Intersecting Solitons
Amoeba and Tropical Geometry

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What are we going to do?

Study of topological solitons: interplay of physics and mathematics

- infinite Grassmanian, Sato theory
- CFT techniques, free fermions; KdV, KP, Toda
- ADHM construction, Donaldson theory, Nekrasov's instanton counting, Seiberg-Witten

$$\vdots$$

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New mathematical tools, called amoeba and tropical geometry, are useful to study solitons!

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New mathematical tools, called amoeba and tropical geometry, are useful to study solitons!

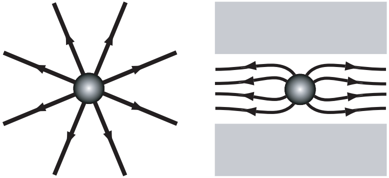
We study 1/4 BPS vortex-instanton system in the Higgs phase 5d $\mathcal{N} = \mathbf{1}$ $\mathbf{U(N_c)}$ SYM with $\mathbf{N_F}$ fundamental Higgs scalars.

Solitons in the Higgs Phase

When Higgs fields get VEV, gauge symmetry is completely broken and we are in a Higgs phase. Monopoles and instantons become composite in the Higgs phase.

- Monopoles

  magnetic flux are squeezed into vortices, and monopoles become composite of monopoles and vortices (Meissner effect)



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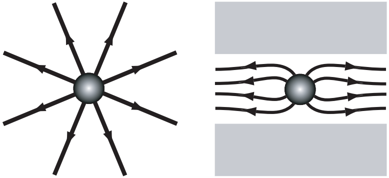
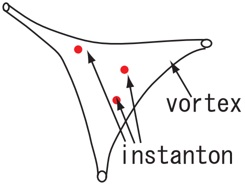


- Instantons
  shrink to point, but can reside in vortex (trapped instantons)

  ⇓

  vortex-instanton system:
  Today's talk



1/4 BPS solitons in the Higgs phase

- Many 1/4 BPS solitons in the Higgs phase of 8 SUSY non-Abelian 8-SUSY gauge theories found recently. Most of them are obtained from solitons in 5d by (Scherk-Schwarz) dimensional reduction. [Eto-Isozumi-Nitta-Ohashi-Sakai '05]

Example: (4+1)d

	0	1	2	3	4
vortex	o	o	o		
vortex	o			o	o
instanton	o				

codim=2

codim=1

⇒

(2+1)d

	0	1	3
domain wall	o	o	
domain wall	o		o
Hitchin charge	o		

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	0	1	2	3	4
vortex	○	○	○		
vortex	○			○	○
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$\Rightarrow$ (2+1)d

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domain wall	○	○	
domain wall	○		○
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- From this viewpoint, 5d soliton is the most important of these classes of solitons, but so far almost no study has been carried out.

We study vortex-instanton system in the Higgs phase of 5d $\mathcal{N}=1$ SYM.

Plan

Vortex-Instanton System in 5d Yang-Mills-Higgs

We consider vortex-instanton system on $\mathbb{R}^{2,1} \times T^2 \simeq \mathbb{R}_t \times (\mathbb{C}^*)^2$. Let $z_1 \equiv x_1 + iy_1,\ z_2 \equiv x_2 + iy_2$ be complex coordinates of $\mathbb{R}^2 \times T^2$.

$$\mathcal{L} = \mathrm{tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu H(\mathcal{D}^\mu H)^\dagger - \frac{g^2}{4}(HH^\dagger - c\mathbf{1}_{N_C})^2\right],$$

(By adding $\tilde{H}$), this is the bosonic part of 5d $\mathcal{N} = 1$ super Yang-Mills.

- W_μ: $SU(N_C)$ gauge field, $F_{\mu\nu}$: field strength
- H^{rA} $(r = 1, \cdots, N_C,\ A = 1, \cdots, N_F(= N_C))$: Higgs fields
- g: gauge coupling,
- c: FI parameter $c \neq 0$: Higgs gets VEV and we are in the Higgs phase

BPS inequality

By Bogomol'nyi completion, we have BPS inequality [Hanany-Tong, Eto-Isozumi-Nitta-Ohashi-Sakai '04]:

$$\mathbf{E} \;\geq\; \frac{8\pi^2}{g^2}\mathbf{I} + 2\pi c\,\mathbf{V},$$

two types of topological charges: instanton charge **I** and vortex charge **V**:

$$\begin{aligned} \mathbf{I} &\equiv -\frac{1}{8\pi^2}\int \mathrm{tr}\,(F\wedge F) = \int ch_2, \\ \mathbf{V} &\equiv -\frac{1}{2\pi}\int \mathrm{tr}\;F\wedge\omega = \int c_1\wedge\omega. \end{aligned}$$

where $\omega \equiv \frac{i}{2}(dz_1\wedge d\bar{z}_1 + dz_2\wedge d\bar{z}_2)$ is the Kähler form on $(\mathbb{C}^*)^2$.

