Brane Tilings

Masahito Yamazaki (Univ. Tokyo, Hongo)

2008/Jun/18 @ Komaba

Based on review: M. Y., Fortscr. Phys. 56 (2008) 555.
(arXiv:0803.4474 [hep-th])

See also K. Kennaway, Int. J. Mod. Phys. A22 (2007) 2977.
(arXiv:0706.1660 [hep-th])

Thanks to my collaborators!

Based on joint works with T. Fujimori, K. Ohta,M. Nitta, Y. Imamura, H. Isono, K. Kimura, K. Ueda, N. Sakai

- K. Ueda and M.Y., math.AG/0605780, math.AG/0606548, math.AG/0703267
- Y. Imamura, H. Isono, K. Kimura and M. Y., Prog. Theor. Phys., Vol 117. no 5, pp923 (hep-th/0702049)
- Y. Imamura, K. Kimura and M. Y., JHEP 0803:058,2008. (arXiv:0801.3528 [hep-th])
- T. Fujimori, K. Ohta, M. Nitta, N. Sakai and M. Y., arXiv:0805.1194 [hep-th]

The questions

Today's topic: brane tilings ('05-); Introduced by Hanany and collaborators [Hanany-Kennaway, Franco-Hanany-Kennaway-Vegh-Wecht '05]

The questions

Today's topic: brane tilings ('05-); Introduced by Hanany and collaborators [Hanany-Kennaway, Franco-Hanany-Kennaway-Vegh-Wecht '05]

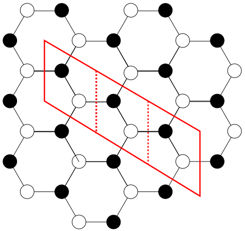
- Part I: Foundation of brane tilings
  (What is brane tiling after all? What is its physical meaning?)

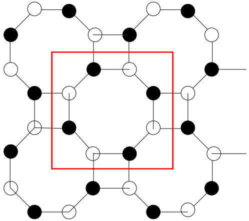
- Part II: Generalizations and application of brane tilings
  (Why study brane tilings? What kind of applications/generalizations do they have?)

Plan

Part I: What is a brane tiling?

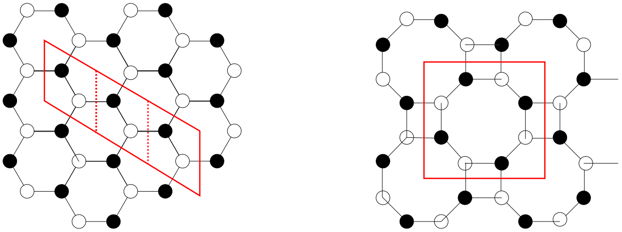
- The answer by Hanany and collaborators: brane tiling is a bipartite graph on $\mathbb{T}^2$:





Part I: What is a brane tiling?

- The answer by Hanany and collaborators: brane tiling is a bipartite graph on $\mathbb{T}^2$:



- Powerful technique to study 4d $\mathcal{N} = 1$ supersymmetric quiver gauge theories corresponding to arbitrary toric Calabi-Yau.
- Successfuly applied to AdS/CFT [Franco et. al. '06]

What is brane tiling?

Our answer: [Imamura-Isono-Kimura-M.Y. '07]

A brane tiling is a physical brane systems consisting of D5-branes and NS5-brane. The bipartite graphs are just convenient ways of representing them!

What is brane tiling?

Our answer: [Imamura-Isono-Kimura-M.Y. '07]

A brane tiling is a physical brane systems consisting of D5-branes and NS5-brane. The bipartite graphs are just convenient ways of representing them!

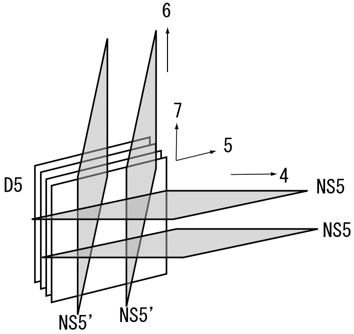
	0	1	2	3	4	5	6	7	8	9
D5	o	o	o	o		o		o		
NS5	o	o	o	o	Σ (2-dim surface)					

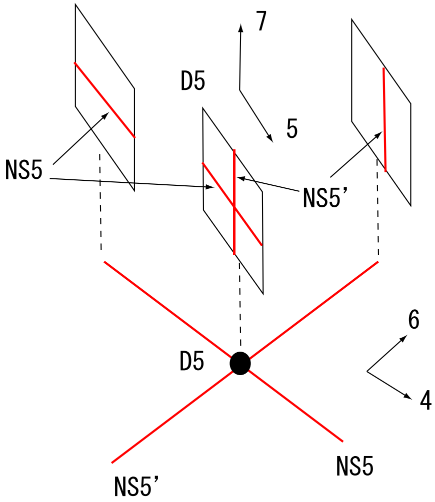
- Since 5, 7-directions are compactified, the theory on D5-branes reduces to 4d gauge theory.
- Due to the presence of NS5-brane, supersymmetry is broken to $\mathcal{N} = 1$.

