Exactly marginal deformation of quiver gauge theories as seen from brane tilings

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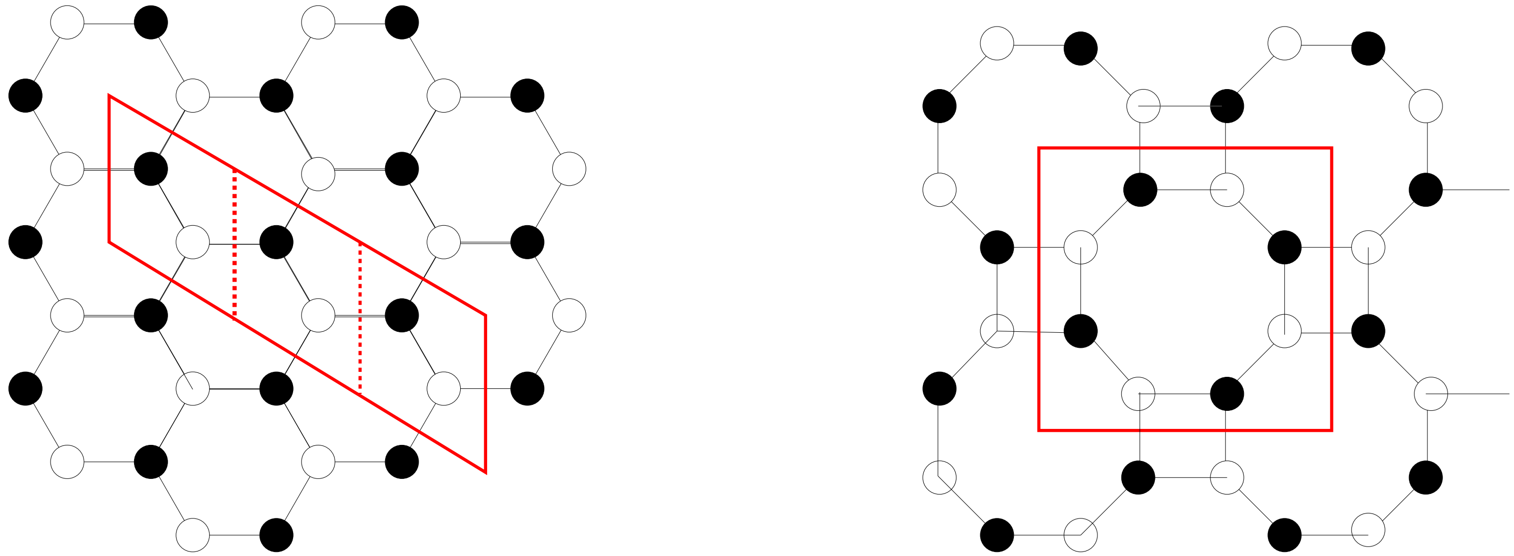
Based on hep-th/0702049

(w/ Y. Imamura, H. Isono and K. Kimura)

Brane Tilings

- "Brane Tiling" (2005∼, by Hanany et. al.):
  A bipartite graph (dimer) on $\mathbb{T}^2$ (+ extra conditions).



- Brane tilings are powerful techniques to study 4d $\mathcal{N} = 1$ superconformal quiver gauge theories.

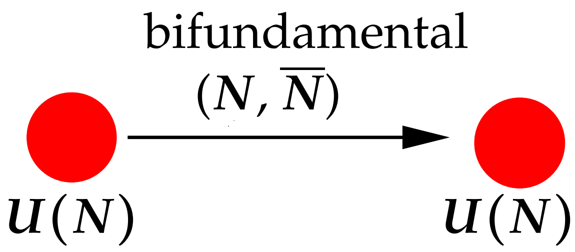
Quiver gauge theories

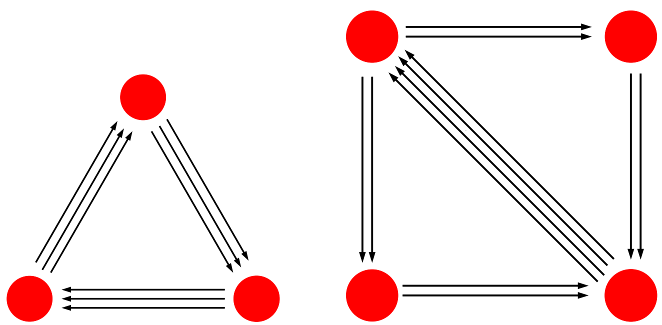
Quiver(箙): "portable case for holding arrows", an oriented graph

- Quivers are useful graphs to remember complicated gauge theories.

We consider 4d gauge theories.

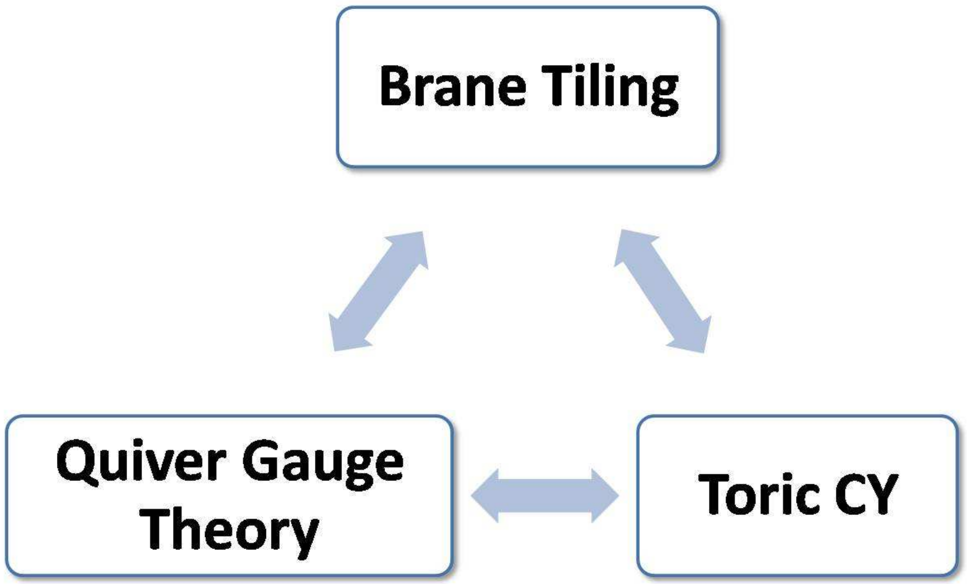
- vertex=gauge group (rank all equal to N)
- arrow=bifundamental





Brane tilings are useful to study quivers

- We can make superconformal quiver gauge theories from CY manifolds. Brane tiling tells you which quiver you have from which CY (and vice versa).



- Applied to AdS/CFT($\mathcal{N} = 1$), and had great success.

Brane tilings are really branes

- So far, brane tilings are simply *graphs*.
- BUT it was later clarified that brane tilings represent physical brane systems of D5-branes and NS5-brane (brane tilings are really branes!).
- On the worldvolume of D5-branes, we have *4d* $\mathcal{N} = \mathbf{1}$ *superconformal quiver gauge theories*.

Meaning of deformation of brane tilings?

- Since now brane tilings are not simply graphs, but really branes, d.o.f. of deformation of branes should have physical meaning.

Question: Physical meaning of deformation of branes in brane tilings?

Answer: exactly marginal deformation in quiver gauge theories

Contents

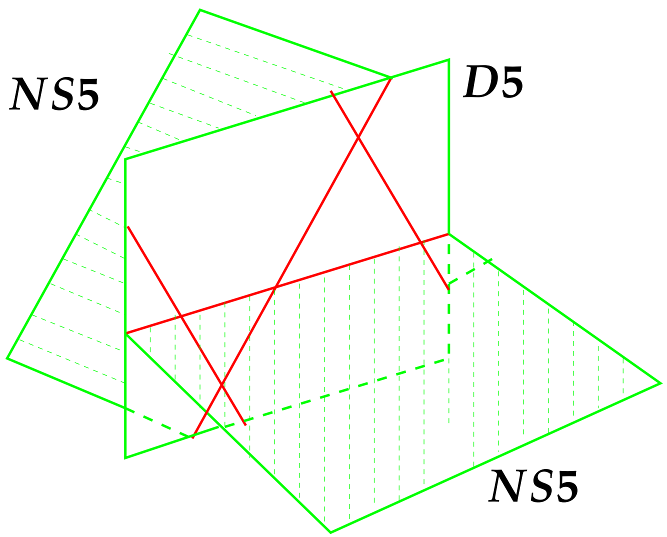
1. Introduction
2. Brane configuration for brane tilings
3. Quiver gauge theory side: marginal deformation
4. Brane tiling side : deformation of branes
    - Strong coupling analysis
    - Weak coupling analysis
5. Comparison of parameters
6. Summary and outlook

Brane configurations

- First we prepare N D5-branes. When N D5-branes coincide, we have $U(N)$ gauge theory.
- In order to obtain (3+1)-dim. theory, we compactify two directions of D5-brane on $\mathbb{T}^2$. Then we have 4d $U(N)$ SYM on D5-brane.
- We still want obtain multiple gauge groups!

