text{if } (\#\text{ of edges} \equiv 0 \bmod 4) \end{cases}$$



Kasteleyn Matrix (2)

2. choose two paths γ_w, γ_z whose homology classes span the basis of $H^1(T^2, \mathbb{Z})$. Assign weight $w, z, \frac{1}{w}$ or $\frac{1}{z}$ to each edge γ_w, γ_z crosses:



3. multiply the two factors of 1 and 2. (i, j)-component of matrix K is given by the sum of edge weights for all edges connecting node i and j.

Then Toric Diagram (up to $SL(2, \mathbb{Z})$ and translation) is the Newton polygon of characteristic polynomial:

$$P(z, w) = \det K(z, w)$$

Newton Polynomial and Newton Polytope

Let $P(z,w)$ be a Laurant Polynomial $\sum_{(i,j)} c_{(i,j)} z^i w^j$.
Newton Polytope Δ of $P(z,w)$ is defined by

$$\Delta = \{(i,j) | c_{(i,j)} \neq 0\} \in \mathbb{Z}^2$$

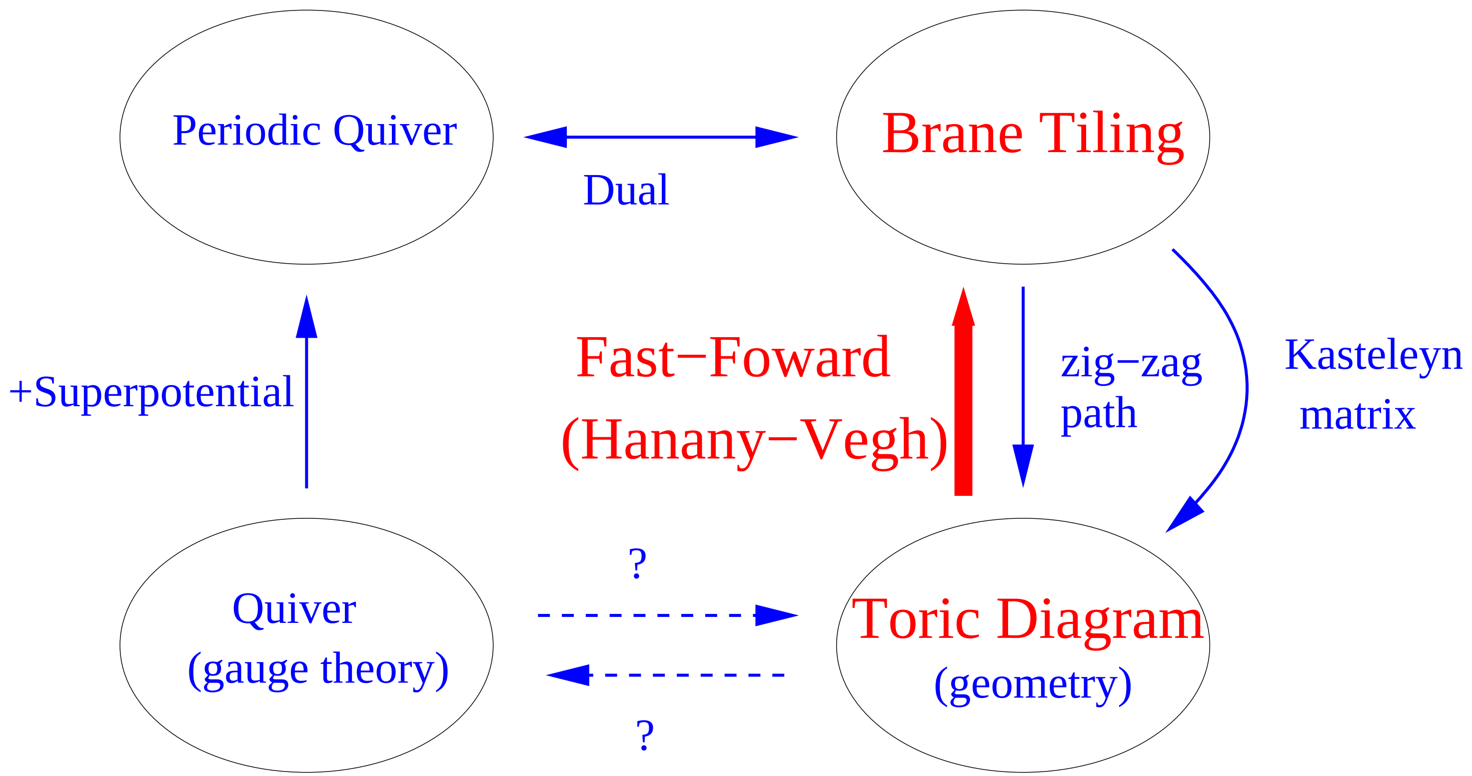
Conversely, let Δ be a (finite) collection of lattice points.
Then the Newton Polynomial of Δ is a Laurant polynomial

$$P(z,w) = \sum_{(i,j)\in\Delta} c_{(i,j)} z^i w^j$$

Plan of This Talk

1. Introduction
2. Origin of Brane Tilings
3. Brane Tiling Algorithm
   - Brane Tiling $\longrightarrow$ Quiver and Superpotential
   - Brane Tiling $\longrightarrow$ Toric Diagram
   - Toric Diagram $\longrightarrow$ Brane Tiling
4. Homological Mirror Symmetry from Brane Tilings and Algae
5. Summary, Outlook and Speculations

Toric Diagram → Quiver



Fast-Forward Algorithm

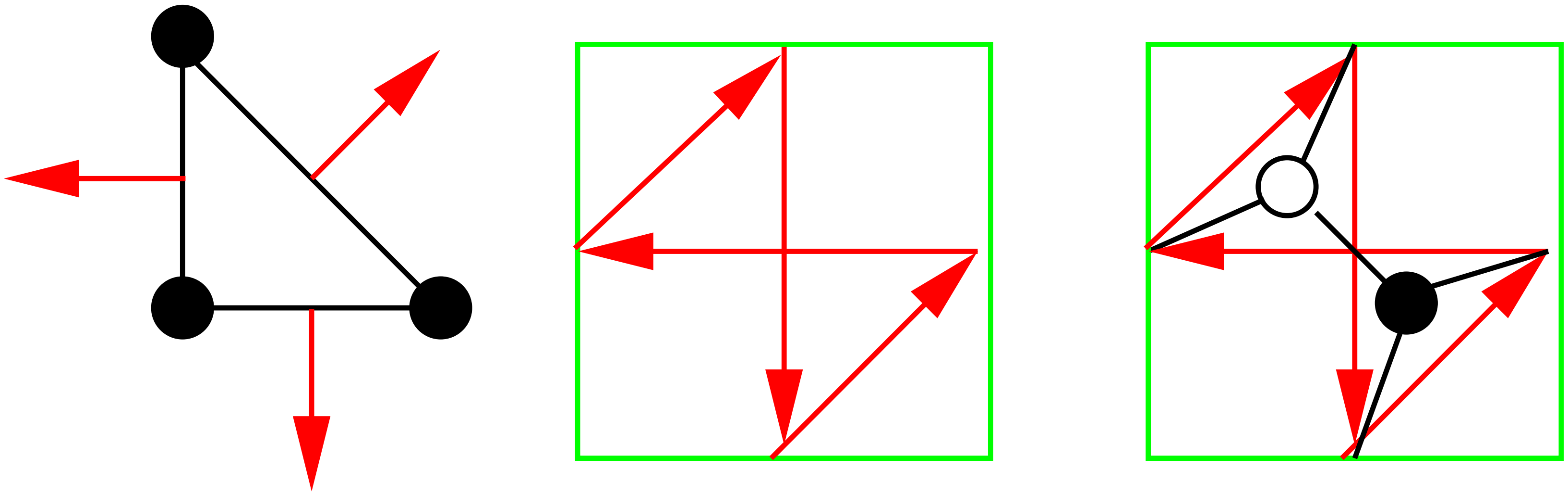
A new method to obtain brane tiling from quiver [Hanany-Vegh].