Moduli Matrix Formalism

BPS equations

$$(a): \quad F_{\bar{z}_1\bar{z}_2} = 0, \qquad (b): \mathcal{D}_{\bar{z}_i} H = 0,$$
$$(c): \quad -2i(F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2}) = \frac{g^2}{2}(HH^\dagger - c\mathbf{1}_{N_C}),$$

- From (a), $\exists N_C \times N_C$ matrix valued function $S(z_i, \bar{z}_i) \in U(N_C)^{\mathbb{C}} = GL(N_C, \mathbb{C})$ such that

$$W_{\bar{z}_i} = -iS^{-1}\partial_{\bar{z}_i} S.$$

- Defining an $N_C \times N_F$ matrix

$$H_0 \equiv SH,$$

(b) reduces to

$$\partial_{\bar{z}_i} H_0 = 0.$$

or H_0 is holomorphic.

- From (c), $\Omega \equiv SS^\dagger$ satisfies "master equation":

$$\partial_{\bar{z}_1}(\Omega\partial_{z_1}\Omega^{-1}) + \partial_{\bar{z}_2}(\Omega\partial_{z_2}\Omega^{-1}) = -\frac{g^2c}{4}\left(1_{N_C} - \Omega_0\Omega^{-1}\right),$$

where we have defined

$$\Omega_0 \equiv \frac{1}{c}H_0H_0^\dagger.$$

Note: we have "gauge symmetry", which we call "**V**-transformation", defined by

$$(H_0,\, S) \;\rightarrow\; (VH_0,\, VS)\,, \qquad V(z) \in GL(N_C, \mathbb{C}).$$

Existence and uniqueness (modulo V-transformation) of solutions is known for arbitrary Kähler manifolds

Moduli matrix formalism [Eto-Isozumi-Nitta-Ohashi-Sakai]

$$\boxed{\mathbf{H_0(z)}\text{: given, } \boldsymbol{\Omega_0} = \frac{1}{\mathbf{c}}\mathbf{H_0 H_0^\dagger}}$$

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$$\mathbf{H_0(z)}\text{: given, } \mathbf{\Omega_0 = \frac{1}{c} H_0 H_0^\dagger}$$

$$\Downarrow$$

Solve $\partial_{\bar{z}_1}(\Omega \partial_{z_1} \Omega^{-1}) + \partial_{\bar{z}_2}(\Omega \partial_{z_2} \Omega^{-1}) = -\frac{g^2 c}{4}\left(\mathbf{1}_{N_C} - \Omega_0 \Omega^{-1}\right)$
to obtain $\mathbf{\Omega \equiv SS^\dagger}$.

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$\mathbf{W_{\bar{z}_i}}$ and $\mathbf{H}$ determined by $\mathbf{W_{\bar{z}_i} = -iS^{-1}\partial_{\bar{z}_i}S}$, $\mathbf{H = S^{-1}H_0}$
modulo gauge equivalence.

We can use $\mathbf{H_0}$ ("moduli matrix") to parametrize moduli space, although it is difficult to solve master equation explicitly.

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- Note: In the strong gauge coupling limit $\mathbf{g \longrightarrow \infty}$, we have explicit solution: $\mathbf{\Omega = \Omega_0 = \frac{1}{c} H_0 H_0^\dagger}$, except for subtlety around $\mathbf{H_0 = 0}$ (to be used later)

Questions

So far, we have explained general formalism, but we do not have a clear physical picture....

- shape of vortex?
- distribution of instanton charge?
- What happens in dimensional reduction ((4+1)d $\longrightarrow$ (2+1)d)?

$\vdots$

1 Introduction

2 Vortex-Instanton Systems

3 Relation with Amoeba and Tropical Geometry
- Vortex Sheet and Amoeba
- Dimensional Reduction and Tropical Geometry
- Topological Charges
- Metric on Moduli Space

4 Conclusions and Discussions

We have composite solitons of vortices and instantons. Let us first concentrate on vortices.

Where is the vortex? How does it look like?

Vortex Sheet

- When $\mathbf{det\,H_0(z_1,z_2)=0}$, gauge symmetry is partially restored and thus this surface is the position of vortex ("'vortex sheet"). From periodicity on $\mathbf{T^2}$,

$$\mathbf{P(u_1,u_2)\equiv det\,H_0(z_1,z_2)=\sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1,n_2}\,u_1^{n_1}u_2^{n_2}.}$$

with $\mathbf{u_i\equiv e^{\frac{z_i}{R_i}}}$ (recall periodicity in $\mathbf{T^2}$: $\mathbf{z_i\sim z_i+2\pi\sqrt{-1}R_i}$).
Still difficult to visualize.....

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  with $u_i \equiv e^{\frac{z_i}{R_i}}$ (recall periodicity in T^2: $z_i \sim z_i + 2\pi\sqrt{-1}R_i$). Still difficult to visualize.....

- Consider the projection of vortex sheet (amoeba) onto two non-compact directions:

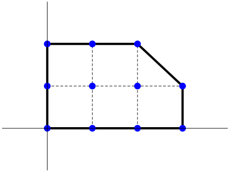
$$\mathcal{A}_P = \left\{ \left(R_1 \log |u_1|, \ R_2 \log |u_2| \right) \in \mathbb{R}^2 \ \middle| \ P(u_1, u_2) = 0 \right\}.$$

  Note here that $R_1 \log |u_1| = x_1$ and $R_2 \log |u_2| = x_2$.