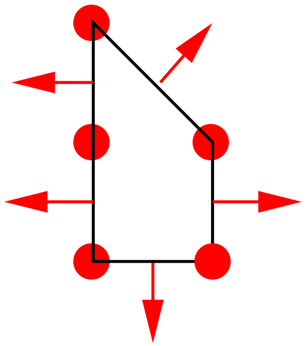
Brane configurations

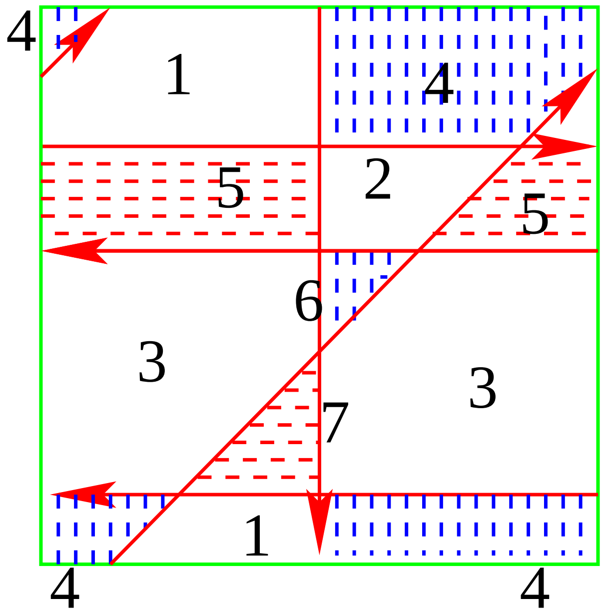
- Stack of N D5-branes are divided by NS5-branes into several regions, then SUSY is broken to $\mathcal{N} = 1$ and we have multiple gauge groups.



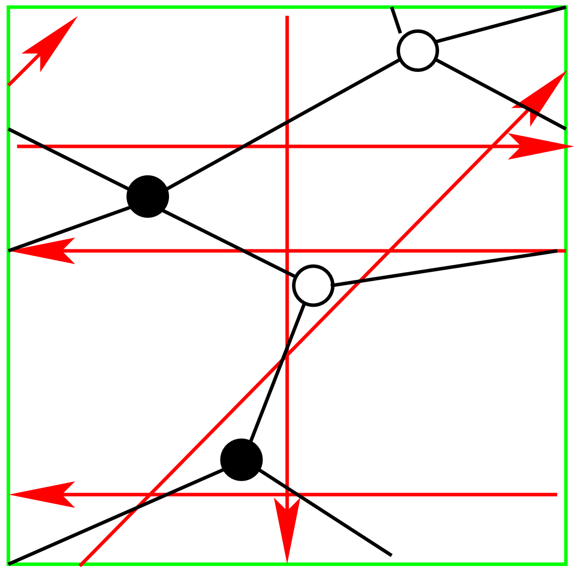
Brane configurations

- Due to conservation of NS-charge, D5-brane actually becomes (N, k)-branes. ($k = 1, 0, -1$ in this talk)





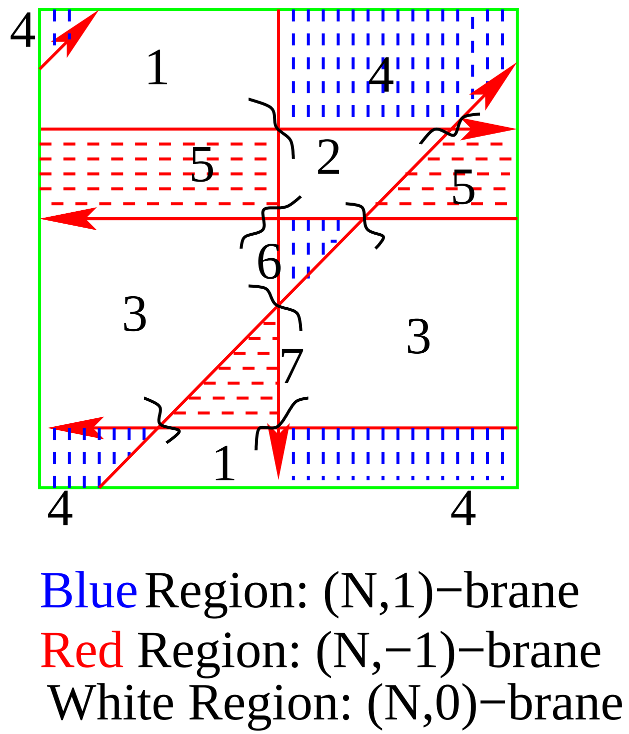
Blue Region: (N,1)−brane
Red Region: (N,−1)−brane
White Region: (N,0)−brane

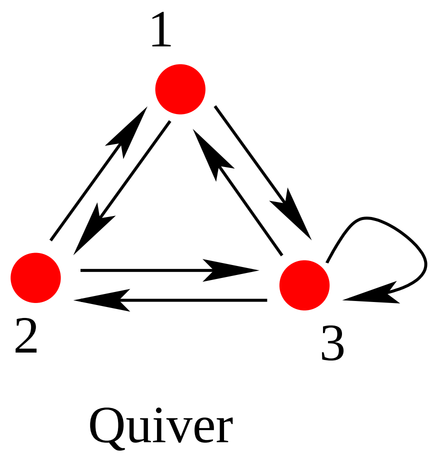


Dimer Model

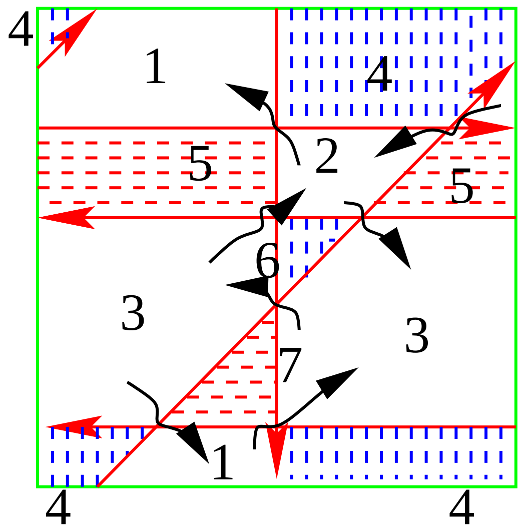
Brane configurations

- $U(N)$ gauge groups lives only on $(N,0)$-branes.
- We have a bifundamental for each intersection pt of $(N,0)$-branes.
- From this we can read off quiver!





Superpotentials



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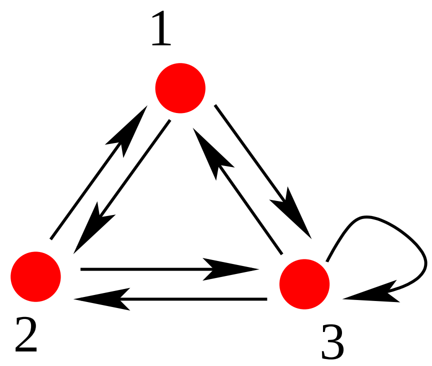

Quiver

$$W = \mathrm{tr} X_{23} X_{33} X_{32} - \mathrm{tr} X_{33} X_{31} X_{13} + \mathrm{tr} X_{13} X_{31} X_{12} X_{21} - \mathrm{tr} X_{23} X_{32} X_{21} X_{12}$$

Superpotentials

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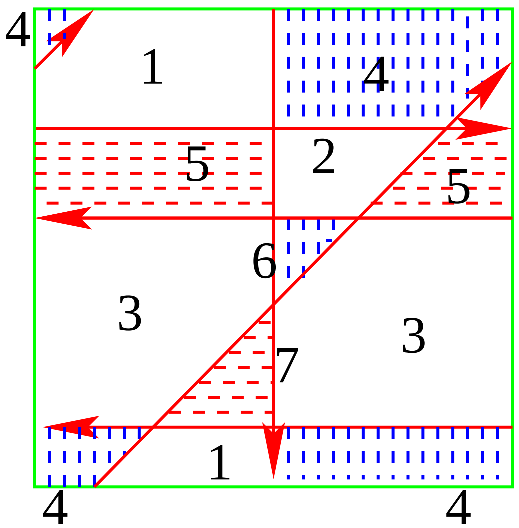
In general superpotential are

$$W = \sum_k \pm h_k \, \mathbf{tr} \prod_{I \in k} \Phi_I,$$

Here $I \in k$ means I is one of corners of the face k

Correspondence with bipartite graphs

Place black vertices on $(N,1)$-brane, white vertices on $(N,-1)$-brane. Connect vertices for each intersection point. Then we have a bipartite graph on $\mathbb{T}^2$.