- Convenient way of obtaining quiver+superpotential from toric diagram (i.e. you don't have to draw amoeba)
- Originally came from analogy with graph theory (zig-zag path, isoradial embedding)

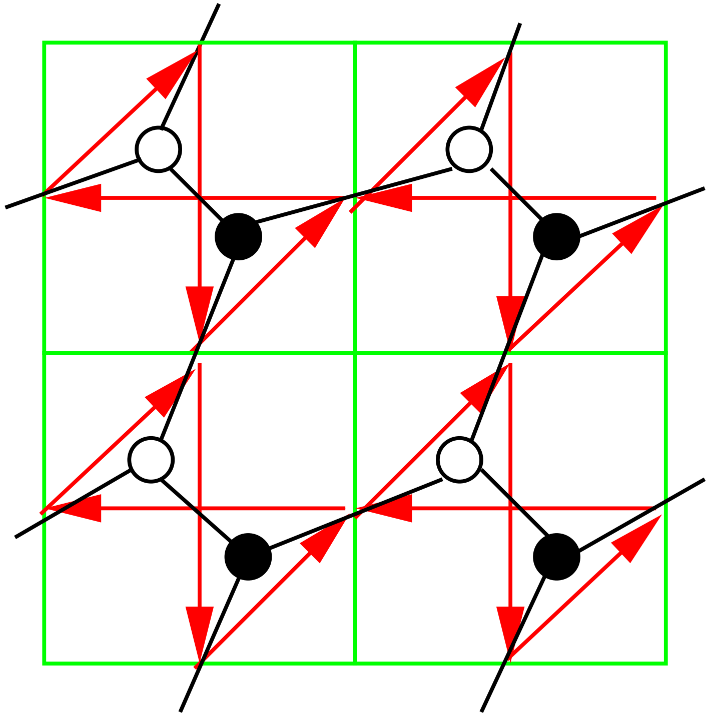
Fast-Forward Algorithm

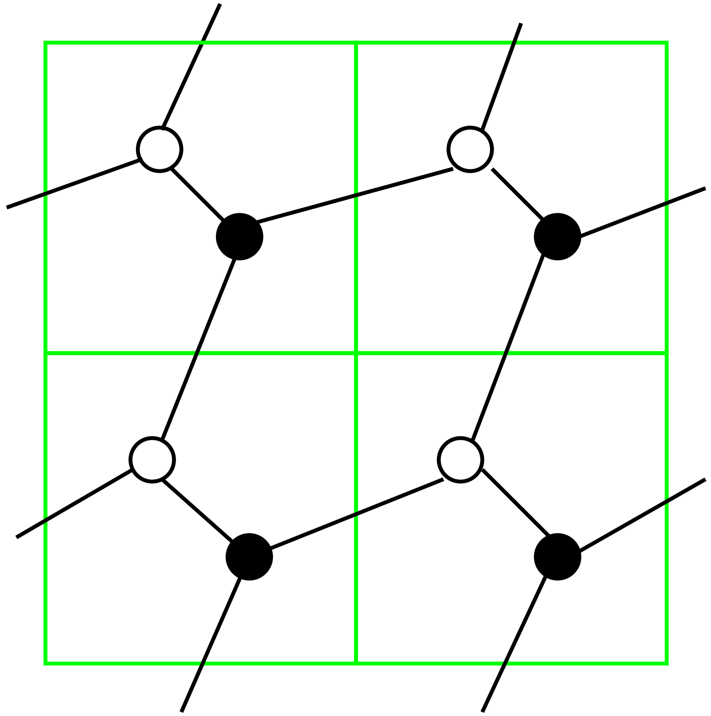
1. Obtain the normals to toric diagram.
2. For each normal (p, q), draw a path on torus whose homology class is $p\alpha + q\beta$, where α and β are basis of $H^1(T^2, \mathbb{Z})$.
3. Construct brane tiling. For each face, we place a black(white) vertex if the edges around the face all points clockwise(counterclockwise) direction. Connect two vertices by an edge for each intersection point of the corresponding two faces.