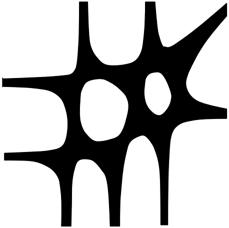
Define "Newton polytope" (a.k.a grid diagram) $\mathbf{\Delta(P)} \subset \mathbb{R}^2$ of a Laurent polynomial $\mathbf{P(u_1, u_2)}$ by

$$\mathbf{\Delta(P)} = \text{conv. hull } \left\{ (\mathbf{n_1}, \mathbf{n_2}) \in \mathbb{Z}^2 \middle| \, \mathbf{a_{n_1, n_2}} \neq \mathbf{0} \right\}.$$

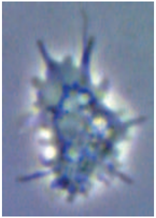
Conversely, $\mathbf{P(u_1, u_2)}$ is called the Newton polynomial of $\mathbf{\Delta}$,



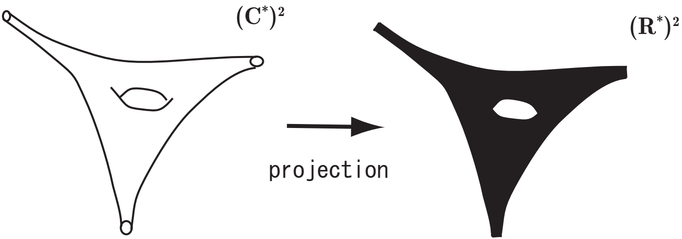
Newton polytope



amoeba

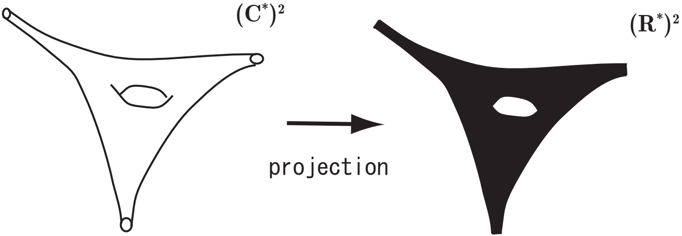


$$\mathbf{P(u_1, u_2)} = \mathbf{a_{0,0}} + \mathbf{a_{1,0} u_1} + \mathbf{a_{2,0} u_1^2} + \mathbf{a_{3,0} u_1^3} + \mathbf{a_{0,1} u_2} + \mathbf{a_{1,1} u_1 u_2} + \mathbf{a_{2,1} u_1^2 u_2} + \mathbf{a_{3,1} u_1^3 u_2} + \mathbf{a_{0,2} u_2^2} + \mathbf{a_{1,2} u_1 u_2^2} + \mathbf{a_{2,2} u_1^2 u_2^2}.$$



amoeba=projection of vortex junctions/webs

tentacles=semi-infinite cylinder of vortex



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- Amoeba: introduced by [Gelfand-Kapranov-Zelevinsky] in the study of hypergeometric functions
- Relation with real algebraic geometry [Mikhalkin], topological string theory [Nekrasov-Okounkov-Vafa], dimer model [Kenyon-Okounkov], instanton counting [Maeda-Nakatsu]...

Dimensional Reduction and Tropical Limit

What happens in dimensional reduction from (4+1)d $\longrightarrow$ (2+1)d? Take $\mathbf{R_1 = R_2 = R \rightarrow 0}$ with fixed $\mathbf{r_{n_1,n_2} \equiv R \log |a_{n_1,n_2}|}$, and neglect all KK modes.

	0	1	2	3	4
vortex	o	o	o		
vortex	o			o	o
instanton	o				

$\Rightarrow$

	0	1	3
domain wall	o	o	
domain wall	o		o
Hitchin charge	o		

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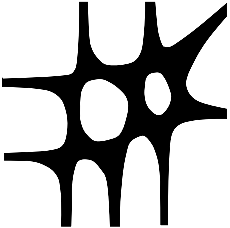
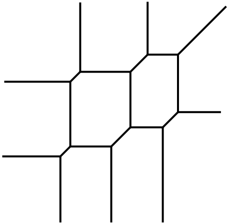


vortex(amoeba)

$\xrightarrow{}$

$\mathbf{R \Rightarrow 0}$



domain wall(tropical variety)

Wilson Loops and Ronkin Functions

Define adjoint scalars $\hat{\Sigma}_1(x_1, x_2)$ by

$$\hat{\Sigma}_i(x_1, x_2) \equiv -\frac{1}{2\pi R_1} \oint \frac{dy_2}{2\pi R_2} \log \left[P \exp \left(i \oint dy_1 \, W_{y_i} \right) \right],$$

This is the Wilson loop along T^2, or zero mode in KK decomposition, and is interpreted as the two dimensional kink profiles.

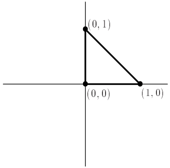
$$\lim_{g \to \infty} \mathrm{tr} \left[\hat{\Sigma}_i(x_1, x_2) \right] = \frac{\partial}{\partial x_i} N_P(x_1, x_2),$$

where $N_P(x_1, x_2)$ (Ronkin function) is given by

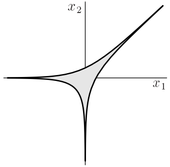
$$N_P(x_1, x_2) = \frac{1}{2\pi R_1} \frac{1}{2\pi R_2} \int_{T^2} d^2y \, \log |P(u_1, u_2)|$$

Wilson loop=derivatives of Ronkin function

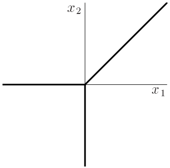
$$P(u_1, u_2) = u_1 + u_2 + 1 = e^{z_1} + e^{z_2} + 1,$$