The D5/NS5 brane configuration

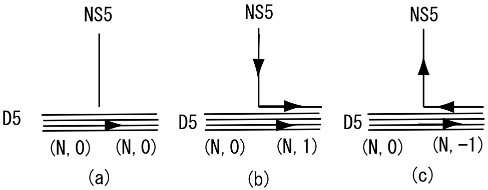
Eg. (conifold [Klebanov-Witten '98]) :

	0	1	2	3	4	5	6	7	8	9
D5	○	○	○	○		○		○		
NS5	○	○	○	○	○	○				
NS5	○	○	○	○			○	○		

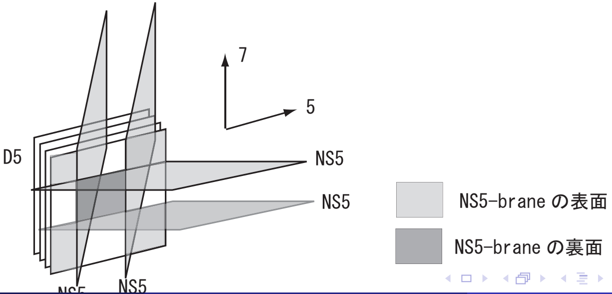



7
D5
5
NS5
NS5'
6
D5
4
NS5'
NS5

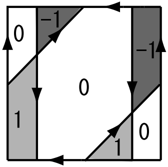
Due to the conservation of NS5-charge, $\mathbb{T}^2$ is divided into many regions, each of which corresponds to either $\mathbf{(N,0)}$, $\mathbf{(N,1)}$ or $\mathbf{(N,-1)}$-branes

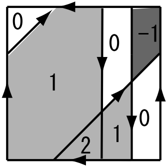


NS5 cylinders now merge into one!



terminology: division of $\mathbb{T}^2$ by cycles of NS5-branes are called the fivebrane diagram.

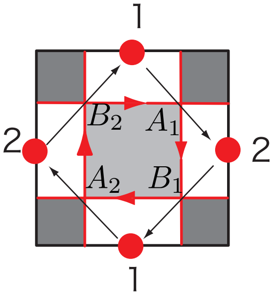




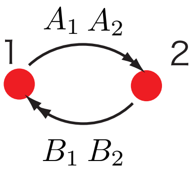
When $(\mathbf{N}, \mathbf{k})$-brane with $|\mathbf{k}| \geq \mathbf{2}$ appear, the gauge theory meaning is not known, and we restrict ourselves to case with $\mathbf{k} = \mathbf{0}, \pm\mathbf{1}$.

The gauge theory on D5-brane is a quiver gauge theory

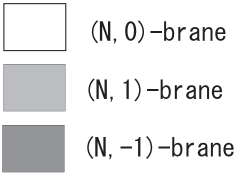
- **SU(N)** gauge group (vertex of quiver) is only on **(N, 0)**-branes, and not on **(N, ±1)**-branes
- bifundamental fields are at the intersection of **(N, 0)**-branes (massless open string)



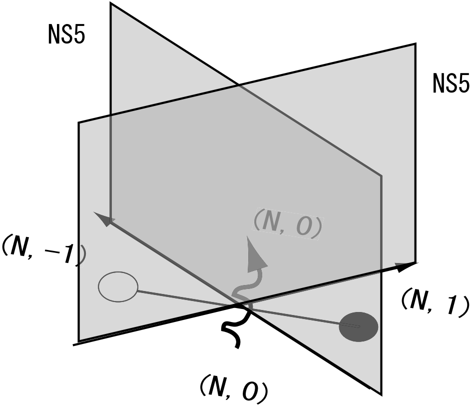
(brane tiling)



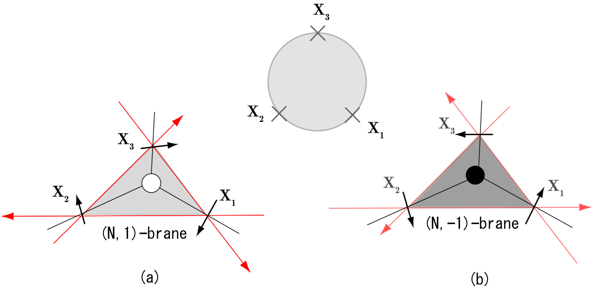
(quiver)



Due to the presence of NS5-branes, we have a chiral spectrum:



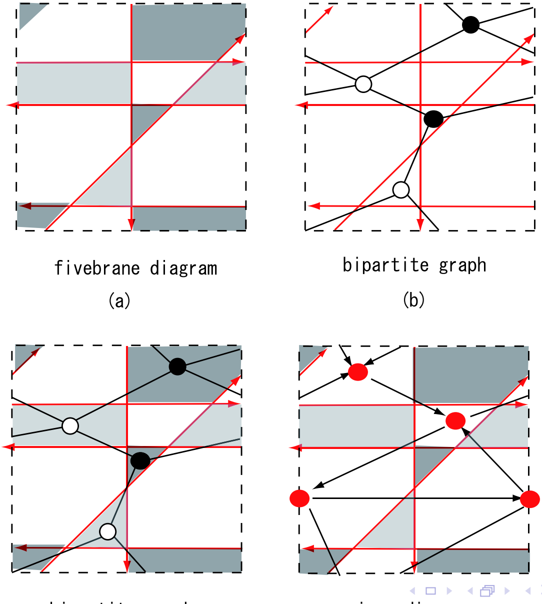
- A term in the superpotential corresponds to a region of $\mathbf{(N,1)}$-brane and $\mathbf{(N,-1)}$-brane.