Example: $\mathbb{C}^3$

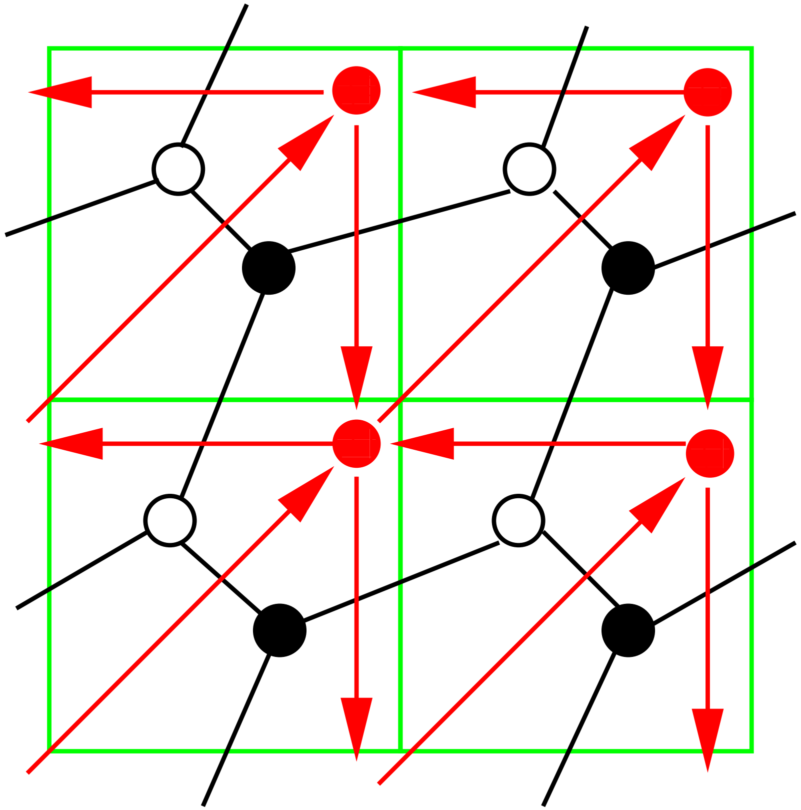


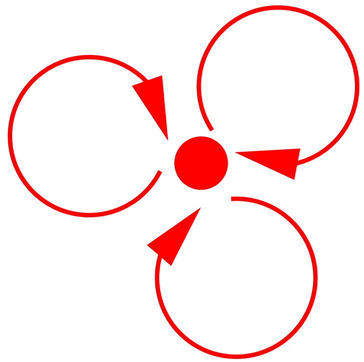
Example: $\mathbb{C}^3$



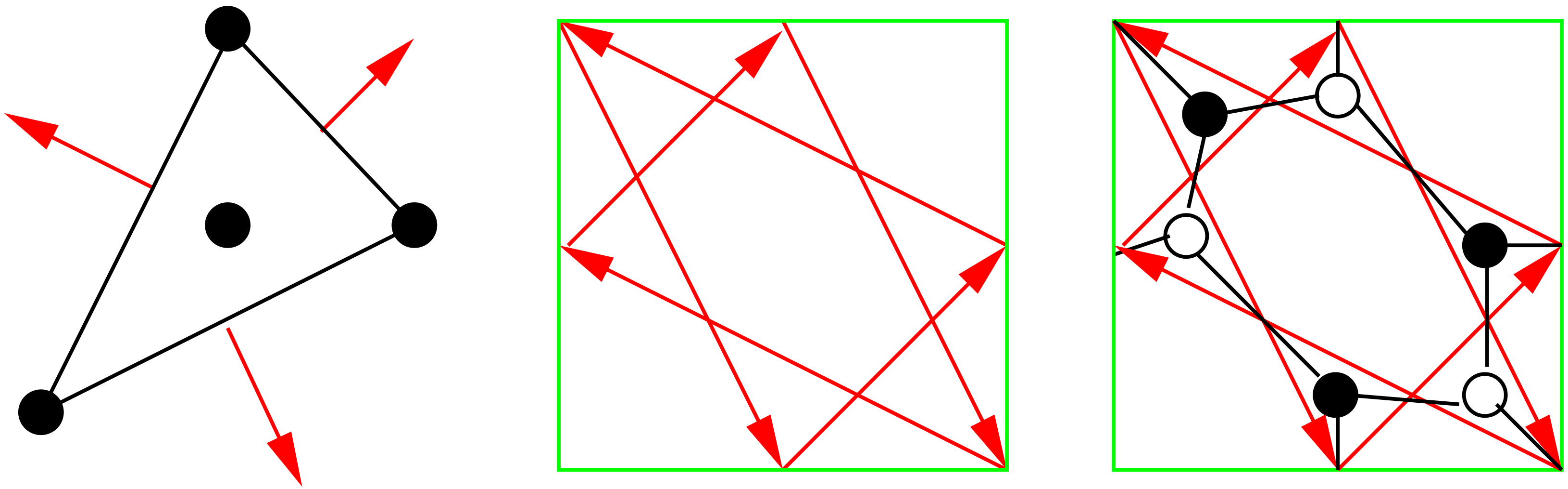


From dimer to quiver and superpotential

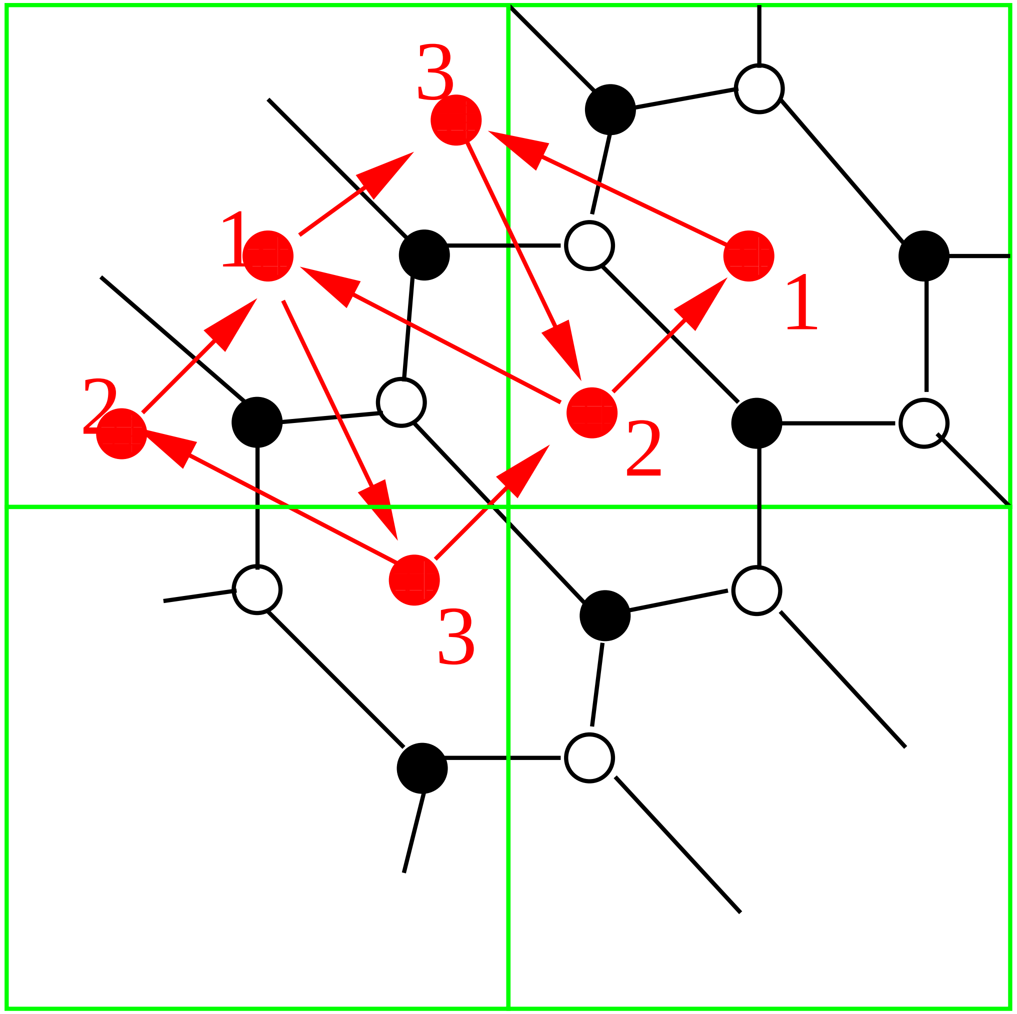


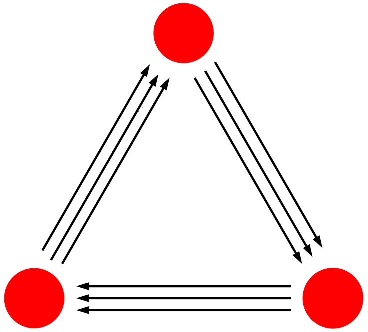


Example: $\mathbb{C}^3/\mathbb{Z}_3$



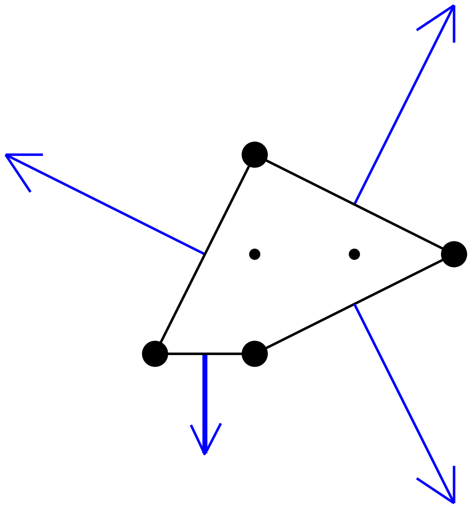
From dimer to quiver and superpotential



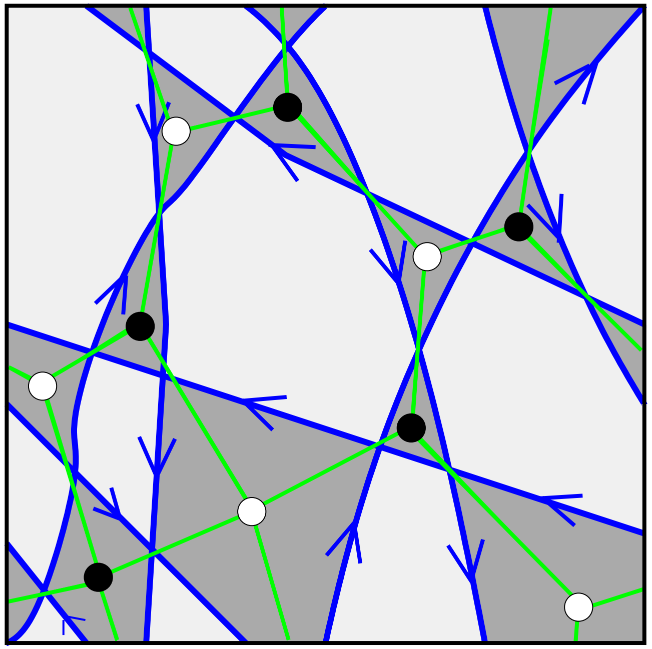


Example: $L^{1,5,2}$

We have now explicit metric on infinity family of toric varieties, which are known as $L^{a,b,c}$.