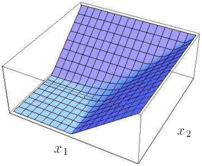
Newton polytope



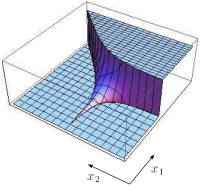
amoeba



tropical variety



Ronkin function



gradient of Ronkin function ($\mathbf{Tr}\hat{\boldsymbol{\Sigma}}$)

Tropical Geometry

In tropical limit ($\mathbf{R} \to \mathbf{0}$),

$$\mathbf{P}(\mathbf{u}_1, \mathbf{u}_2) = \sum_{(\mathbf{n}_1,\mathbf{n}_2)\in\mathbb{Z}^2} \mathbf{a}_{\mathbf{n}_1,\mathbf{n}_2} \mathbf{u}_1^{\mathbf{n}_1} \mathbf{u}_2^{\mathbf{n}_2}$$

$$\Rightarrow \mathbf{F}_{\mathbf{P}}(\mathbf{x}_1, \mathbf{x}_2) = \max_{(\mathbf{n}_1,\mathbf{n}_2)} (\mathbf{n}_1\mathbf{x}_1 + \mathbf{n}_2\mathbf{x}_2 + \mathbf{r}_{\mathbf{n}_1,\mathbf{n}_2}),$$

or $\mathbf{u}_1 + \mathbf{u}_2 \to \mathbf{x}_1 \oplus \mathbf{x}_2 \equiv \max(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{u}_1\mathbf{u}_2 \to \mathbf{x}_1 \otimes \mathbf{x}_2 \equiv \mathbf{x}_1 + \mathbf{x}_2$.
Namely, commutative ring $(\mathbb{R}, +, \times)$ is replaced by commutative semiring (ring w.o/ additive inverse) $(\mathbb{R}, \oplus, \otimes)$ (dequnatization, ultradiscretizaiton),

Tropical Geometry

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"ordinary" geometry $\Leftrightarrow$ commutative ring
"tropical" geometry $\Leftrightarrow$ commutative semiring

Tropical Geometry

In tropical limit ($\mathbf{R} \to \mathbf{0}$),

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Namely, commutative ring $(\mathbb{R}, +, \times)$ is replaced by commutative semiring (ring w.o/ additive inverse) $(\mathbb{R}, \oplus, \otimes)$ (dequnatization, ultradiscretizaiton),

> "ordinary" geometry ⇔ commutative ring
> "tropical" geometry ⇔ commutative semiring

- New, active area of research in mathematics
- Various applications (enumeration of curves, mirror symmetry, computational biology, celluar automata...)

Questions still remain

- shape of vortex? OK
- What happens in dimensional reduction ((4+1)d $\longrightarrow$ (2+1)d)? OK
- **distribution of instanton charge? Not Yet**

The key to answer this question is the topological charges.

BPS inequality (remainder)

BPS inequality:

$$\mathbf{E} \geq \frac{8\pi^2}{g^2}\mathbf{I} + 2\pi c\,\mathbf{V},$$

two types of topological charges: instanton charge **I** and vortex charge **V**:

$$\mathbf{I} \equiv -\frac{1}{8\pi^2}\int \mathrm{tr}\,(F \wedge F) = \int ch_2,$$

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where $\omega \equiv \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ is the Kähler form on $(\mathbb{C}^*)^2$.

Vortex Charge

We can take $\mathbf{g} \to \infty$ since the topological charges are independent of the gauge coupling constant $\mathbf{g}$. Then

$$-\frac{1}{2\pi}\,\mathrm{tr}\ \mathbf{F} = \frac{1}{4\pi}\mathbf{dd_c}\log\det\Omega \ \rightarrow\ \frac{1}{2\pi}\mathbf{dd_c}\log|\mathbf{P}|,$$

where $\mathbf{d_c} \equiv -\mathbf{i}(\partial - \bar{\partial})$. By using the Poincaré-Lelong formula

$$\int_{(\mathbb{C}^*)^2} \frac{1}{2\pi}\mathbf{dd_c}\log|\mathbf{P}| \wedge \alpha = \int_{\mathbf{X}} \alpha, \quad \mathbf{X} = \{\mathbf{P}(\mathbf{u_1},\mathbf{u_2}) = \mathbf{0}\} \text{ in } (\mathbb{C}^*)^2,$$

the vortex charge can be evaluated as

$$\mathbf{V} \ = \ -\,\mathbf{c}\int_{(\mathbb{C}^*)^2} \mathrm{tr}\ \mathbf{F}\wedge\omega \ = \ 2\pi\mathbf{c}\int_{\mathbf{X}}\omega \ = \ 2\pi\mathbf{c}\,\mathrm{Area}(\mathbf{X}).$$

$\mathbf{V}$ is uniformly distributed along vortex sheets $\mathbf{X}$, and the total vortex charge is given by the area of the vortex sheets multiplied by the tension $2\pi\mathbf{c}$.