(a) with $\mathbf{(N,1)}$-brane contributes operators of the form $\mathrm{tr}(\mathbf{X_1X_2X_3})$ to the superpotential, and $\mathbf{(N,-1)}$-brane as in (b) contributes $\mathrm{tr}(\mathbf{X_1X_3X_2})$ term.

$$\mathbf{W} = +\mathrm{tr}(\mathbf{A_1B_1A_2B_2}) - \mathrm{tr}(\mathbf{A_1B_2A_2B_1})$$

The relation with bipartite graphs and quiver diagrams



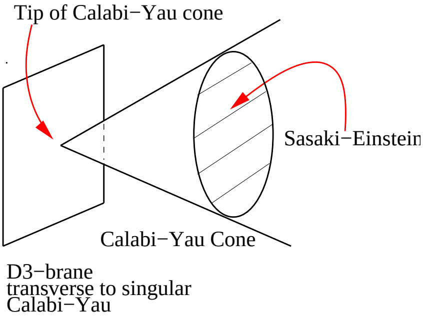
Summary of correspondence

fivebrane	bipartite graph	quiver diagram	quiver gauge theory
$(\mathbf{N},\mathbf{1})$-brane (light gray)	white vertex	(face)	superpotential term (+ sign)
$(\mathbf{N},-\mathbf{1})$-brane (dark gray)	black vertex	(face)	superpotential term (− sign)
$(\mathbf{N},\mathbf{0})$-brane	face	vertex	gauge group
open string	edge	edge	bifundamental

Relation with toric Calabi-Yau cones

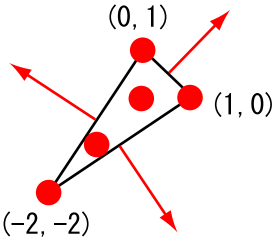
- The D5/NS5 system represented by brane tilings are equivalent to the geometry of toric Calabi-Yau (+ D3-brane) by T-duality along $\mathbb{T}^2$(57).

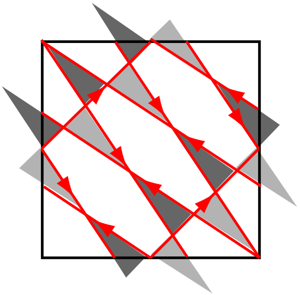
D5-brane ⟶ D3-brane
NS5-brane ⟶ toric Calabi-Yau



Rule

write NS5-cycle with winding $(\mathbf{p}, \mathbf{q})$ for each primitive normal $(\mathbf{p}, \mathbf{q})$ to the toric diagram





Q: We have a NS5 cycle for each primitive normal of the toric diagram. Why?

Q: We have a NS5 cycle for each primitive normal of the toric diagram. Why?

The key is Buscher's rule ('87,'88)

$$ds_{10}^2 = ds_9^2 + g_{99}(dx^9 + v_1)^2.$$

Decompose

$$B = b_2 + b_1 \wedge (dx^9 + v_1).$$

Then Buscher's T-duality rule gives

$$v_1' = b_1, \quad b_1' = v_1, \quad b_2' = b_2 + b_1 \wedge v_1.$$

Roughly,

Buscher's rule

metric ↔ B-field

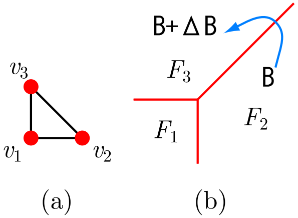
By applying T-duality along 57-directions to toric Calabi-Yau cone, the metric becomes flat and instead we have the B-field

$$\mathbf{B}^{\alpha} = \mathbf{v_1} \wedge (\mathbf{d}\theta_1^{\alpha} + \mathbf{v_1}) + \mathbf{v_2} \wedge (\mathbf{d}\theta_2^{\alpha} + \mathbf{v_1}),$$

where $\mathbf{v_1}$ and $\mathbf{v_2}$ are gauge fields in 57-directions, respectively. This B-field depends on the facet α, and jumps as we go from one facet α to another $\alpha + 1$:

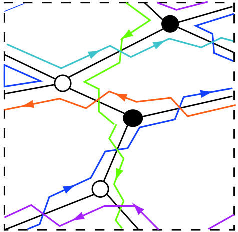
$$\mathbf{B}^{\alpha} \rightarrow \mathbf{B}^{\alpha+1} = \mathbf{B}^{\alpha} - (\mathbf{\Delta p}\ \mathbf{v_1} + \mathbf{\Delta q}\ \mathbf{v_2}) \wedge \mathbf{d}\theta_3,$$

This means the presence of NS5-brane at the intersection of NS5-branes, i.e. at the primitive normals of the toric diagram.