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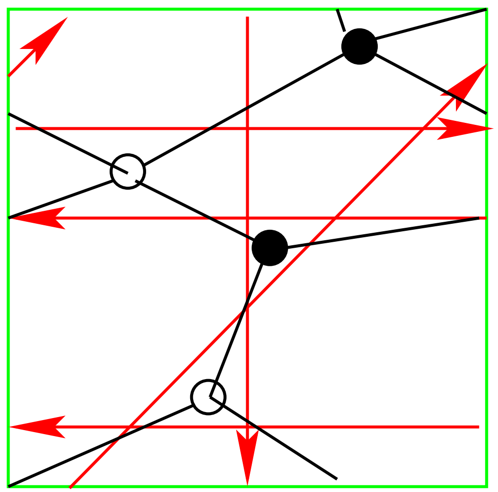


Dimer Model

Quiver (or its lift to $\mathbb{T}^2$) are dual to bipartite graphs.

D5/NS5-system

- Summarizing, the brane configuration is:

	0	1	2	3	4	5	6	7	8	9
D5	○	○	○	○		○		○		
NS5	○	○	○	○	Σ (2-dim surface)					

  - D5-brane worldvolume: $\mathbf{R}^4 \times T^2$
  - NS5-brane worldvolume: $\mathbf{R}^4 \times \Sigma$

  - Projection of NS5 to 5 and 7 directions: brane tiling (coamoeba)
  - Projection of NS5 to 4 and 6 directions: (p,q)-webs (amoeba)

Brane tilings are really branes!

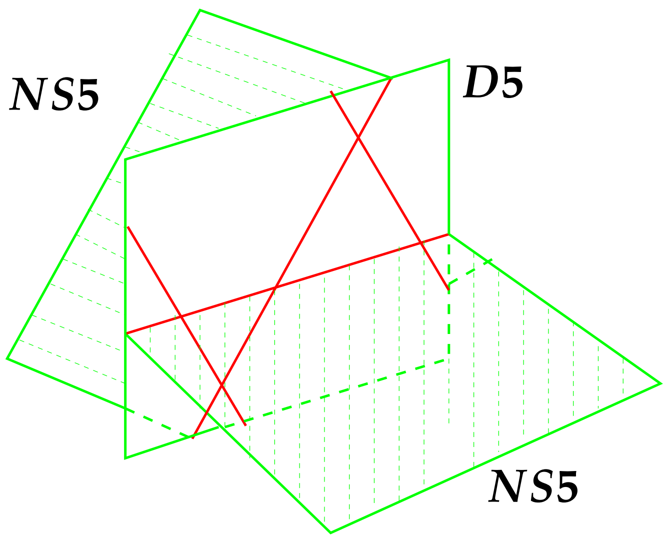
Important Point

Brane tilings represent (projection of) configurations of physical D5-branes and NS5-branes, which realizes quiver gauge theories in string theory.

This is important not only conceptually, but also for practial applications!

A note on string coupling

- Actually, we have so far talked about the limit $g_{str} \longrightarrow \infty$.
- Real shape of branes: difficult to determine in general (we need to solve EOM), but can be analyzed when $g_s \longrightarrow \mathbf{0}$ and $g_s \longrightarrow \infty$.
- When $g_s \longrightarrow \infty$, $T_{D5} \gg T_{NS5}$, thus D5-branes become flat and NS5-branes are orthogonal to D5-branes.



Weak coupling

Consider the weak coupling limit $g_s \longrightarrow 0$. Then

$$T_{NS5} \gg T_{D5}$$

Then NS5-brane worldvolume Σ is a holomolophic curve $W(x,y) = 0$ in $\left(\mathbb{C}^{\times}\right)^2$, where

- $x = \exp(x_4 + ix_5), y = \exp(x_6 + ix_7)$
- $W(x,y)$ is a Newton Polynomial of the toric diagram

$$W(x,y) = \sum_{(i,j)\in\Delta} c_{(i,j)} x^i y^j$$

where $\Delta \in \mathbb{Z}^2$ is the toric diagram.

Contents

Exactly Marginal Deformations

- Consider deformation of 4d $\mathcal{N} = 1$ superconformal quiver gauge theories preserving conformality.
- Parameters in theory: gauge coupling g_a and superpotential coupling h_k
- Conformal manifold : $\{\beta_a = \beta_k = 0\} \subset \{g_a, h_k\}$
- Usually only isolated solutions, but when we have SUSY, β-functions are not linearly independent and marginal deformation exists.

NSVZ β-function

- Gauge coupling g_a

$$\beta_a \equiv \mu \frac{d}{d\mu} \frac{1}{g_a^2} = \frac{N}{1 - g_a^2 N / 8\pi^2} \left[3 - \frac{1}{2} \sum_{I \in a} (1 - \gamma_I) \right].$$

(sum over fields coupled to $SU(N)_a$)

- Superpotential coupling h_k

$$\beta_k \equiv \mu \frac{d}{d\mu} h_k = -h_k \left[3 - \sum_{I \in k} \left(1 + \frac{1}{2} \gamma_I \right) \right].$$

NSVZ β-function

Rewrite g_a^2 by $g_a'^2$, which is given by

$$d\left(\frac{1}{g_a'^2}\right) = \left(1 - \frac{g_a^2 N}{8\pi^2}\right) d\left(\frac{1}{g_a^2}\right).$$

Then

$$\frac{1}{N}\beta'_a \equiv \mu\frac{d}{d\mu}\frac{1}{N g_a'^2} = 3 - \frac{1}{2}\sum_{I\in a}(1-\gamma_I).$$

$$-\frac{\beta_k}{h_k} \equiv \mu\frac{d}{d\mu}\log h_k = 3 - \sum_{I\in k}\left(1 + \frac{1}{2}\gamma_I\right).$$

RG-invariant combination

Since β-functions are not independent, we search for linear combination (coefficients S_a and S_k) which vanish.

$$\begin{aligned}\beta[S_A] &\equiv \sum_a S_a \frac{1}{N}\beta'_a - \sum_k S_k \frac{\beta_k}{h_k} \\ &= \sum_a S_a \left[3 - \frac{1}{2}\sum_{I\in a}(1-\gamma_I)\right] + \sum_k S_k \left[3 - \sum_{I\in k}\left(1+\frac{1}{2}\gamma_I\right)\right] \\ &= 0.\end{aligned}$$

This says $f^{(I)} = \sum_a S_a^{(I)} \frac{1}{N g_a'^2} - \sum_k S_k^{(I)} \log h_k$ is RG invariant and thus parametrize conformal manifold.

The number of parameters is $d-1$

Since we have linear equation in γ_I, we have

$$\sum_{a\in I} S_a = \sum_{k\in I} S_k.$$

In other words, we can assign number b_μ to each cycle μ such that

$$S_A - S_B = b_\mu.$$

Here S_A and S_B are adjacent to each other with cycle μ.

- Thus the number of parameters is $d-2+1 = d-1$ ($d-2$ chocies of b_μ, and extra one comes from overroll shift of S_A)

Special marginal deformations

We have following two special marginal deformations, irrespective of the details of bipartite graphs.

$$s_A^{(1)} = 1 \quad \forall A, \quad b_\mu^{(1)} = 0 \quad \forall \mu.$$

RG-inv. quantity、

$$f^{(1)} = \sum_a \frac{1}{N g_a'^2} - \sum_k \log h_k \sim 1/(N g_{\mathrm{diag}}^2)$$

which is roughly gauge coupling g_{diag} of diagonal $SU(N)$:
$f^{(1)} \sim 1/(N g_{\mathrm{diag}}^2)$.

β-like deformation

Another is

$$s_A^{(2)} = Q_{\mathrm{NS5}} \quad \text{for } (N, Q_{\mathrm{NS5}}) \text{ face } A, \quad b_\mu^{(2)} = 1 \quad \forall \mu.$$

RG-inv. quantity is

$$f^{(2)} = \sum_k \pm \log h_k.$$

Depends only on superpotential couplings, not on gauge couplings. Generalization of β-deformation in $\mathcal{N} = 4$ case

- These two marginal deformations will play important roles in later discussions.

Inclusion of θ-angle

So far: real parameter, but
superpotential couplings : complex gauge couplings:
complex parameter with θ-angles

$$\tau_a = \frac{\theta_a}{2\pi} + \frac{4\pi i}{g_a^2}.$$

Thus we have $d-1$ complex parameters.