Instanton Charge

$$\mathbf{I} \equiv -\frac{1}{8\pi^2}\int \mathrm{tr}\,(\mathbf{F}\wedge\mathbf{F}) = \int \mathbf{ch}_2,$$

Instanton charge $\mathbf{I}$ is divided into two contributions:

$$\mathbf{I} = \underbrace{-\mathbf{I}_{\mathrm{intersection}}}_{\text{binding energy}} + \underbrace{\mathbf{I}_{\mathrm{instanton}}}_{\text{point-like instantons}}$$

- intersection charge

$$\mathbf{I}_{\mathrm{intersection}} \equiv \frac{1}{8\pi^2}\int \mathrm{tr}\ \mathbf{F}\wedge\mathrm{tr}\ \mathbf{F} = \frac{1}{2}\int \mathbf{c}_1\wedge\mathbf{c}_1,$$

  Binding energy of solitons (negative contribution to energy)

- instanton number

$$\mathbf{I}_{\mathrm{instanton}} \equiv \int \mathbf{c}_2.$$

  Number of point-like instantons (positive contribution to energy)

Intersection Charge

Next consider the intersection charge. By taking $\mathbf{g} \to \infty$, the intersection charge density $\mathcal{I}_{\text{intersection}}$ becomes a complex Monge-Ampère measure $(\mathbf{dd_c} \log |\mathbf{P}|)^2$ on $(\mathbb{C}^*)^2$

$$\mathcal{I}_{\text{intersection}} = \frac{1}{8\pi^2} \mathrm{tr}\ \mathbf{F} \wedge \mathrm{tr}\ \mathbf{F} \ \to \ \frac{1}{8\pi^2} \mathbf{dd_c} \log |\mathbf{P}| \wedge \mathbf{dd_c} \log |\mathbf{P}|.$$

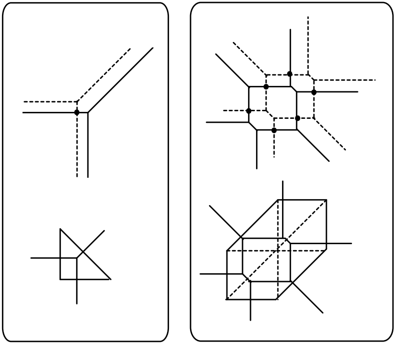
Then the intersection charge is evaluated again by using Poincaré-Lelong formula,

$$\mathbf{I}_{\text{intersection}} = \frac{1}{4\pi} \int_{\mathbf{X}} \mathbf{dd_c} \log |\mathbf{P}| = \frac{1}{2} \#(\mathbf{X}, \mathbf{X}).$$

but this naive evaluation is the self-intersection number and divergent!
By using suitable reguralization, we have $\mathbf{I}_{\text{intersection}} = 2\mathrm{Area}(\boldsymbol{\Delta}(\mathbf{P}))$.
(For $\mathbf{N}_{\mathrm{C}} = \mathbf{1}, \mathbf{N}_{\mathrm{F}} = \mathbf{2}$, this is rigorously proven).

Intersection Charge in the Tropical Limit

In the tropical limit $\mathbf{R_1}, \mathbf{R_2} \to \infty$, the intersection charge is given by the intersection number of the tropical varieties of $\mathbf{P_1}$ and $\mathbf{P_2}$, and it is easy to see that the number is given by $\mathbf{2Area(\Delta(P))}$ (tropical Berenstein theorem).



Non-Abelian case: Instanton Number

So far, only the trace part ($\mathbf{U(1)}$ part) is discussed.

$$\mathbf{I} = \underbrace{-\mathbf{I}_{\text{intersection}}}_{\text{binding energy}} + \underbrace{\mathbf{I}_{\text{instanton}}}_{\text{point-like instantons}},$$

$$\mathbf{I}_{\text{intersection}} \equiv \frac{1}{8\pi^2}\int \mathrm{tr}\ \mathbf{F} \wedge \mathrm{tr}\ \mathbf{F} = \frac{1}{2}\int \mathbf{c}_1 \wedge \mathbf{c}_1,$$

$$\mathbf{I}_{\text{instanton}} = \int \mathbf{c}_2 = \frac{1}{8\pi^2}\left[\mathrm{Tr}\ \mathbf{F} \wedge \mathrm{Tr}\ \mathbf{F} - \mathrm{Tr}(\mathbf{F} \wedge \mathbf{F})\right]$$

- Only the $\mathbf{U(1)}$ part matter for intersection charge.
- In order to discuss instanton number, we have to go to non-Abelian case.

Moduli Matrix in Non-Abelian Case

Consider the simplest case $\mathbf{N_C = N_F = 2}$.

1/2 BPS vortex moduli space

- we have orientational moduli (NG mode)
  $\mathbf{U(2)_C \times SU(2)_F \xrightarrow{\text{vacuum}} SU(2)_{C+F} \xrightarrow{\text{vortex}} U(1)_{C+F}}$.
  $\mathcal{M}_{\text{orientation}} \equiv \mathbb{C}\mathbf{P}^1 \simeq \mathbf{SU(2)_{C+F}/U(1)_{C+F}} \subset \mathcal{M}$
- The moduli matrix for the vortex at $\mathbf{z_1 = 0}$ [Eto:'05]

$$\mathbf{H_0} = \sqrt{\mathbf{c}} \begin{pmatrix} \mathbf{1} & \mathbf{b} \\ \mathbf{0} & \mathbf{z_1} \end{pmatrix} \sim \sqrt{\mathbf{c}} \begin{pmatrix} \mathbf{z_1} & \mathbf{0} \\ \mathbf{1/b} & \mathbf{1} \end{pmatrix},$$

where $\sim$ represents the $\mathbf{V}$-equivalence relation. These two moduli matrices provide two patches $\mathbf{b}$ and $\mathbf{1/b}$ of $\mathbb{C}\mathbf{P}^1$.