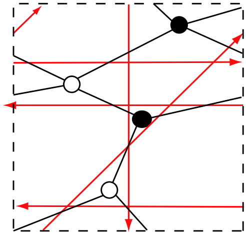


Zia-zag paths and fivebrane diagrams

Hanany-Vegh ('05) discovered a relation between 'zig-zag paths' of the bipartite graphs and the toric diagram.



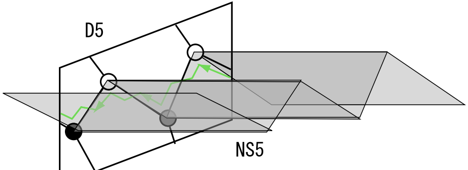
zig−zag paths



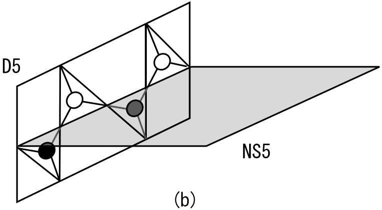
NS5−cycles

In our explanation, this is not a conjecture and automatically follows from our construction since 'zig-zag paths' is equivalent to cycles of NS5-branes.

So far as topological information of branes are concerned, we can use zig-zag paths, but the real brane configuration is different.



(a)



(b)

Notes on string coupling

Actually, our analysis so far is in the strong coupling limit.

- Real shape of branes: difficult to determine in general string coupling constant g_s (we need to solve EOM), but can be analyzed in the limit of $g_s \to 0$ and $g_s \to \infty$:

Notes on string coupling

Actually, our analysis so far is in the strong coupling limit.

- Real shape of branes: difficult to determine in general string coupling constant g_s (we need to solve EOM), but can be analyzed in the limit of $g_s \to 0$ and $g_s \to \infty$:
- (Strong coupling limit) Take $g_s \to \infty$. Then

$$T_{D5} \gg T_{NS5}$$

  and D5-branes become flat $\mathbb{T}^2$ and NS5-branes orthogonal to D5.
- (Weak coupling limit) Take $g_s \to 0$. Then

$$T_{NS5} \gg T_{D5}$$

  and N5-brane worldvolume Σ becomes a holomolophic curve $\Sigma : W(x, y) = 0$ in $\left(\mathbb{C}^\times\right)^2$,

Weak coupling limit

Consider the weak coupling limit $\mathbf{g_s} \longrightarrow \mathbf{0}$ (the decoupling limit). Then

$$\mathbf{T_{NS5} \gg T_{D5}}$$

Then NS5-brane worldvolume $\boldsymbol{\Sigma}$ is a holomolophic curve $\mathbf{W(x,y) = 0}$ in $(\mathbb{C}^\times)^2$, where

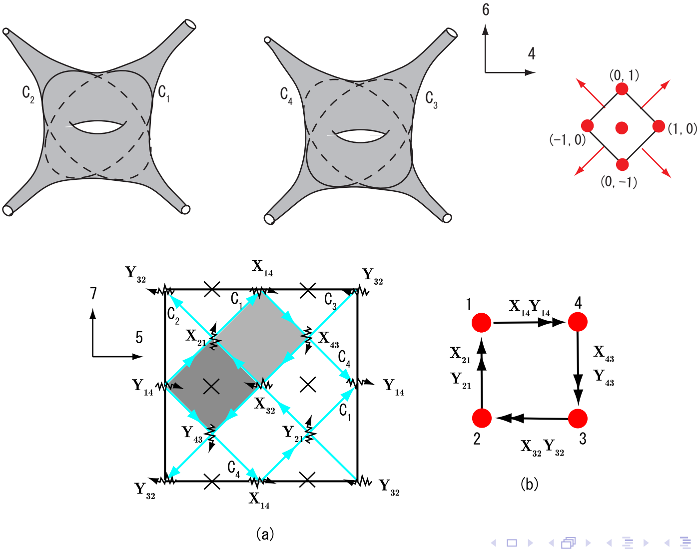
- $\mathbf{x = \exp(x_4 + ix_5), y = \exp(x_6 + ix_7)}$
- $\mathbf{W(x,y)}$ is a Newton Polynomial of the toric diagram

$$\mathbf{W(x,y) = \sum_{(i,j)\in\Delta} c_{(i,j)} x^i y^j}$$

where $\boldsymbol{\Delta} \in \mathbb{Z}^2$ is the toric diagram.

- **SU(N)** gauge groups live on D5-brane discs $\mathbf{D_a}$'s, whose intersection with NS5-brane $\mathbf{\Sigma}$ gives one-cycles $\mathbf{C_a}$ on $\mathbf{\Sigma}$.