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Analysis in the strong coupling limit

- D.o.f of deformation of brane tilings? we can change the position of cycles of NS5-brane in $\mathbb{T}^2$. Thus we have d(=# of NS5 cycles) parameters.
- But we actually have to subtract 2, since $\mathbb{T}^2$ is translationally invariant.
- We have to preserve $\mathcal{N} = 1$ SUSY Condition

$$\sum_{\text{faces}} (\text{NS5-charge}) \times (\text{Area}) = 0.$$

- We thus have $d - 2 - 1 = d - 3$ parameters.

Analysis in the weak coupling limit

$\mathcal{N} = 1$ SUSY should be preserved.
The SUSY preserved on each D5-brane is

$$\epsilon_2 = (Z/|Z|)\epsilon_1$$

Here Z is the central charge of D5-brane:

$$Z = \int_{\mathrm{D5}} d\log x \wedge d\log y = \oint_{\partial\mathrm{D5}} \log x \frac{dy}{y}$$

Here $\partial\mathrm{D5}$ is the 1-cycle, coming from the intersection with NS5-brane.
Since these should coincide,

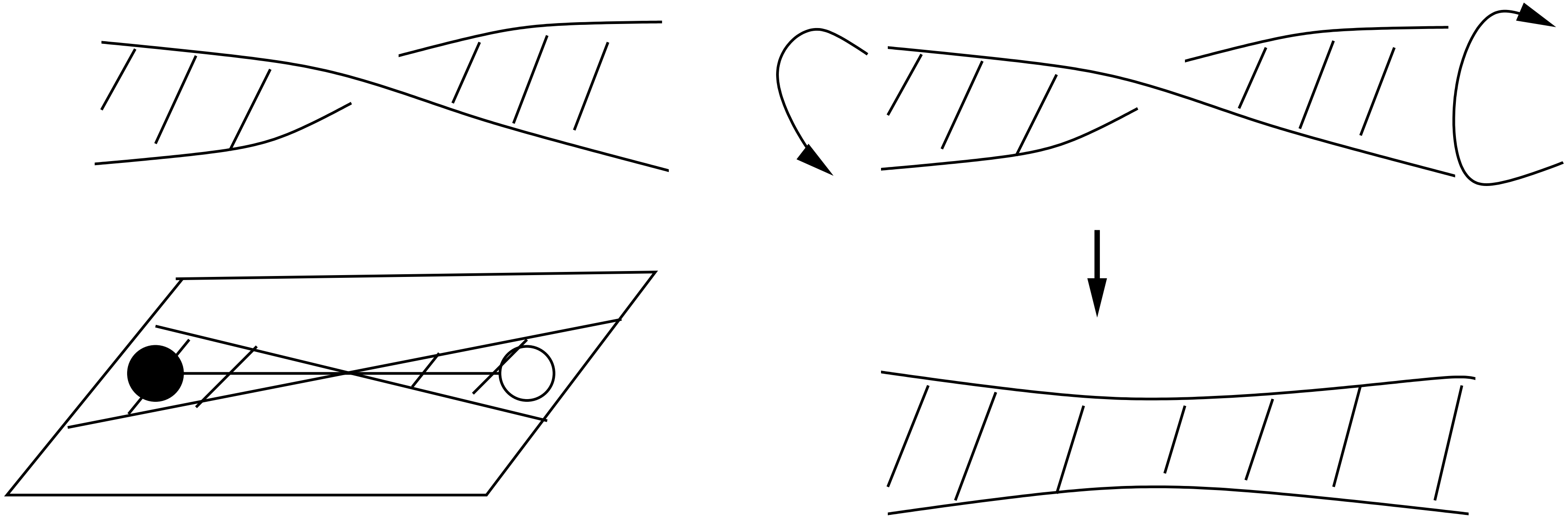
$$\arg Z_1 = \arg Z_2 = \cdots = \arg Z_{n_g}.$$

Counting number of parameters

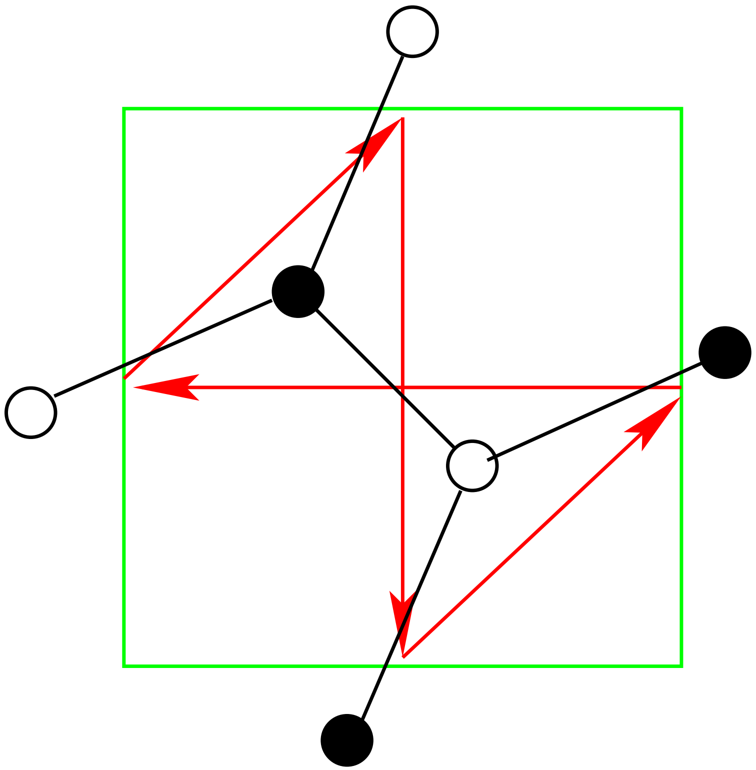
- First, 3 out of all the coefficients of Newton polynomial can be eliminated.
- Furthermore, BPS condition gives $2S-1$ relations since it is known that we have $2S$ gauge groups.
- Thus
$$2(I+d-3)-(2S-1)=d-3,$$
Here I is the number of internal lattice points of toric diagram and we have used Pick's theorem
$$S=I+\frac{d}{2}-1.$$

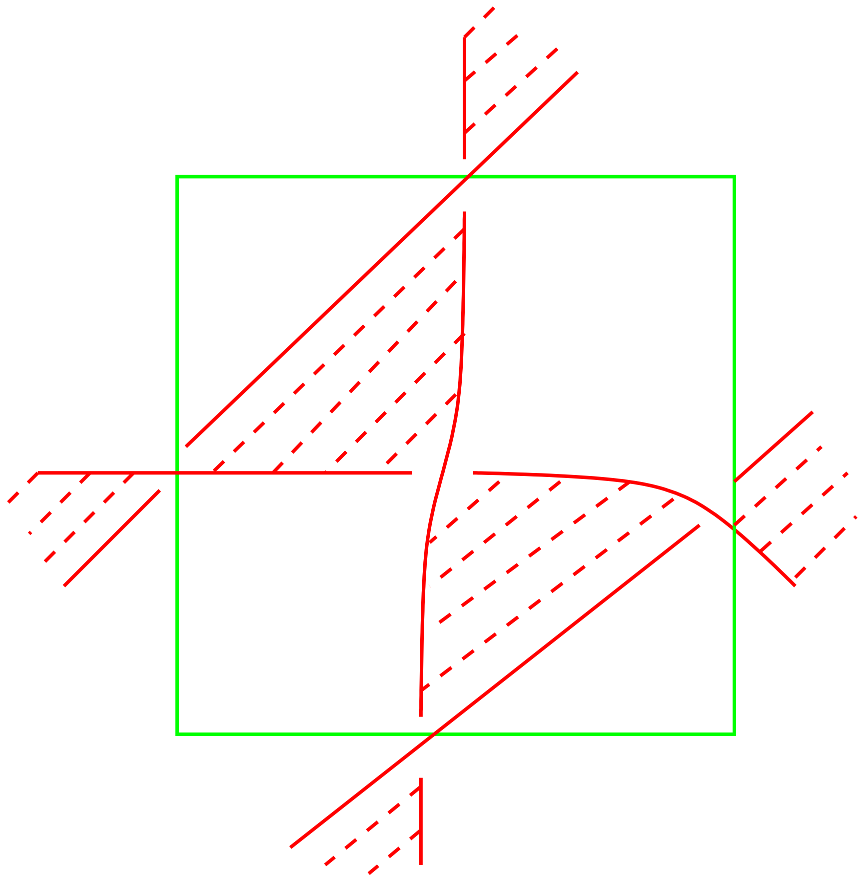
Untwist

If we want to know the shape of moduli, not simply dimension, we need to know the boundary of D5-brane on NS5.
Actually, we can read off that cycle from the figure of strong coupling. This is called *untwisting* [Feng-He-Kennaway-Vafa].