1/4 BPS moduli space

$$H_0 = \begin{pmatrix} 1 & b(u_1, u_2) \\ 0 & P(u_1, u_2) \end{pmatrix},$$

where $P(u_1, u_2)$ and $b(u_1, u_2)$ are Laurent polynomials

$$P(u_1, u_2) = \sum a_{n_1,n_2} u_1^{n_1} u_2^{n_2}, \qquad b(u_1, u_2) = \sum b_{n_1,n_2} u_1^{n_1} u_2^{n_2},$$

In $g \to \infty$ limit,

$$\Omega \;\to\; \Omega_0 = \frac{1}{c} H_0 H_0^\dagger = \begin{pmatrix} 1 + |b|^2 & b\overline{P} \\ P\overline{b} & |P|^2 \end{pmatrix}.$$

By appropriately taking care of the subtlety of $g \to \infty$ limit, instanton number is given by

$$I = \int ch_2 \;=\; \frac{1}{8\pi^2} \int \Big(dd_c \log|P| \wedge dd_c \log(1 + |b|^2) - dd_c \log|P| \wedge dd_c \log|P| \Big),$$

instanton number

■ small radius limit $\mathbf{R \longrightarrow 0}$

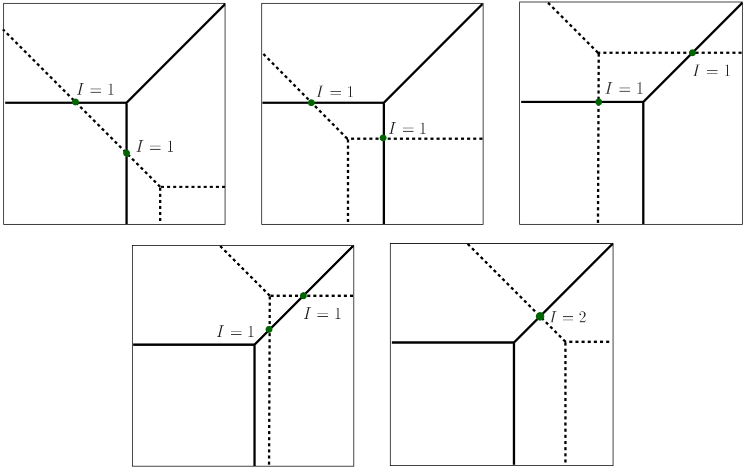
$$\frac{R}{2}\log(1+|b|^2) \longrightarrow \tilde{F}_b(x_1,x_2) \equiv \max_{(n_1,n_2)}(n_1x_1+n_2x_2+s_{n_1,n_2}),$$

where $s_{0,0}=\frac{R}{2}\log(1+|b_{0,0}|^2)$ and $s_{n_1,n_2}=R\log|b_{n_1,n_2}|$ for $(n_1,n_2)\neq 0$ are fixed in the limit. Then

$$I_{\mathrm{instanton}} \quad\longrightarrow\quad \int_{\mathbb{R}^2} d^2x\,\epsilon_{ij}\epsilon_{kl}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}F_P(x_1,x_2)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}F_b'(x_1,x_2).$$

Instanton number density is localized at the intersection of the tropical variety of the Laurent polynomial P and the lines on which the piece-wise linear function $\tilde{F}_b(x_1,x_2)$ is not differentiable.

The instanton number density in the small radius limit for $\mathbf{P} = \mathbf{u_1} + \mathbf{u_2} + 1$ and $\mathbf{b} = \mathbf{b_{1,1}u_1u_2} + \mathbf{b_{1,0}u_1} + \mathbf{b_{0,0}}$. Solid line: tropical variety of $\mathbf{P}$, dashed line: the lines on which $\tilde{\mathbf{F}}_{\mathbf{b}}(\mathbf{x_1}, \mathbf{x_2})$ is not differentiable.



Metric on Moduli Space

We can also discuss metric of the moduli space. This is important when we become dynamical issues.

- Moduli parameters: coefficients $\mathbf{a}_{n_1,n_2}$ of $\mathbf{P}$.

$$\mathbf{P}(u_1, u_2) = \sum_{(n_1,n_2)\in V(Q)} a_{n_1,n_2} u_1^{n_1} u_2^{n_2}.$$

For $N_C = N_F = 1$, the metric of the moduli space is given by

$$\begin{aligned} K_{i\bar{j}} &= c \int \Bigg(\frac{\omega^2}{2!} \partial_i \bar{\partial}_j \log \det \Omega \\ &+ 2l^2 \, \omega \wedge i \, \mathrm{tr} \left[\bar{\partial}(\Omega \partial \Omega^{-1}) \bar{\partial}_j (\Omega \partial_i \Omega^{-1}) - \bar{\partial}(\Omega \partial_i \Omega^{-1}) \bar{\partial}_j (\Omega \partial \Omega^{-1}) \right] \Bigg) \end{aligned}$$

- Only the coefficients $\mathbf{a}_{n_1,n_2}$ corresponding to internal lattice points are normalizable.
- Coefficients corresponding to externall lattice points correspond to the motion of external legs