Example: $\mathbb{P}^1 \times \mathbb{P}^1$



Reading off quiver gauge theories is completely analogous to the strong coupling case.

- D-branes $\mathbf{C_a}$: gauge groups
- intersection point of $\mathbf{C_a}$: bifundamentals
- discs surrounded by $\mathbf{C_a}$'s: term in the superpotential

In fact, we can explain everything in the weak coupling limit.

Reading off quiver gauge theories is completely analogous to the strong coupling case.

- D-branes $\mathbf{C_a}$: gauge groups
- intersection point of $\mathbf{C_a}$: bifundamentals
- discs surrounded by $\mathbf{C_a}$'s: term in the superpotential

In fact, we can explain everything in the weak coupling limit.

Problem: how to determine D5-brane cycles $\mathbf{C_a}$ from toric data?

Reading off quiver gauge theories is completely analogous to the strong coupling case.

- D-branes $\mathbf{C_a}$: gauge groups
- intersection point of $\mathbf{C_a}$: bifundamentals
- discs surrounded by $\mathbf{C_a}$'s: term in the superpotential

In fact, we can explain everything in the weak coupling limit.

Problem: how to determine D5-brane cycles $\mathbf{C_a}$ from toric data?

Answer: topological date of $\mathbf{C_a}$'s are given by the method of untwisting, relating weak coupling to strong coupling

Untwisting

Advantages and disadvantages

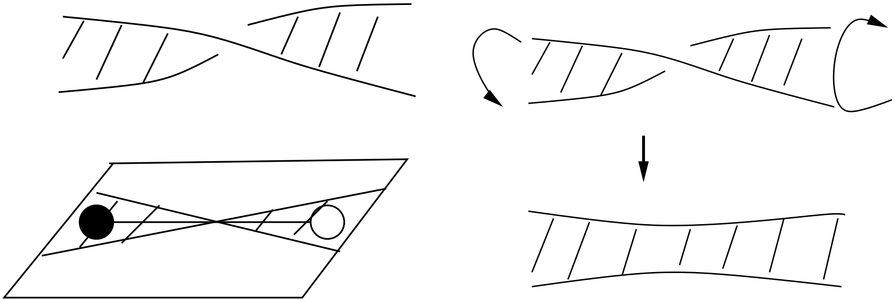
- Strong coupling limit: direct relation with toric data
- Weak coupling limit: gauge theory meaning clear

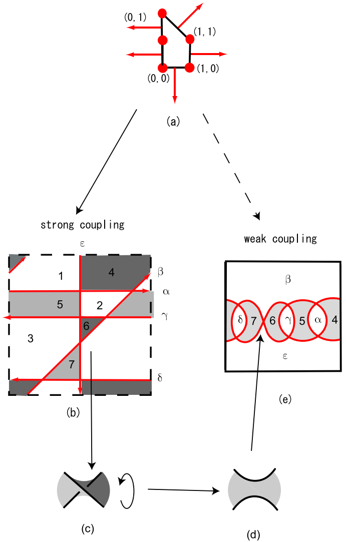
Untwisting

Advantages and disadvantages

- Strong coupling limit: direct relation with toric data
- Weak coupling limit: gauge theory meaning clear

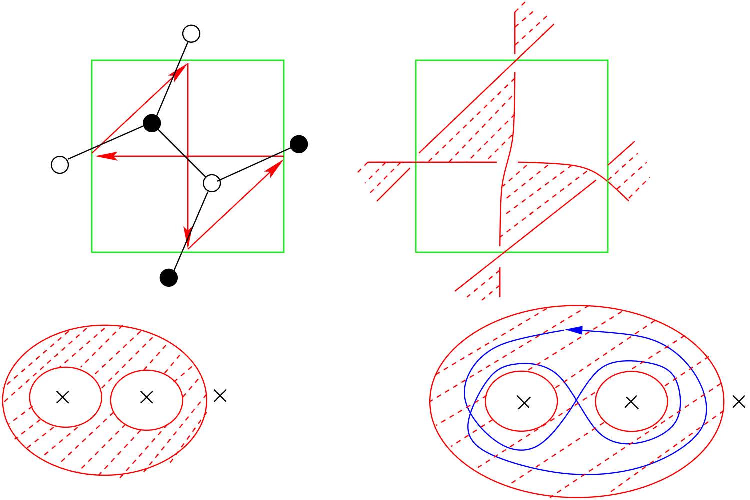
There exists a method to relate weak coupling to strong coupling, which is known as untwisting [Feng-He-Kennaway-Vafa '05].