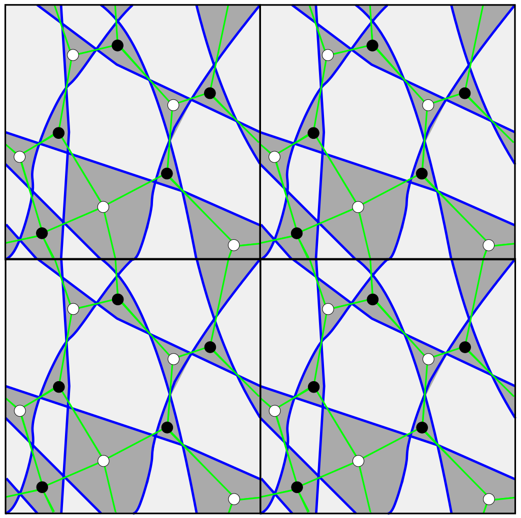


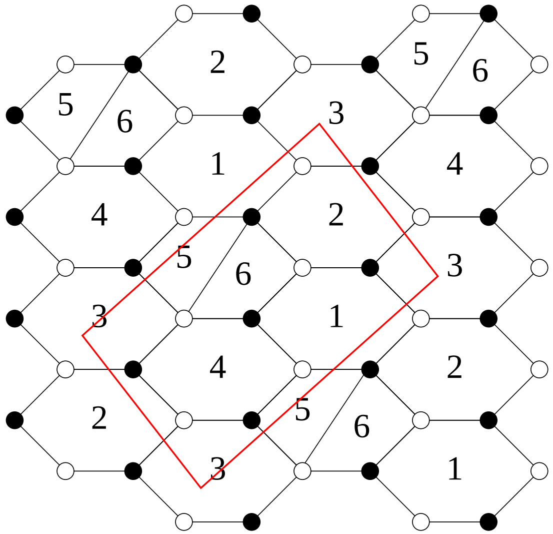
toric diagram

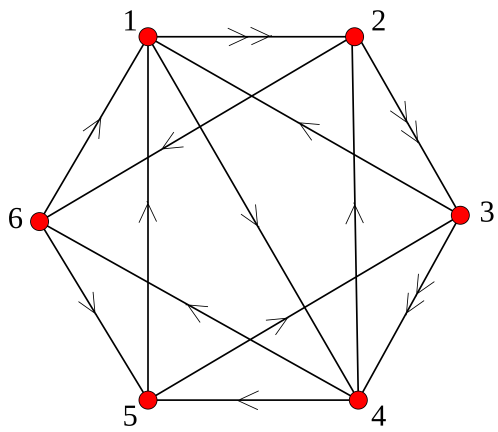


brane tiling

Example: $L^{1,5,2}$ (2)



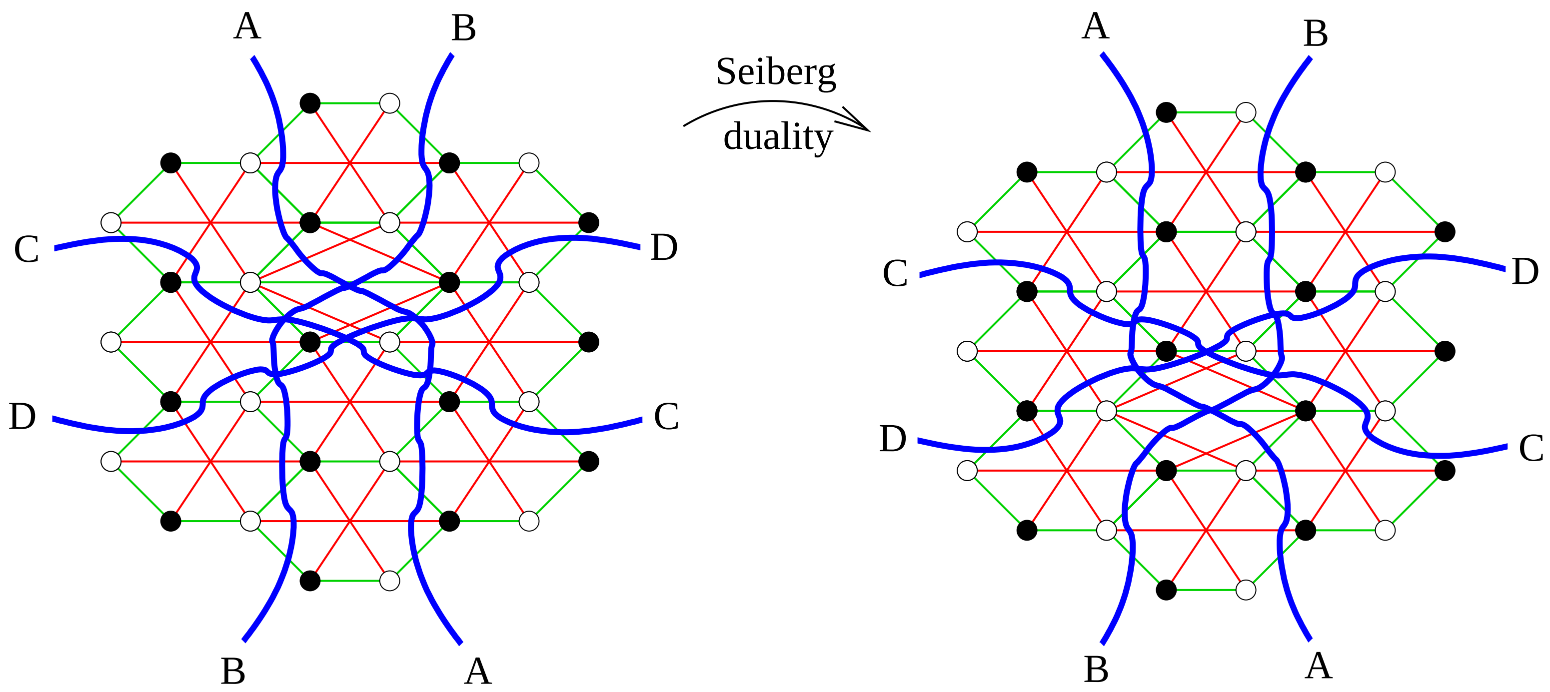




Seiberg duality in Hanany-Vegh

We have ambiguities in drawing paths on torus in Fast-Forward Algorithm (Yang-Baxter type move)

Conjecture: This ambiguity is nothing but Seiberg duality.



Mathematical Proof of the Algorithm

Δ: toric diagram (triangle or parallelogram, that is, the case of abelian orbifold of $\mathbb{C}^3$ or $\mathbb{P}^1 \times \mathbb{P}^1$)   In this case

- It is possible to give a precise mathematical formulation of Fast-Foward algorithm.
- The correctness of Fast-Forward Algorithm can be proven rigorously [Ueda-Y, math.AG/0605780, 0606548]
- In the triangle case ($\mathbb{C}^3/A$, A: abelian subgroup of $SL_3(\mathbb{C})$), the quiver obtained from Hanany-Vegh algorithm coincides with the McKay quiver of A.