Consider the case where the loop sizes are much larger than the radius of torus $\mathbf{R} \equiv \mathbf{R_1} = \mathbf{R_2} \gg \mathbf{l}$. Then

$$\begin{aligned} \mathbf{K} &\approx 8\pi^2 \mathbf{cR} \lim_{\mathbf{R}\to 0} \int \mathbf{d^2xR} \left(\mathbf{N_P}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{a}, \bar{\mathbf{a}}) - \mathbf{N_{\widetilde{P}}}(\mathbf{x_1}, \mathbf{x_2}) \right) \\ &= 8\pi^2 \mathbf{cR} \int \mathbf{d^2x} \left(\mathbf{F_P}(\mathbf{x_1}, \mathbf{x_2}) - \mathbf{F_{\widetilde{P}}}(\mathbf{x_1}, \mathbf{x_2}) \right), \end{aligned}$$

where $\mathbf{F_P}$ and $\mathbf{F_{\widetilde{P}}}$ are piece-wise linear functions defined by

$$\begin{aligned} \mathbf{F_P} &= \lim_{\mathbf{R}\to 0} \mathbf{RN_P}(\mathbf{x_1}, \mathbf{x_2}) = \max_{(\mathbf{n_1},\mathbf{n_2})\in \mathbf{V(Q)}} (\mathbf{n_1x_1} + \mathbf{n_2x_2} + \mathbf{r_{n_1,n_2}}), \\ \mathbf{F_{\widetilde{P}}} &= \lim_{\mathbf{R}\to 0} \mathbf{RN_{\widetilde{P}}}(\mathbf{x_1}, \mathbf{x_2}) = \max_{(\mathbf{n_1},\mathbf{n_2})\in \mathbf{V_{ex}(Q)}} (\mathbf{n_1x_1} + \mathbf{n_2x_2} + \mathbf{r_{n_1,n_2}}), \end{aligned}$$

with $\mathbf{r_{n_1,n_2}} \equiv \mathbf{R} \log |\mathbf{a_{n_1,n_2}}|$.

Consider the case where the loop sizes are much larger than the radius of torus $\mathbf{R} \equiv \mathbf{R_1} = \mathbf{R_2} \gg \mathbf{l}$. Then

Ronkin function

$$\mathbf{K} \approx 8\pi^2 \mathbf{c}\mathbf{R} \lim_{\mathbf{R}\to 0} \int \mathbf{d}^2\mathbf{x}\mathbf{R} \left(\boxed{\mathbf{N_P}(\mathbf{x_1},\mathbf{x_2},\mathbf{a},\bar{\mathbf{a}})} - \mathbf{N_{\tilde{P}}}(\mathbf{x_1},\mathbf{x_2}) \right)$$
$$= 8\pi^2 \mathbf{c}\mathbf{R} \int \mathbf{d}^2\mathbf{x} \left(\boxed{\mathbf{F_P}(\mathbf{x_1},\mathbf{x_2})} - \mathbf{F_{\tilde{P}}}(\mathbf{x_1},\mathbf{x_2}) \right),$$

tropical polynomial

where $\mathbf{F_P}$ and $\mathbf{F_{\tilde{P}}}$ are piece-wise linear functions defined by

$$\mathbf{F_P} = \lim_{\mathbf{R}\to 0} \mathbf{R}\mathbf{N_P}(\mathbf{x_1},\mathbf{x_2}) = \max_{(\mathbf{n_1},\mathbf{n_2})\in \mathbf{V}(\mathbf{Q})} (\mathbf{n_1}\mathbf{x_1} + \mathbf{n_2}\mathbf{x_2} + \mathbf{r_{n_1,n_2}}),$$
$$\mathbf{F_{\tilde{P}}} = \lim_{\mathbf{R}\to 0} \mathbf{R}\mathbf{N_{\tilde{P}}}(\mathbf{x_1},\mathbf{x_2}) = \max_{(\mathbf{n_1},\mathbf{n_2})\in \mathbf{V_{ex}}(\mathbf{Q})} (\mathbf{n_1}\mathbf{x_1} + \mathbf{n_2}\mathbf{x_2} + \mathbf{r_{n_1,n_2}}),$$

with $\mathbf{r_{n_1,n_2}} \equiv \mathbf{R}\log|\mathbf{a_{n_1,n_2}}|$.

Consider the case where the loop sizes are much larger than the radius of torus $\mathbf{R} \equiv \mathbf{R_1} = \mathbf{R_2} \gg \mathbf{l}$. Then

Ronkin function

$$\mathbf{K} \approx 8\pi^2 \mathbf{cR} \lim_{\mathbf{R}\to 0} \int \mathbf{d^2 x R} \left(\mathbf{N_P(x_1, x_2, a, \bar{a})} - \mathbf{N_{\tilde{P}}(x_1, x_2)} \right)$$

$$= 8\pi^2 \mathbf{cR} \int \mathbf{d^2 x} \left(\mathbf{F_P(x_1, x_2)} - \mathbf{F_{\tilde{P}}(x_1, x_2)} \right),$$

constant term

tropical polynomial

where $\mathbf{F_P}$ and $\mathbf{F_{\tilde{P}}}$ are piece-wise linear functions defined by