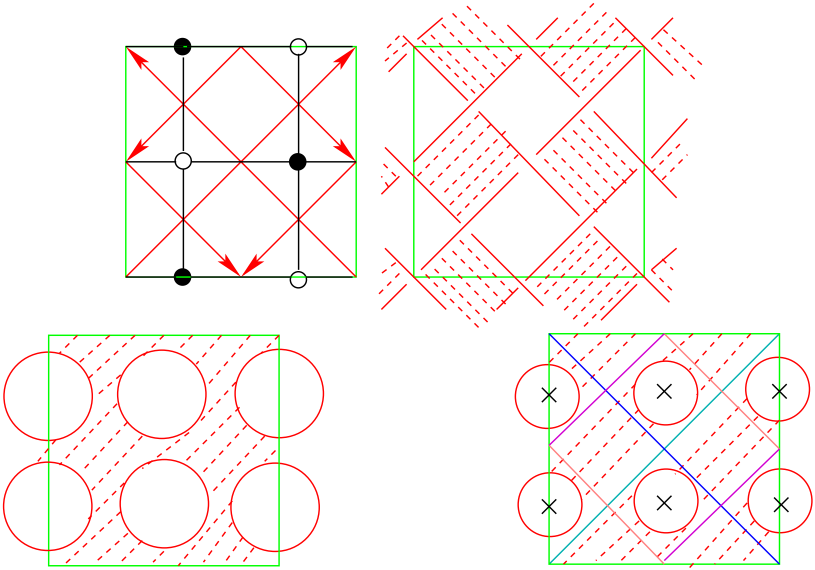


- The zig-zag path, or the winding cycle of $\mathbb{T}^2$ in bipartite graph, is turned into punctures of $\mathbf{\Sigma}$ in the weak coupling limit. Physically, this corresponds to semi-infinite cylinder of NS5-brane.
- The face of bipartite graph on $\mathbb{T}^2$ in the strong coupling limit is mapped to a disc $\mathbf{D_a}$ bounding 1-cycle $\mathbf{C_a}$ of $\mathbf{\Sigma}$. Physically, this corresponds to a stack of **N** D5-branes.

Example of untwisting: $\mathbb{C}^3$



Example of untwisting: $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$



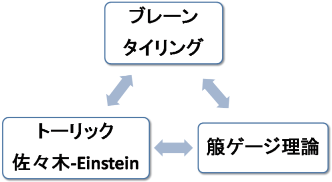
Plan

AdS/CFT

- AdS/CFT correspondence: IIB on $\mathbf{AdS_5} \times \mathbf{X_5}$ ($\mathbf{X_5}$: Sasaki-Einstein manifold) is dual to 4d $\boldsymbol{\mathcal{N} = 1}$ superconformal quiver gauge theories. [Morrirson-Plesser, Acharya-Fugyerioa-O'Farrill-Hull-Spence '98]

AdS/CFT

- AdS/CFT correspondence: IIB on $\mathbf{AdS_5} \times \mathbf{X_5}$ ($\mathbf{X_5}$: Sasaki-Einstein manifold) is dual to 4d $\mathcal{N} = \mathbf{1}$ superconformal quiver gauge theories. [Morrirson-Plesser, Acharya-Fugyerioa-O'Farrill-Hull-Spence '98]



- Brane tiling give the relation between gravity side (toric Sasaki-Einstein manifold) and gauge theory (quiver) for infinitely many examples.

Of course, this is NOT a derivation of AdS/CFT! We should try to verify the proposal.
Difficulty: superconformal field theory is strongly coupled

One possible check: use [Gubser '98]

$$\mathrm{vol}(X_5) = \frac{\pi^3}{4}\frac{1}{a}$$

where **a** is the central charge of 4d superconformal quiver gauge theory.

'proven' for arbitary toric Calabi-Yau [Butti-Zaffaroni '05]

- volume: computed the volume minimization (localization) [Martelli-Sparks '05, '06]
- central charge: computed by a-maximization [Intriligator-Wecht '03]

Equivalence with mirror Calabi-Yau with intersecting D6-branes

Take T-duality along 8-directions. Then

D5-brane $\longrightarrow$ D6-brane
NS5-brane $\longrightarrow$ mirror Calabi-Yau

here NS5-brane surface $\mathbf{\Sigma : W(x,y) = 0}$ $(\mathbf{x}, \mathbf{y} \in \mathbb{C}^\times)$ is replaced by

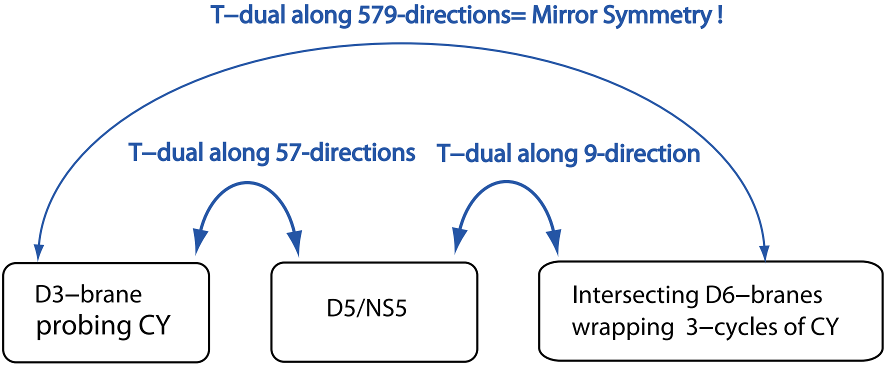
$$\mathbf{W(x,y) = uv},$$

with $\mathbf{u}, \mathbf{v} \in \mathbb{C}$, which is the mirror Calabi-Yau [Hori-Iqbal-Vafa '00]
Thus we have intersecting D6-brane systems with mirror Calabi-Yau [Feng-He-Kennaway-Vafa '05]