Mathematical Proof of the Algorithm

Δ: toric diagram (triangle or parallelogram, that is, the case of abelian orbifold of $\mathbb{C}^3$ or $\mathbb{P}^1 \times \mathbb{P}^1$)

- X: singular Calabi-Yau obtained from Δ
- $Y \to X$: crepant resolution of X ($K_Y = f^*(K_X)$, $K_{X,Y}$: canonical sheaf)
- $\mathcal{P}/\mathcal{I}$: path algebra of quiver with relations obtained by Hanany-Vegh algorithm

Then

$$D^b \,\mathbf{coh}\, Y \cong D^b \,\mathbf{mod}\, \mathcal{P}/\mathcal{I}$$

Mathematical Proof of the Algorithm

$\mathbf{\Delta}$: toric diagram (triangle or parallelogram, that is, the case of abelian orbifold of $\mathbb{C}^{\mathbf{3}}$ or $\mathbb{P}^{\mathbf{1}} \times \mathbb{P}^{\mathbf{1}}$)   Then

$$D^b \, \mathbf{coh} \, Y \cong D^b \, \mathbf{mod} \, \mathcal{P}/\mathcal{I}$$

Physical interpretation of this fact is:

$$\mathcal{M} = X(\text{Calabi-Yau})$$

where $\mathcal{M}$ is the vacuum moduli space (D-term condition + F-term condition) of quiver gauge theory obtain from Fast-Forward algorithm. Recall $\underbrace{\mathbb{R}^{\mathbf{4}}}_{\text{D3-brane}} \times \underbrace{C(X_{\mathbf{5}})}_{\text{CY}}$

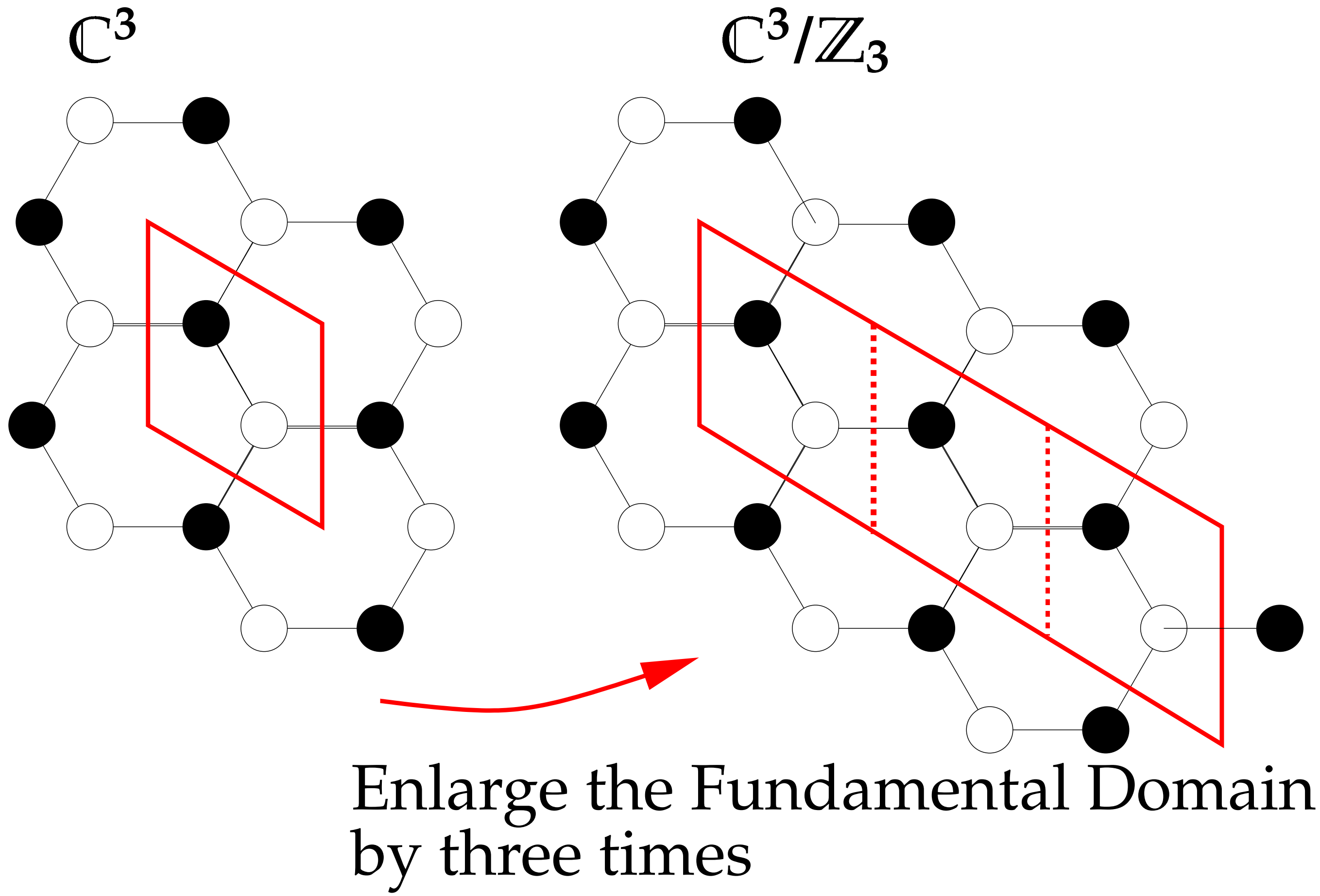
Advantage of brane tiling

Bonus of brane tiling: it is possible to give simple graphical representations to various operations:

- orbifold
- Seiberg duality
- integrating out massive fields

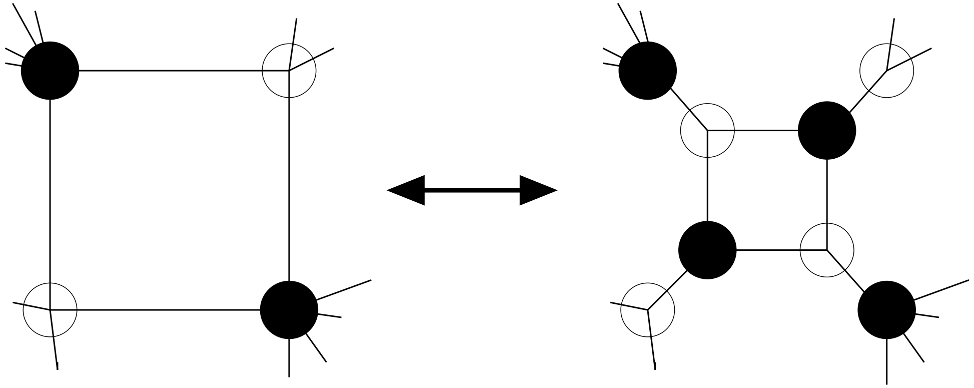
Orbifolding

Orbifolding by $\mathbb{Z}_n$ simply corresponds to enlarging the fundemental domain by n. Example: $\mathbb{C}^3/\mathbb{Z}_3$

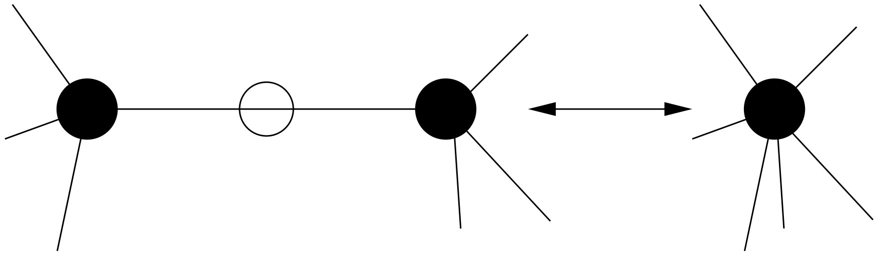


Seiberg Duality

Seiberg duality has simple graphical representation:

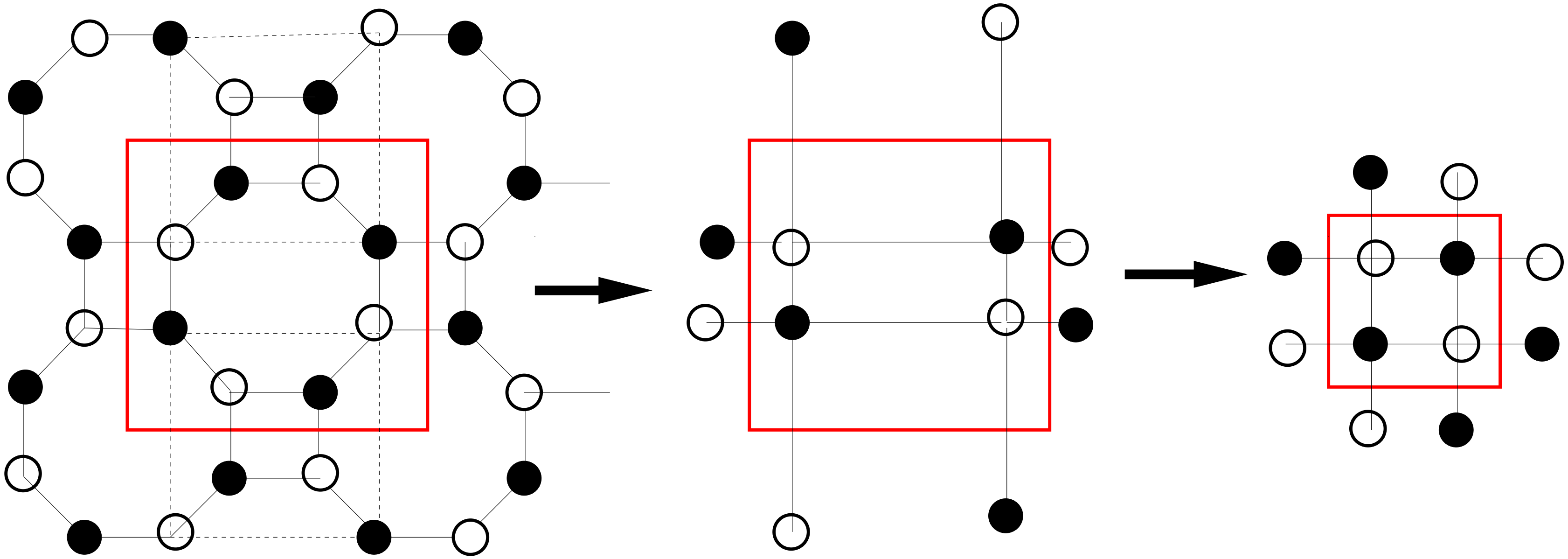


Sometimes you can integrate out massive fields after Seiberg duality.



Example of Seiberg Duality: F0 case

Hirzebruch 0 has two phases



Toric Duality is Seiberg Duality?

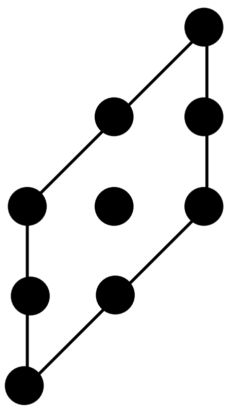
In general, the correspondence with Quiver and Fan is many-to-one. That is, several different quivers correspond to the same fan. This phenomena is dubbed "toric duality" [Beasley-Plesser].

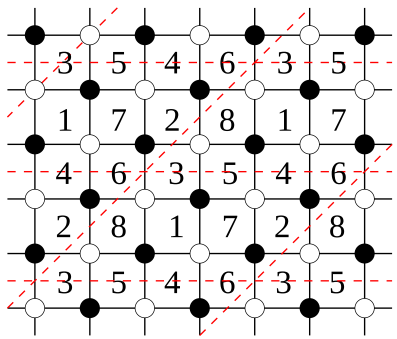
Conjecture

Toric Duality=Seiberg duality

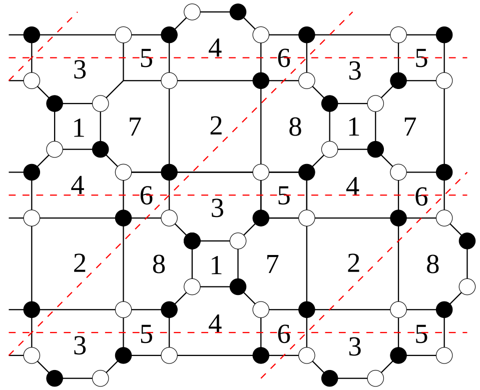
Example: Pseudo del Pezzo 5

There are four phases for PdP5, each connected by chain of Seiberg dualities.

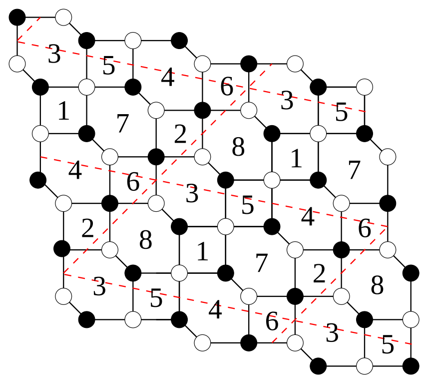




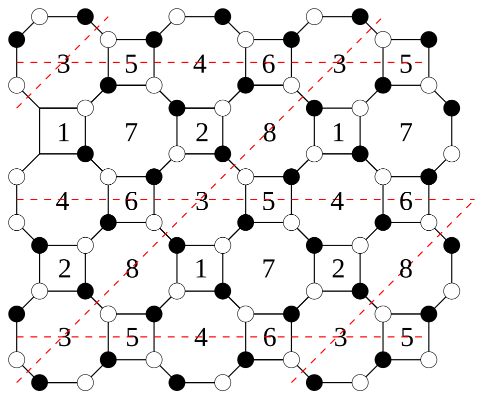
Model I



Model II



Model III



Model IV

Plan of This Talk

1. Introduction
2. Origin of Brane Tilings
3. Brane Tiling Algorithm
   - Brane Tiling $\longrightarrow$ Quiver and Superpotential
   - Brane Tiling $\longrightarrow$ Toric Diagram
   - Toric Diagram $\longrightarrow$ Brane Tiling
4. Homological Mirror Symmetry from Brane Tilings and Algae
5. Summary, Outlook and Speculations

Mirror Symmetry

Mirror Symmetry:

$$\text{B-model on CY } \mathcal{X} = \text{A-model on CY } \mathcal{W}$$

How to give a mathematical formulation of mirror symmetry?

Homological Mirror Symmetry

Homological Mirror Symmetry [Kontsevich]:

$$\underbrace{D^b \operatorname{coh} Y}_{B-model} \cong \underbrace{D^b \mathfrak{Fuk}^{\rightarrow} W}_{A-model},$$

$D^b \operatorname{coh} Y$: (derived category of coherent sheaves) B-brane
$\mathfrak{Fuk}^{\rightarrow} W$: (directed) Fukaya category A-brane:
Proven for T^2, projective plane, weighted projective plane, toric del Pezzo etc.