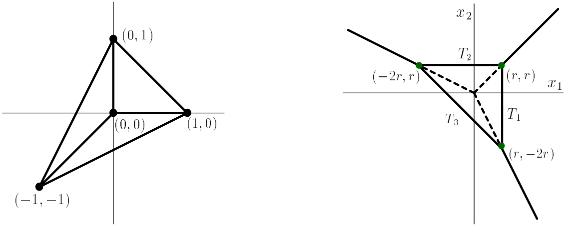
$$\mathbf{F_P} = \lim_{\mathbf{R}\to 0} \mathbf{RN_P(x_1, x_2)} = \max_{\mathbf{(n_1,n_2)\in V(Q)}} \mathbf{(n_1 x_1 + n_2 x_2 + r_{n_1,n_2})},$$

$$\mathbf{F_{\tilde{P}}} = \lim_{\mathbf{R}\to 0} \mathbf{RN_{\tilde{P}}(x_1, x_2)} = \max_{\mathbf{(n_1,n_2)\in V_{ex}(Q)}} \mathbf{(n_1 x_1 + n_2 x_2 + r_{n_1,n_2})},$$

with $\mathbf{r_{n_1,n_2}} \equiv \mathbf{R \log |a_{n_1,n_2}|}$.

Example

$$\mathbf{P(u_1, u_2) = u_1 + u_2 + u_1^{-1} u_2^{-1} + a_{0,0}}.$$

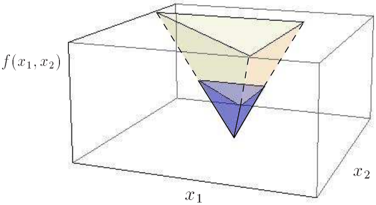


(a) Newton polytope (b) wall web (tropical variety)

In this case, only one normalizable moduli parameter $\mathbf{a_{0,0}}$, which is related to the size of the loop.

The asymptotic Kähler potential is proportional to the volume of the tetrahedron surrounded by four planes

$$\mathbf{K} \approx 12\pi^2 \mathbf{c} \mathbf{R}\, \mathbf{r}^3.$$



When we write $\mathbf{a} = \mathbf{e}^{\mathbf{r}/\mathbf{R}+\mathbf{i}\theta}$, effective Lagrangian is given by

$$\mathbf{L}_{\text{eff}} = 18\pi^2 \mathbf{crR} \left(\dot{\mathbf{r}}^2 + \mathbf{R}^2 \dot{\theta}^2 \right).$$

This can be interpreted as the kinetic energy associated with the motion of the three walls composing the loop $(\sum_{i=1}^{3} \frac{m_i}{2} v_i^2)$.

Conclusion

- We have studied composite solitons consisting of intersections (webs) of vortices and instantons. They are a 1/4 BPS soliton in the Higgs phase of $\mathcal{N} = \mathbf{1}$ supersymmetric Yang-Mills theory on $\mathbb{R} \times (\mathbb{C}^*)^2$.
- These solitons are important because they reduce to solitons in other dimensions by dimensional reduction.

Conclusion

- We have studied composite solitons consisting of intersections (webs) of vortices and instantons. They are a 1/4 BPS soliton in the Higgs phase of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory on $\mathbb{R} \times (\mathbb{C}^*)^2$.
- These solitons are important because they reduce to solitons in other dimensions by dimensional reduction.

amoeba and tropical geometry arise quite naturally, and they provide powerful techniques to study solitons and gauge theory!

Dictionary

soliton/gauge theory	amoeba/tropical geometry
moduli matrix	Newton polynomial
projection of vortex sheet	amoeba
dimensional reduction	tropical limit
domain wall	tropical variety
Wilson loop along $\mathbf{T}^2$	derivative of Ronkin function
intersection charge	complex Monge-Ampère measure

Discussions

Possible generalizations include:

- Generalization to arbitrary gauge group? (cf. [Eto-Fujimori-Gudnason-Konishi-Nitta-Ohashi-Vinci]) non-Abelian/non-commutative tropical geometry?
- Extention to $(\mathbb{C}^*)^n$ or on general Kähler manifolds [Mundet i Rierra]

Actually, amoeba and tropical geometry appears in various topics, and the relation between them is not clear:

- Connection with dimer models, brane tilings?
  - Ronkin function is the thermodynamic limit of partition function of dimer model
  - 'Good' amoeba (Harnack curve) appears from spectral curve of dimer model [Kenyon-Okounkov]
  - coamoeba/alga? (projection onto $\mathbf{T}^2$) [Feng-He-Kennaway-Vafa, Ueda-Yamazaki]

More Discussions

- Relation with topological string theory? (cf.[Okounkov-Reshetikhin-Vafa]: asymptotic form of plane partition gives amoeba)
- D-brane realization? (cf. cylinder case: kinky D-brane [Lambert-Tong,Eto-Fujimori-Isozumi-Nitta-Ohashi-Ohta-Sakai])?
- Relation with 1/4 BPS dyons? String web? (cf. [Lunin '08, Ray '08]: Ronkin function appear in SUGRA solution) Relation with black hole entropy?
- Relation with Nekrasov's partition function (we are back to Coulomb phase when $\mathbf{c} \to \mathbf{0}$, vortices disappear and instantons remain), symplectic Gromov-Witten invariant (Donaldson+Gromov-Witten) [Baptista]?

$\vdots$

Welcome to the world of "TROPICAL PHYSICS"!