Example of untwist: $\mathbb{C}^3$





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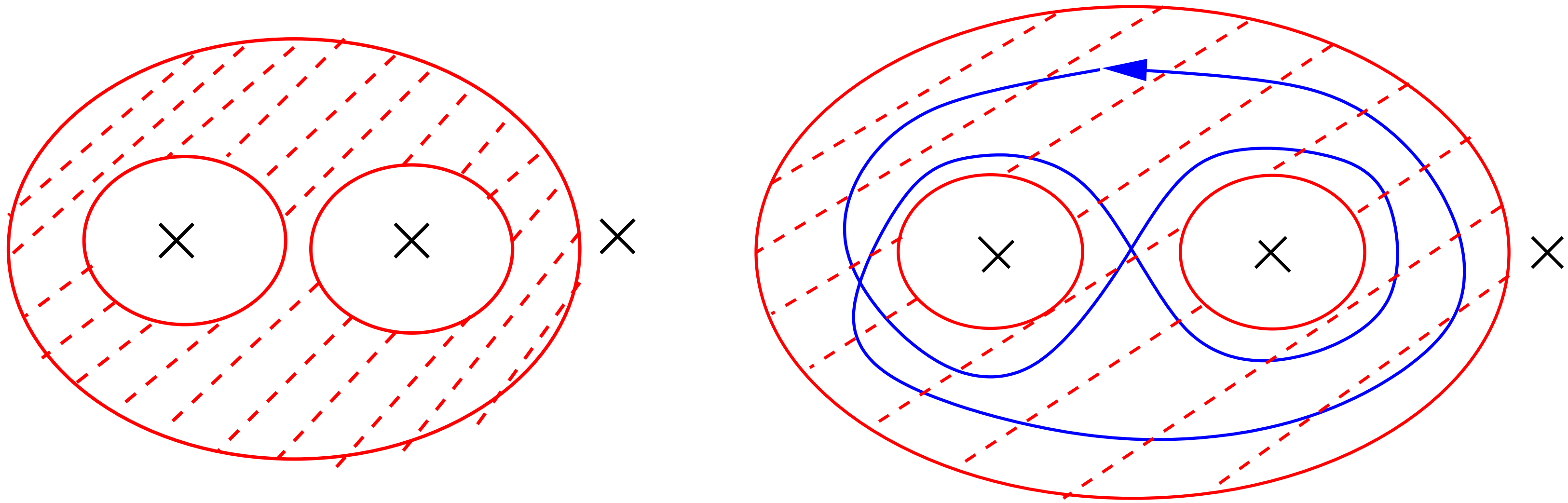
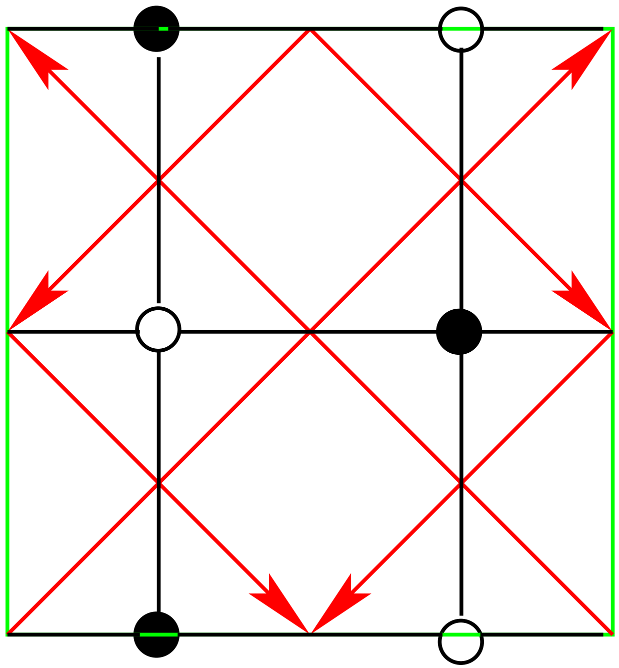
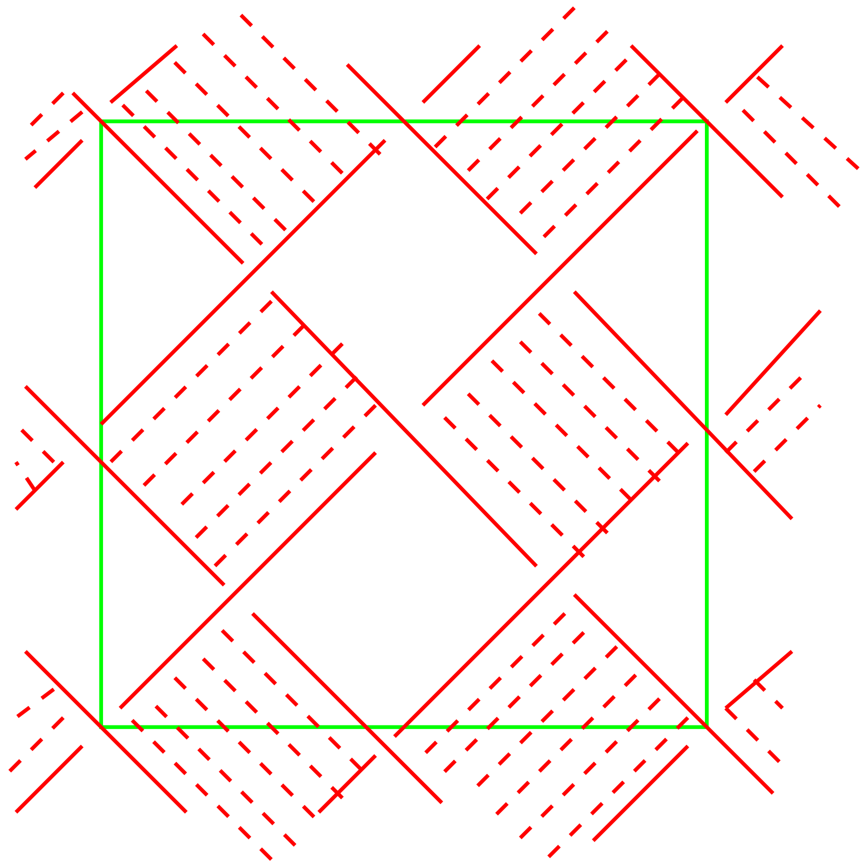


Example of untwist: $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$





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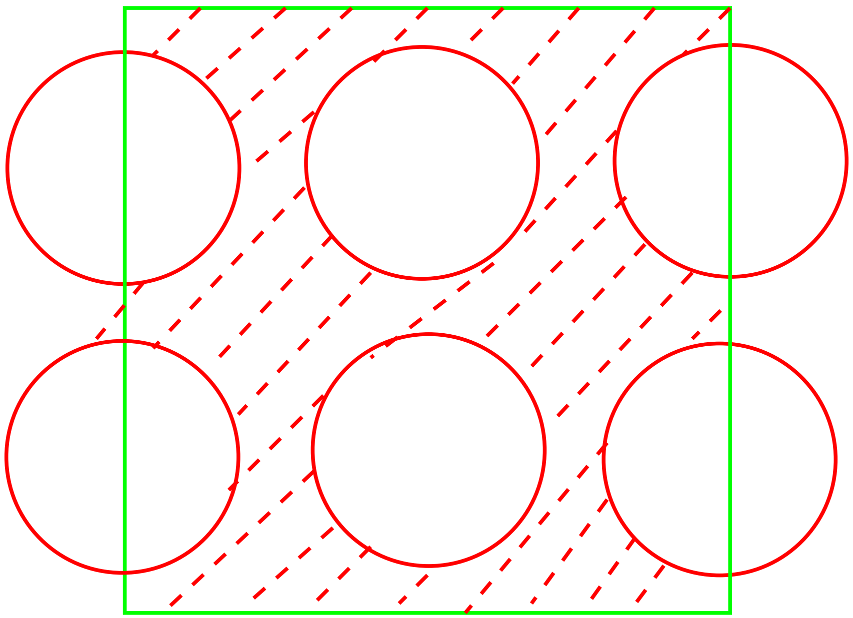
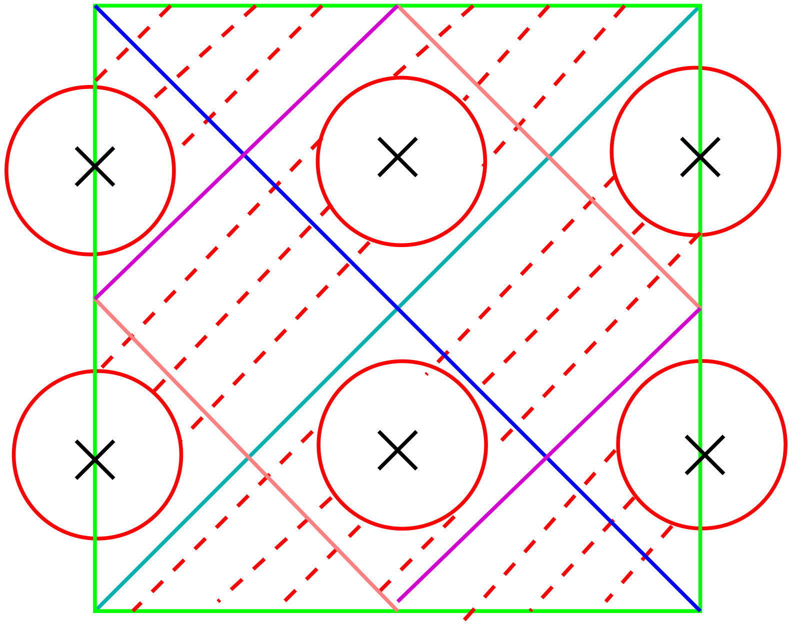