(Homological) mirror symmetry

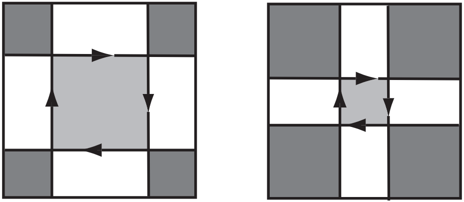
- (Homological) Mirror Symmetry: useful for studying Calabi-Yau manifolds themselves. Completely rigorous mathematical proof is given from physics intuition [Ueda-M.Y. '06, '07]

- (Homological) Mirror Symmetry: useful for studying Calabi-Yau manifolds themselves. Completely rigorous mathematical proof is given from physics intuition [Ueda-M.Y. '06, '07]



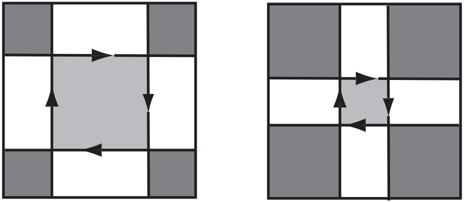
Deformation of brane systems and marginal deformation of $\mathcal{N} = 1$ superconformal quiver gauge theories

- The degrees of freedom of deformation of branes have a physical meaning as the marginal deformation of corresponding superconformal quiver gauge theory [Imamura-Isono-Kimura-M.Y. '07]



Deformation of brane systems and marginal deformation of $\mathcal{N} = 1$ superconformal quiver gauge theories

- The degrees of freedom of deformation of branes have a physical meaning as the marginal deformation of corresponding superconformal quiver gauge theory [Imamura-Isono-Kimura-M.Y. '07]



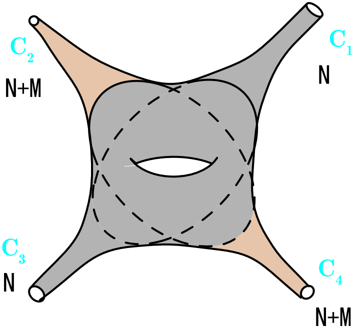
- This is probably the first discussion of real branes, not just graphs

More precisely,

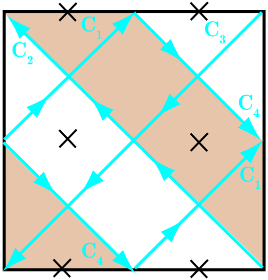
$$\begin{aligned} \text{diagonal gauge coupling} &\leftrightarrow \mathbf{C_{57} + ie^{-\phi}} \\ \beta\text{-like deformation} &\leftrightarrow \mathbf{C + iB_{57}} \\ \text{other } \mathbf{d-3} \text{ marginal deformations} &\leftrightarrow \text{geometric deformation of branes} \end{aligned}$$

Fractional branes

We can change the rank of the gauge groups by including fractional branes.



(a)

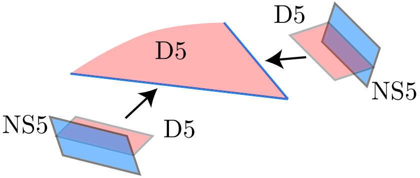


(b)

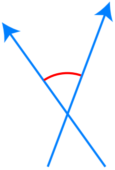
Anomaly cancellation in gauge theory follows from charge conservation [Imamura '06]

Flavor Branes

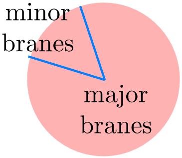
We can also consider flavor branes, which are D7-branes wrapping 4-cycles of Calabi-Yau in D3-brane setup. By T-duality, they become D5-branes spaned between two NS5-branes. [Imamura-Kimura-M.Y. '08]



(a)



(b)



(c)

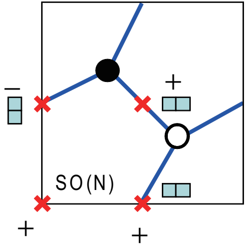
Orientifold of brane tilings

We can also include orientifold planes [Franco et. al. '07]. If you preserve SUSY then you can include O5-planes and O7-planes

	0	1	2	3	4	5	6	7	8	9
D5	○	○	○	○		○		○		
NS5	○	○	○	○		Σ				
O5	○	○	○	○	○		○			
O7	○	○	○	○	○			○	○	○

Consider the case of O5-planes, which are represented as points on $\mathbb{T}^2$.