- It is known that B-model side can be computed from exceptional collctions (Bondal's theorem)
- A-model side is much more difficult ot compute

Homological Mirror Symmetry from Algae

Brane Tilings and Algae are useful for computing Fukaya category, and thus proving Homological Mirror Symmetry! [Ueda-Y, math.AG/0606548], e.g.

$$D^b \operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1 \cong D^b \mathfrak{Fuk}^{\to} W_1,$$

with $W_1(x, y) = x - \frac{1}{x} + y + \frac{1}{y}$

This method makes it easy to prove the orbifolds (thus, ∞ # of sequences) of $\mathbb{P}^1 \times \mathbb{P}^1$, which is a new result in mathematics.

A-model side: Mirror for Toric CY

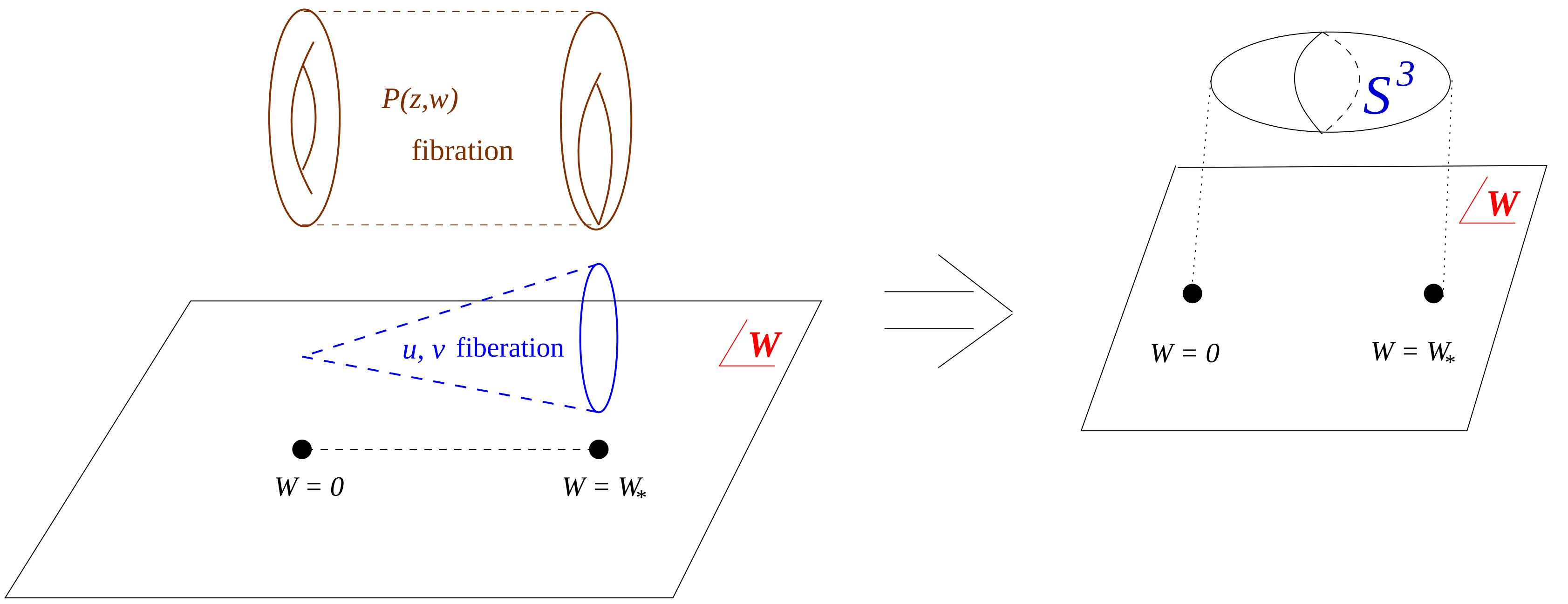
Mirror geometry $\mathcal{W}$ for toric CY 3-fold $\mathcal{M}$ is known (Hori-Vafa, hep-th/0002222), and is a double fibration over W-brane:

$$
\begin{aligned}
W &= P(z,w) = \sum_{(p,q)\in Q} c_{(p,q)} z^p w^q \\
W &= uv
\end{aligned}
$$

where $w, z \in \mathbb{C}^\times$ and $u, v \in \mathbb{C}$.

Vanishing Cycles

Mirror fiber degenerates at critical points of W and at $W = 0$ and we have "vanishing cycles" ($\cong S^3$) which span the basis of $H^3(\mathcal{W};\mathbb{Z})$:



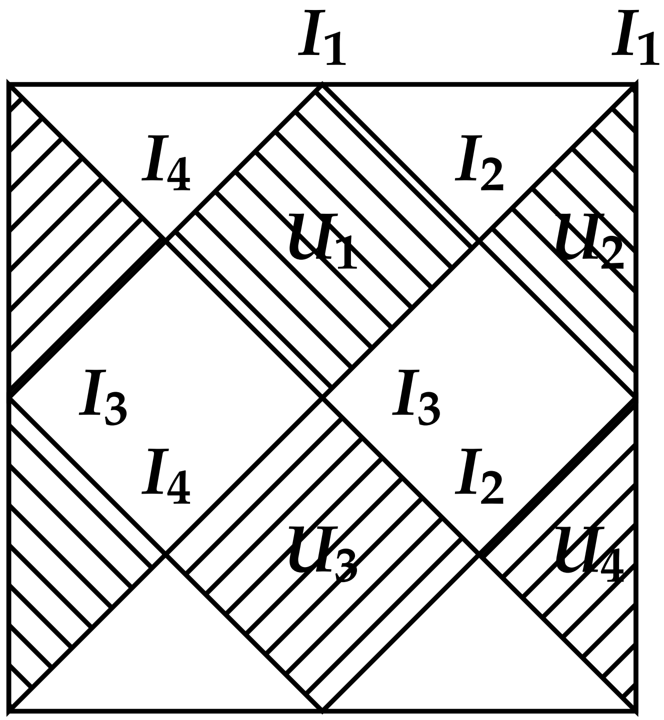
D-branes and (directed) Fukaya Cagetory

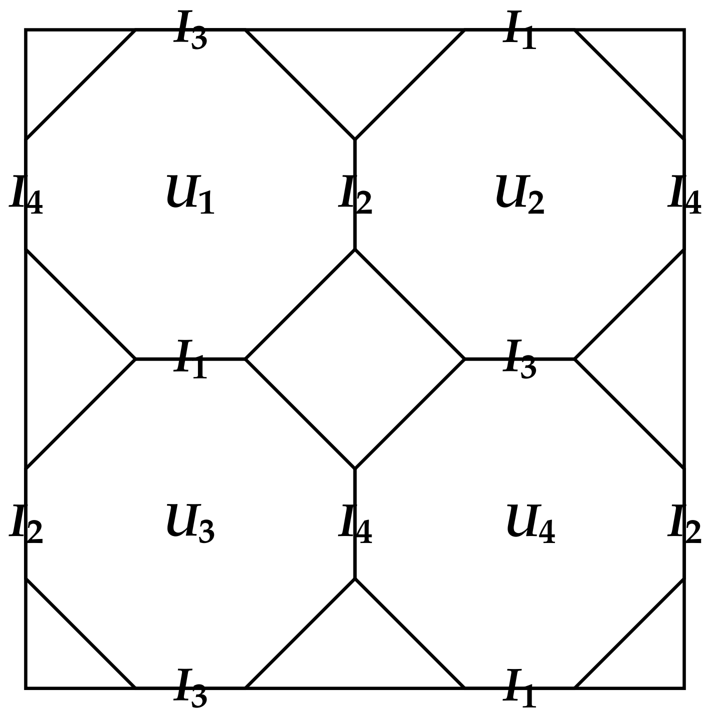
Objects vanishing cycles $\{C_i\}_{i=1}^{N}$ (D-branes)