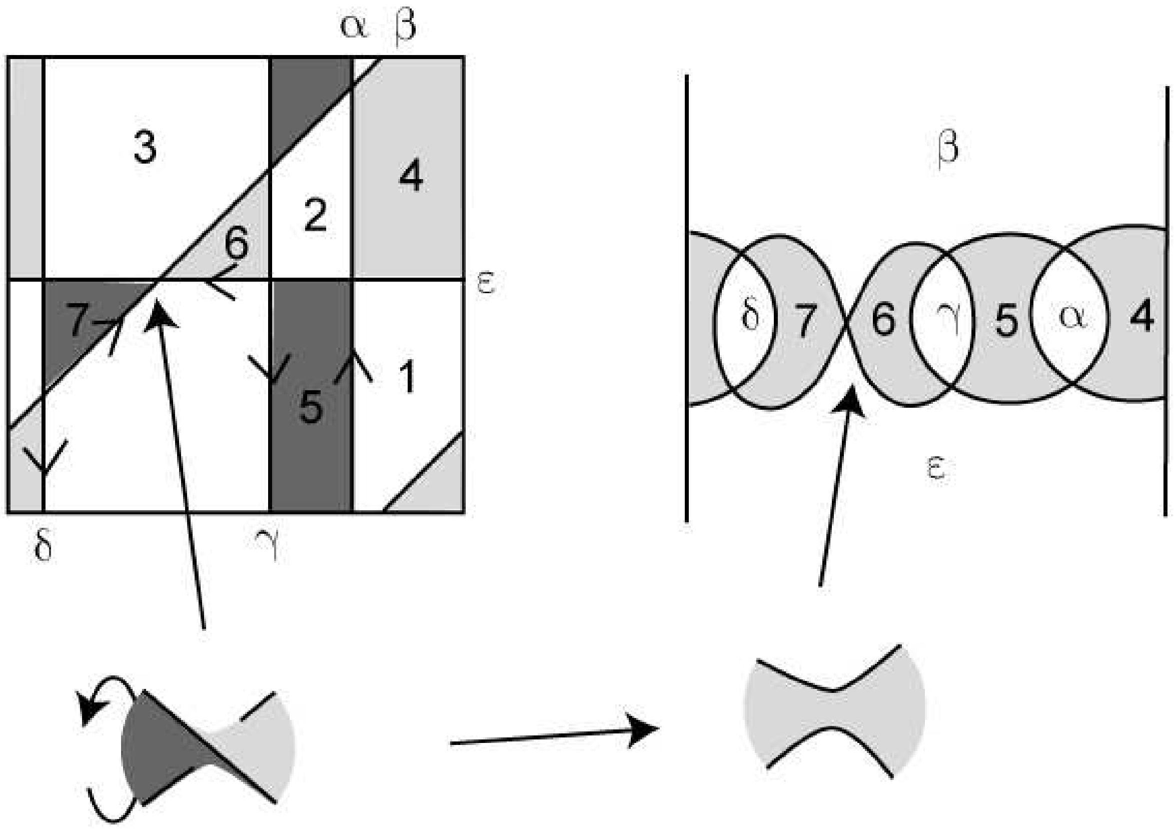


Result of Untwisting

- The surface Σ we obtain after twisting is the curve of NS5-brane (i.e. $W(x, y) = 0$). Therefore, by untwisting, we can go to weak coupling.
- Winding cycles of $\mathbb{T}^2$ are mapped to boundaries of Σ
- D5-branes (regions of $\mathbb{T}^2$) are mapped to winding cycles of Σ.

Example of untwist

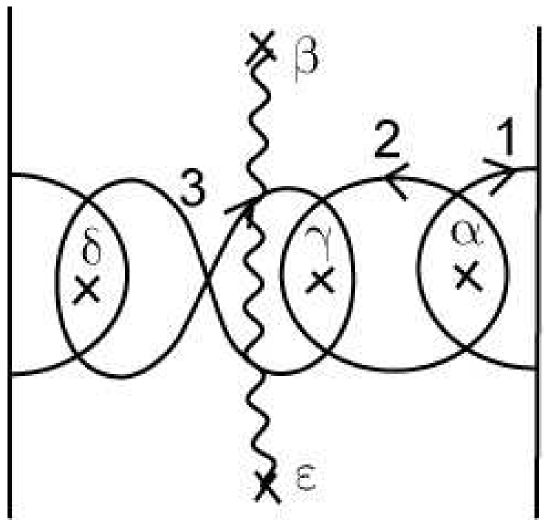
$$P(x, y) = y(x - x_\alpha) + (x - x_\gamma)(x - x_\delta), x_\alpha = -1$$



(a) (b)

Example of untwist

The cycles of D5 after untwist is



Example of untwist

central charge Z is

$$Z_1 = \oint_1 \log x \frac{dy}{y} \quad = \quad 2\pi i(\mathrm{Log}\, x_\alpha - \mathrm{Log}\, x_\delta),$$

$$Z_2 = \oint_2 \log x \frac{dy}{y} \quad = \quad 2\pi i(\mathrm{Log}\, x_\gamma - \mathrm{Log}\, x_\alpha),$$

$$Z_3 = \oint_3 \log x \frac{dy}{y} \quad = \quad 2\pi i(\mathrm{Log}\, x_\delta - \mathrm{Log}\, x_\gamma + 2\pi i),$$

and thus BPS condition is

$$1 = |x_\alpha| = |x_\delta| = |x_\gamma|,$$

which means moduli is $(S^1)^2$.

Complexification of parameters

So far, we have $d - 3$ parameters from deformation of brane tilings. These parameters should also be complexified, just as in gauge theory side.

- Answer: Wilson line of $U(1)$ on NS5 cycles

$$W_\mu = \oint_{\alpha_\mu} A.$$

  Checked by SUSY transformation of NS5 fields

- We thus have $d - 3$ complex parameters

Contents

Comparison of parameters

As we have seen, we have

- gauge theory side : $d-1$ complex parameters
- brane tiling side: $d-3$ complex parameters (both in strong and weak coupling limit)

Relation between them? How to account for discrepancies in number?

Identification of parameters

$$\begin{array}{rcl} \text{diagonal gauge coupling} & \leftrightarrow & C_{57} + ie^{-\phi}, \\ \beta\text{-like deformation} & \leftrightarrow & C + iB_{57}, \\ \text{other } d-3 \text{ deformations} & \leftrightarrow & \\ \text{(geometric deformation)} & + & i\,\text{(Wilson line)} \end{array}$$

- We can make more detailed identification of parameters (explicit formulas relating them can be written down).

Summary of brane tilings

- Brane tilings, which are bipartite graphs on $\mathbb{T}^2$, represent physical system of D5-branes and NS5-brane.

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- Brane tilings, which are bipartite graphs on $\mathbb{T}^2$, represent physical system of D5-branes and NS5-brane.
- The brane system realize a large class of 4d $\mathcal{N} = 1$ superconformal quiver gauge theories in string theory.
- Brane tilings are powerful techniques to study quiver gauge theories. Clarifies relation with CY, gives quiver and superpotential, and has application to AdS/CFT...

Summary of our work

- We have studied marginal deformation of a large class of $\mathcal{N} = 1$ superconformal quiver gauge theories described by brane tilings.

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Summary of our work

- We have studied marginal deformation of a large class of $\mathcal{N} = 1$ superconformal quiver gauge theories described by brane tilings.
- Generically, we have $d - 1$ complex parameters, where $d =$ (the number of NS5-brane cycles) = (perimeter of the corresponding toric diagram).
- Out of $d - 1$ complex parameters, $d - 3$ correspond to deformation of branes in brane tilings and $U(1)$ Wilson line, and other two (diagonal coupling and β-like deformation） correspond to background fields.