Gauge groups and matter representations are determined from 'T-parities' (different from RR-charges) of O5-planes



- $+$ on edge: symm. repr. ☐☐
- – on edge: anti-symm. repr. ⊟
- $+$ on face: **SO(N)**
- – on face: **SP(N/2)**

Many rules to determine T-parities were proposed by [Franco et. al. '07], and derived and elaborated by [Imamura-Kimura-M.Y. '08]

Charge conservation of branes and cancellation of gauge anomalies

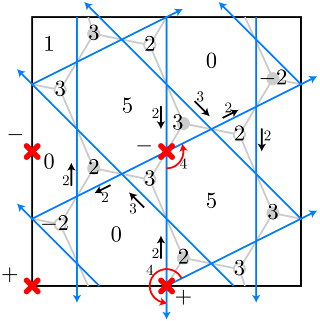
- Orientifolded theories often have anomalies since

$$\mathbf{d}_{\tiny\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}} = (\mathbf{N}-4)\mathbf{d}_{\square}, \quad \mathbf{d}_{\square\square} = (\mathbf{N}+4)\mathbf{d}_{\square}.$$

  and thus we have to include flavor branes.
- We have shown [Imamura-Kimura-M.Y. '08] that anomaly cancellation in gauge theory is derived from the condition of charge conservation in string theory

Models of Dynamical SUSY breaking

You can make models of dynamical SUSY breaking (e.g. (3,2)-model [Affleck-Din-Seiberg]) from string theory [Franco et. al, '07]!



You need to consider "flow of NS5-brane charge" [Imamura-Kimura-M.Y. '08]!

Summary

- Brane tilings are physical brane systems consisting of D5-branes and NS5-branes. Part of their information is concisely summarized as by the fivebrane diagram, or a bipartite graph on $\mathbb{T}^2$.

Summary

- Brane tilings are physical brane systems consisting of D5-branes and NS5-branes. Part of their information is concisely summarized as by the fivebrane diagram, or a bipartite graph on $\mathbb{T}^2$.
- On the D5-brane worldvolume, we have 4d $\mathcal{N} = 1$ superconformal quiver gauge theory.

Summary

- Brane tilings are physical brane systems consisting of D5-branes and NS5-branes. Part of their information is concisely summarized as by the fivebrane diagram, or a bipartite graph on $\mathbb{T}^2$.
- On the D5-brane worldvolume, we have 4d $\mathcal{N} = 1$ superconformal quiver gauge theory.
- By T-duality, brane tilings are equivalent to toric Calabi-Yau cones probed by D3-branes.

Summary

- Brane tilings are physical brane systems consisting of D5-branes and NS5-branes. Part of their information is concisely summarized as by the fivebrane diagram, or a bipartite graph on $\mathbb{T}^2$.
- On the D5-brane worldvolume, we have 4d $\mathcal{N} = 1$ superconformal quiver gauge theory.
- By T-duality, brane tilings are equivalent to toric Calabi-Yau cones probed by D3-branes.

Brane tilings are powerful because they apply to arbitrary toric Calabi-Yaus, and thus to a large class of quiver gauge theories.

Generalizations:

- Fractional branes
- Flavor branes
- Orientifold ...

Applications:

- AdS/CFT correspondence
- mirror symmetry ...

Outlook

- Real meaning of dimers (they are exactly solvable in 2d!)? Partion function of dimer model, perfect matchings?
- Dynamics of $\mathcal{N} = \mathbf{1}$ SYM, such as gaugino condensation, χSB, confinement? a-maximization?

Outlook: relation with many topics

- Model building [Wijnholt, Uranga and many other people,...; also Berenstein-Pinansky '06]: it is possible to realize models almost equivalent to standard models and GUT. Yukawa couplings from D-brane instantonts? (although probably F-theory is better)
- Mathematical applications mirror symmetry, Sasaki-Einstein geometry
- Tropical geometry, amoeba: appears for example in vortex-instanton systems in 5d N=1 SYM [Fujimori-Nitta-Ohta-Sakai-M.Y. '08], string junctions [Lunin '08]
- Relation with dimer models in crystal melting (topological A-model)? Should be related [Feng-He-Kennaway-Vafa '05], but how exactly? Relation with Donaldson-Thomas invariant [Szendroi '07]
- Generalization to non-toric case, such as non-toric del Pezzos?
- Generalization to CY4 (AdS4/CFT3?) Relation with BLG theory? [S. Lee and collaborators '06-'07, cf. recent paper by Hosomichi et. al.]

⋮

What can be learn from “brane tilings” in the future?

The strong coupling limit:

$$\mathbf{g_{str}} \to \infty, \mathbf{l_s} \to \mathbf{0}, \mathbf{R} \to \mathbf{0}, \quad \text{with } \frac{\mathbf{R^2}}{\mathbf{g_{str}} \mathbf{l_s^2}} \text{ kept fixed } . \tag{1}$$

The weak coupling limit:

$$\mathbf{g_{str}} \to \mathbf{0}, \mathbf{l_s} \to \mathbf{0}, \mathbf{R} \to \mathbf{0}, \quad \text{with } \frac{\mathbf{R^2}}{\mathbf{g_{str}} \mathbf{l_s^2}} \text{ kept fixed } . \tag{2}$$