Morphism intersection point of C_i and C_j (open strings)

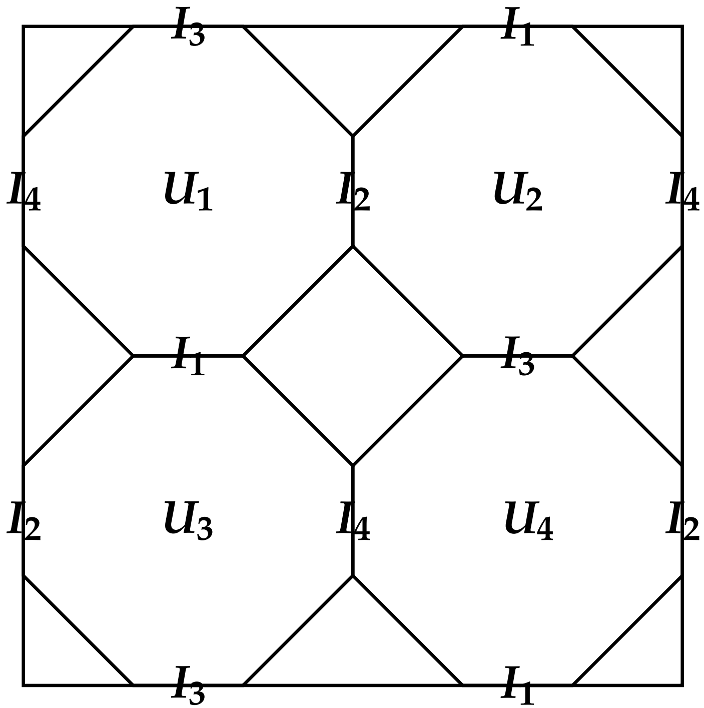
Composition of morphism (interaction of open strings, term in the superpotential, disk amplitude in holomorphic Chern-Simons)

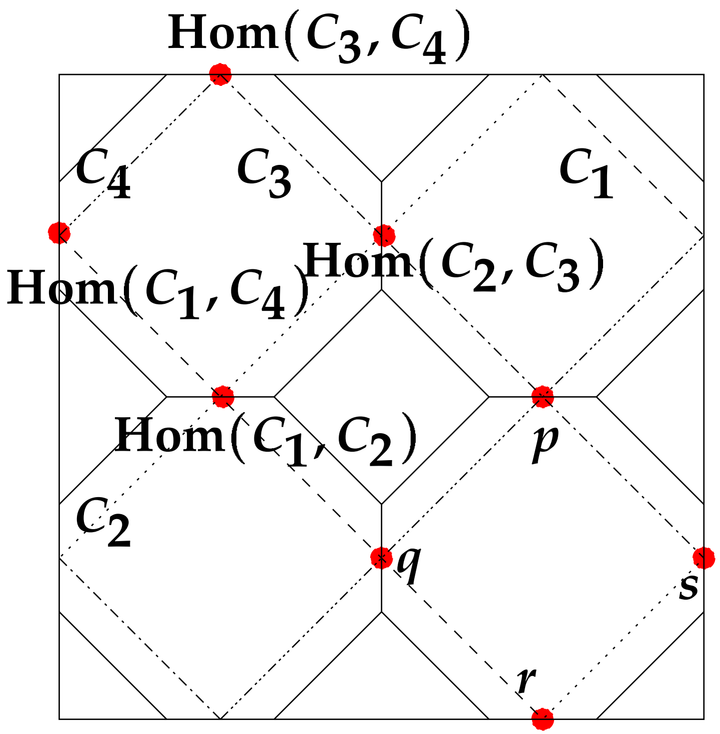
Fukaya Cagetory from Alga





Fukaya Cagetory from Alga





$$m_3(p, q, r) = s$$

Plan of This Talk

1. Introduction
2. Origin of Brane Tilings
3. Brane Tiling Algorithm
   - Brane Tiling $\longrightarrow$ Quiver and Superpotential
   - Brane Tiling $\longrightarrow$ Toric Diagram
   - Toric Diagram $\longrightarrow$ Brane Tiling
4. Homological Mirror Symmetry from Brane Tilings and Algae
5. Summary, Outlook and Speculations

Summary

- T-dualizing D3-brane probing singular Calabi-Yau, we have NS5/D5-brane system. Projection of NS5-brane gives alga, whose "tropical-like" reduction gives brane tiling, a bipartite graph on a torus.
- Brane tiling gives concise combinatorial methods to translate quiver $\leftrightarrow$ toric diagram
- Brane tiling algorithm is verified in numerous examples, and in particular successfully applied to AdS/CFT.
- Mathematical formulation and proof of the algorithm partially given
- Taking further T-dual, we have mirror D6-branes, and we can use brane tiling and alga for concise way of proving homological mirror symmetry, including the orbifold case.

Further developments

- Inclusion of fractional branes, cascading
  [Franco-Hanany-Saad-Uranga...]
- Proof of equivalence with older algorithms
  [Franco-Vegh]
- Relation with exceptional collections
  [Hanany-Herzog-Vegh]
- Effect of partial resolution and complex deformations, deformation to non-toric case
  [Garcia-Etxebarria-Saad-Uranga, Butti-Forcella-Zaffaroni]

$\vdots$

Further developments

- Inclusion of D7-branes (flavor brane)
- Dynamical SUSY breaking [Franco-Uranga]
- Gauge mediation [Garcia-Etxebarria-Saad-Uranga]
- Interpretation of anomalies [Imamura]
- Extension to non-toric case [Butti-Forcella-Zaffaroni]
- Counting of single-trace and multi-trace BPS operators [Benvenuti et. al.]

$\vdots$

Suggested Basic References

Basically Hanany and his students/collaborators + some other people (e.g. Butti, Zaffaroni)

- Franco et al, hep-th/0504110, "Brane Dimers and Quiver Gauge Theories"
- Franco et al, hep-th/0505211, "Gauge theories from toric geometry and brane tilings"
- Hanany-Vegh, hep-th/0511063, "Quivers, tilings, branes and rhombi."
- Feng-He-Kennaway-Vafa, hep-th/0511287, "Dimer models from mirror symmetry and quivering amoebae"
- Imamura, to appear

Some of the figures in this talk are taken from these papers.

Outlook and Speculations

- Mathematical formulation and proof, application to homological mirror: more general cases (e.g. toric del Pezzo, work in progress with K. Ueda)
- Intuitive understanding of other brane tiling algorithms from D-brane perspective?
- Further clarification with derived category (e.g. Aspinwall-Katz). Relation with tachyon condensation?
- Relation of various Seiberg dualities (mutation, tilting, flip/flop, Picard-Lefshetz, YB-type move)? Is it possible to obtain all phases of quiver? Really Seiberg duality?
- Toric Dual=Seiberg Dual? Enumeration of all phases? non-toric phase?

Outlook and Speculations

- More direct relation with BPS state counting as in topological vertex, instanton couting in SYM (cf. amoeba, tropical geometry?)
- Unified Perspective for Amoeba and Alga? "Complexified Amoeba"?
- Higher dimensional generalization? Application to $AdS_3 \times X_7$ (X_7: 7-dim. 3-Sasaki)
- Phenomenological Applications: Dynamical SUSY breaking, Gauge/Anomaly mediation etc.

Outlook and Speculations

- Moduli space of $\mathcal{N} = \mathbf{1}$ gauge quiver gauge theory. Beyond brane tiling: real shape of branes? (work in progress with Y.Imamura, H. Isono, and K. Kimura) Seiberg duality? beta function?