Brane tilings have many problems yet

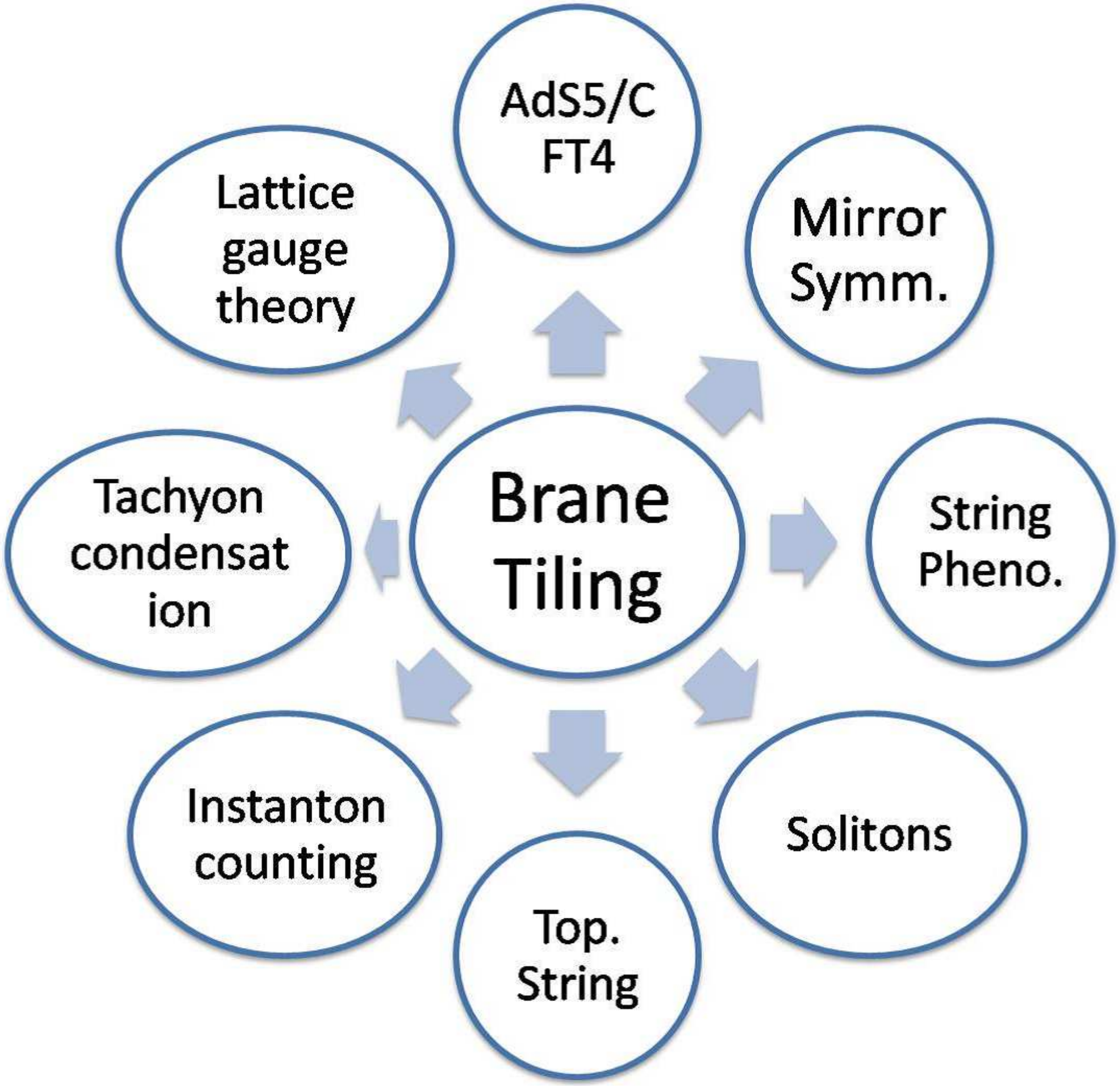
- Non-conformal case, fractional brane, cascading; flavor D7
- How to understand R-charge? a-maximization?
- 3d dimer? $\mathcal{N} = 2$ SCFT? AdS4/CFT3? [Lee and others]
- The shape of brane tilings in general g_s, work in progress
- Extension to orientifold [Franco et. al.], work in progress
- How to understand dynamics of $\mathcal{N} = 1$ theories, such as gaugino condensation, χ-SB? Find new phenomena in quiver gauge theories?

Relation with various topics

- Application to mathematics, rigorous proof of homological mirror symmetry [Ueda-Y]
- Application to phenomenological model building. DSB, metastable vacuum, gauge mediation
- Inclusion of other dimension of branes? Solitons in quiver gauge theories?
- Relation with amoeba, tropical geometry. Unexpected relation with soliton systems, work in progress
- Relation with crystal melting, 3d Young diagrams, Nekrasov's formula, topological strings..., work in progress
- Equivalance with approach using derived category? Tachyon condensation, stability?

⋮

A crazy dream of a "brane-tilinger"?



More detailed correspondence

First, BPS condition is (when axion is zero)

$$\int_{\mathcal{F}} Q(z) d^2 z = 0.$$

Here $Q(z)$ gives NS5-charge at z in $\mathbb{T}^2$. $Q(z)$ can be written as follows

$$Q(x^5, x^7) = \sum_{\mu=1}^{d} q_{\cdot\mu}(x^5, x^7).$$

Here q_μ is a step function at cycle μ

BPS condition : parameter ζ_μ

Define

$$\zeta_\mu = \int_{\mathcal{F}_0} q_\mu(x^5, x^7) dx^5 dx^7,$$

then

$$\sum_{\mu=1}^{d} \zeta_\mu = 0,$$

Gauge/superpotential coupling

$SU(N)_a$ gauge coupling can be read off from DBI action of D5-brane and

$$\frac{1}{Ng_a'^2} \sim \frac{1}{Ng_a^2} \sim \frac{A_a'}{4\pi N g_{\rm str}\alpha'}$$

Here A_a' is a area of face a.
superpotential coupling is comes from string worldsheet wrapping $(N, \pm 1)$ face

$$-\log|h_k| \sim -\log\left(e^{-A_k'/(2\pi\alpha')}\right) = \frac{A_k'}{2\pi\alpha'}$$

Gauge/superpotential coupling

Thus

$$\frac{1}{N{g'_a}^2} \sim A_a, \quad -\log|h_k| \sim A_k.$$

RG-invariant quantity

$$f^{(I)} = \sum_A S_A^{(I)} A_A = \int_{\mathcal{F}} S^{(I)}(x^5, x^7) dx^5 dx^7,$$

Here $S^{(I)}(x^5, x^7)$ is the value of S_A at (x^5, x^7). Since $S_A - S_B = b_\mu$,

$$S^{(I)}(x^5, x^7) = \sum_{\mu=1}^{d} b_\mu^{(I)} q_\mu(x^5, x^7) + c,$$

$$f^{(I)} = \sum_{\mu=1}^{d} b_\mu^{(I)} \zeta_\mu + c.$$

Correspondence with background fields

First, diagonal gauge coupling corresponds to c, which is、

$$c = f^{(1)} \sim \frac{1}{N g_{\text{diag}}^2} \sim \frac{A_{\text{tot}}}{\alpha' e^{\phi}}.$$

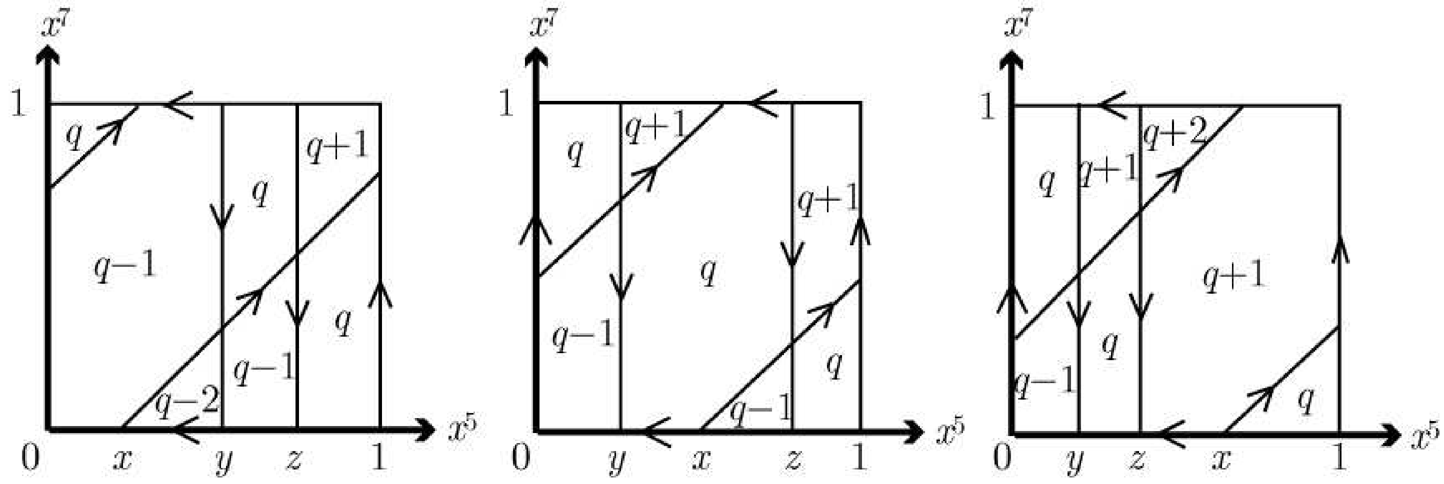
Second, β-like deformation corresponds to

$$f^{(2)} = \sum_{\mu=1}^{d} \zeta_\mu + c.$$

$\sum_{\mu=1}^{d} \zeta_\mu = 0$ by BPS condition if axion is zero, so making this non-zero corresponds to axion.

Appearance of $(N, 2)$-brane

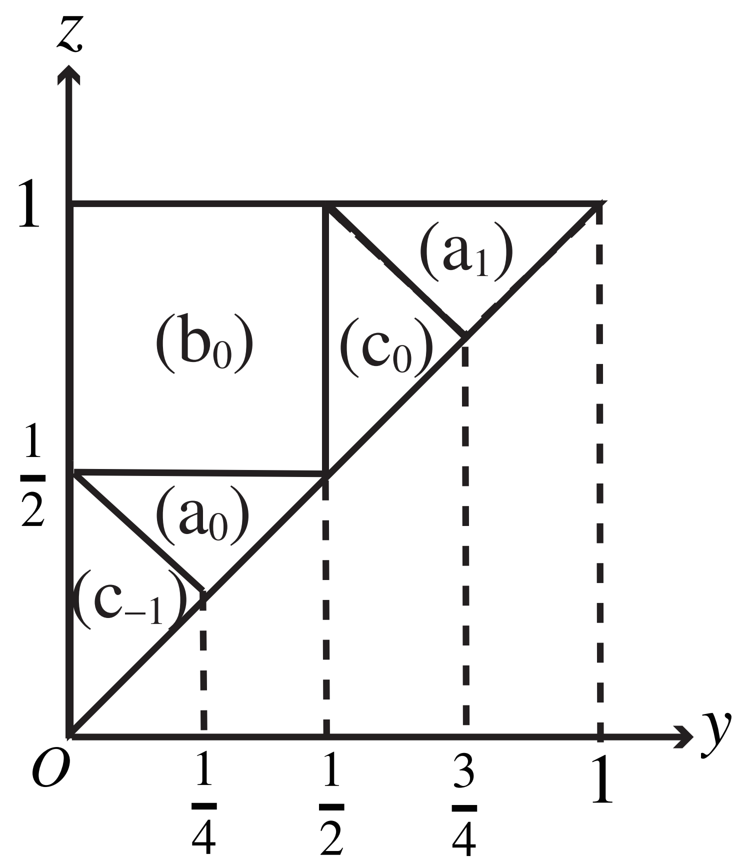
In general, we have (N, k)-branes with $|k| \geq 2$, whose gauge-theoretic meaning not understood 例：



(a) (b) (c)

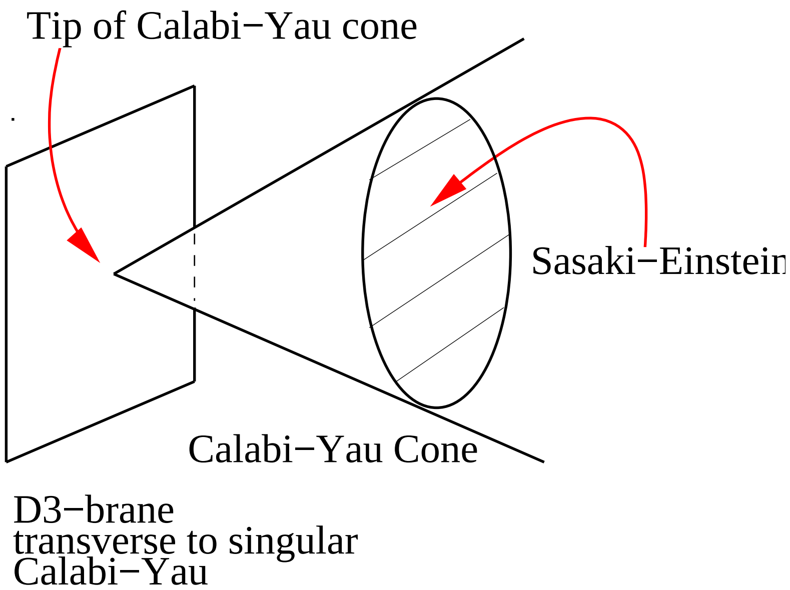
"phase diagram" in the strong coupling limit

In SPP case,



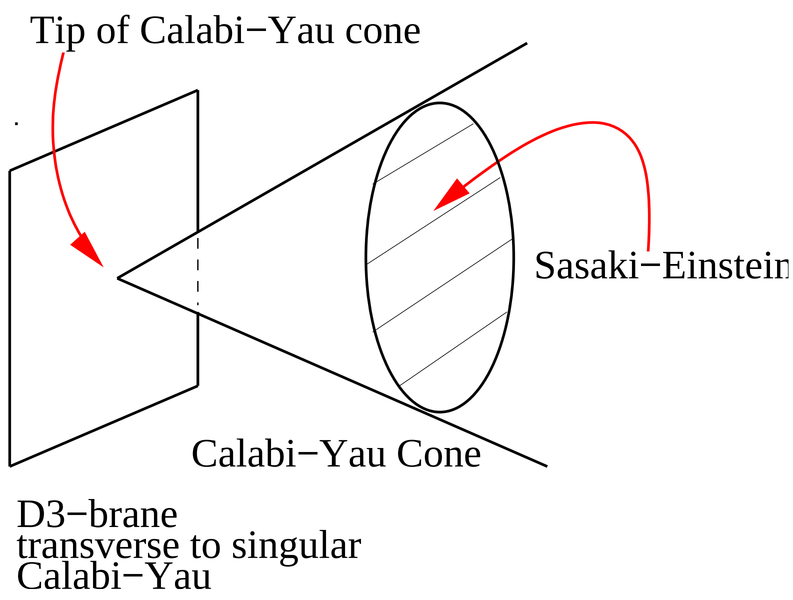
D3-brane probing Calabi-Yau

- Consider non-compact CY with cone-singularity
- Assume CY is toric (specified by toric diagram)
- D3-branes are transverse to CY, and placed at the apex of CY cone



D3-brane probing Calabi-Yau

- It is long belived that we have 4d $\mathcal{N} = \mathbf{1}$ superconformal quiver gauge theories
- Q: which CY corresponds to which quiver?
- A: brane tiling tells us!



Application to AdS/CFT

Most studied case of AdS/CFT:

$$\text{IIB on } AdS_5 \times S^5 \Leftrightarrow \mathcal{N} = 4 \text{ SYM}$$

We want to reduce SUSY to $\mathcal{N} = 1$! We replace S^5 by Sasaki-Einstein mfd X_5.

Locally

$$AdS_5 \times X_5 \sim \mathbb{R}^4 \text{ ''} \times \text{'' } C(X_5)$$

$$\underbrace{\overbrace{\left(\frac{du^2}{u^2} + u^2 ds_4^2\right)}^{AdS_5} + ds_{X_5}^2}_{\text{direct product}} = \underbrace{u^2 ds_4^2 + \frac{1}{u^2}\left(du^2 + u^2 ds_{X_5}^2\right)}_{\text{warped product}}$$

AdS/CFT ($\mathcal{N} = 1$ case)

AdS/CFT ($\mathcal{N} = 1$ version)

IIB on $AdS_5 \times X_5$ (X_5: Sasaki-Einstein, $C(X)$: toric CY) is dual to $\mathcal{N} = 1$ quiver gauge theory

Brane tiling gives gauge theory dual for each toric CY!

Simplest prediction:

$$Vol(X_5) = \frac{\pi^3}{4} \frac{1}{a}$$

where a is the central charge. Now checked for all toric CYs (under certain assumptions)

Example of Brane Tiling: $L^{1,7,3}$