4d Superconformal Index on Lens Space

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4d Superconformal Index on Lens Space

(Benini - Nishioka - Y
1109.0283
+ work in progress)

4d Superconformal Index ①

on Lens Space ②

4d SCI : index on $S^1 \times S^3$

supercharge

$$I = \mathrm{Tr}\left[(-1)^F e^{-\beta\{\overset{\swarrow}{Q}, Q^\dagger\}} \qquad\qquad\qquad\qquad \right]$$

4d SCI : index on $S^1 \times S^3$

fugacities

$$I = I(t, y)$$

$$= \mathrm{Tr}\left[(-1)^F e^{-\beta\{Q, Q^\dagger\}} t^{\underbrace{2(E+j_2)}} y^{\underbrace{2j_1}}\right]$$

Operators commuting with Q

4d SCI : index on $S^1 \times S^3$

fugacities

$$I = I(t, y)$$

energy

$$= \mathrm{Tr}\left[(-1)^F e^{-\beta\{Q, Q^\dagger\}} t^{2(E+j_2)} y^{2j_1}\right]$$

spin under

$\mathrm{Spin}(4) \simeq SU(2) \times SU(2)$

lens space $L(p,q)$ (p,q : coprime)

$$\boxed{L(p,q) = S^3/\mathbb{Z}_p}$$

$$S^3 \ni \left\{ (z_1, z_2) \in \mathbb{C}^2 \,\middle|\, |z_1|^2 + |z_2|^2 = 1 \right\}$$

$$\left[\begin{array}{l} (z_1, z_2) \sim (w^q z_1, w^{-1} z_2) \\ w^p = 1 \end{array} \right]$$

lens space $L(p,q)$ $(p,q$: coprime$)$

$$\boxed{L(p,q) = S^3/\mathbb{Z}_p}$$

ji ⤳

q=1

$$\begin{tikzcd}\mathbb{Z}_p \curvearrowright S^1 \arrow[r] & S^3 \arrow[d] \\ & S^2\end{tikzcd}$$

Hopf fibration

$$\boxed{L(p,1) \xrightarrow{p\to\infty} S^2, \quad L(1,1) = S^3}$$

motivation?

Motivation 1 : "unification"

$I_{4d}[S^1 \times S^3]$

KMMR ('05)
Romelsberger ('05)

$I_{3d}[S^1 \times S^2]$

Kim ('09)
Imamura-Yokoyama ('11)

$Z_{3d}[S^3_b]$ ('09)

Kapustin-Willett-Yaakov
Hama-Hosomichi-Lee ('11)

Motivation 1 : "unification"

$$\begin{tikzcd}
I_{4d}[S^1 \times S^3] \arrow[ddr, "S^1 \to b"] & \\
I_{3d}[S^1 \times S^2] & \\
& Z_{3d}[S^3_b]
\end{tikzcd}$$

Dolan Spiridonov Vartanov
Gadde Yan ('10)
Imamura

Motivation 1 : "unification"

$$\begin{tikzcd}
 & & I_{4d}[S^1 \times L(p,1)] \arrow[dll, bend right, "p=1"'] \\
I_{4d}[S^1 \times S^3] \arrow[drrr, bend left, "S^1 \to 0"] & & & \\
I_{3d}[S^1 \times S^2] & & & Z_{3d}[S^3_b]
\end{tikzcd}$$

Motivation 1 : "unification"

$$\begin{tikzcd}
 & I_{4d}[S^1 \times L(p,1)] \arrow[dl, bend right, "p=1"'] \arrow[ddr, "S^1 \to 0"] & \\
I_{4d}[S^1 \times S^3] \arrow[ddrr, "S^1 \to 0"] & & \\
 & & Z_{3d}[L(p,1)_b] \arrow[d, "p=1"] \\
I_{3d}[S^1 \times S^2] & & Z_{3d}[S^3_b]
\end{tikzcd}$$

Motivation 1 : "unification"

$$\begin{tikzcd}
 & I_{4d}[S^1 \times L(p,1)] \arrow[dl, bend right, "p=1"'] \arrow[dr, "S^1 \to 0"] \arrow[ddl, "p\to\infty \ \text{Benini Nishioka Y}"] & \\
I_{4d}[S^1 \times S^3] \arrow[drr, "S^1 \to 0"] & & Z_{3d}[L(p,1)_b] \arrow[d, "p=1"] \\
I_{3d}[S^1 \times S^2] & & Z_{3d}[S^3_b]
\end{tikzcd}$$

Motivation 1 : "unification"

$$\begin{tikzcd}
 & I_{4d}[S^1 \times L(p,1)] \arrow[dl, "p=1"'] \arrow[dr, "S^1 \to 0"] \arrow[ddl, "p \to \infty"] & \\
I_{4d}[S^1 \times S^3] \arrow[drr, "S^1 \to 0"] & & Z_{3d}[L(p,1)_b] \arrow[d, "p=1"] \arrow[ddl, "p \to \infty"'] \\
I_{3d}[S^1 \times S^2] \arrow[dr, "S^1 \to 0"'] & & Z_{3d}[S^3_b] \\
 & Z_{2d}[S^2] &
\end{tikzcd}$$

Y, in progress

Motivation 2 : new identities

$$\Gamma(z; x, y) = \prod_{j,k \geq 0} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}$$

Motivation 2 : new identities

$$\Gamma_{p,q,M}(z;x,y) = \prod_{\substack{j,k\geq 0 \\ qj-k\equiv M \pmod{p}}} \frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k}$$

the result

(for $L(p, q = 1)$)

$$I_{S^1 \times L(p,q)}(t,y)$$

$$= \sum_m \int da\, e^{iB_m(a;t,y)}$$

$$\times \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \hat{I}_m(e^{ina}; t^n, y^n)\right]$$

$$I_{S^1 \times L(p,q)}(t,y)$$

$$= \sum_{m} \int da\, e^{iB_m(a;t,y)} \times \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \hat{I}_m\left(e^{ina}; t^n, y^n\right)\right]$$

integral over the Cartan

$$I_{S^1\times L(p,q)}(t,y)$$

$$=\sum_m \int da\, e^{iB_m(a;t,y)}$$

$$\times \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\hat{I}_m(e^{ina};t^n,y^n)\right]$$

"plethystics form"

$$\Gamma_{p,q,M}(z;x,y)$$

$$I_{S^1 \times L(p,q)}(t,y)$$

$$= \sum_{m} \int da \, e^{iB_m(a;t,y)}$$

$$\times \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \hat{I}_m(e^{ina}; t^n, y^n) \right]$$

Sum over holonomies

$$V = (\omega^{m_1}, \omega^{m_2}, \cdots \omega^{m_r}) \quad \left[\begin{array}{c} \pi_1(S^3/\mathbb{Z}_p) = \mathbb{Z}_p \\ \omega^p = 1 \end{array} \right]$$

zero-point contribution

$$I_{S' \times L(p,q)}(t,y)$$

$$= \sum_m \int da\, e^{iB_m(a;t,y)}$$

$$\times \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \hat{I}_m(e^{ina}; t^n, y^n) \right]$$

$$\hat{I} = \hat{I}_{\text{vect}} + \hat{I}_{\text{chiral}}$$

$$\hat{I}_{\text{vect}} = \sum_{\rho \in \text{Adj}} \left[(t^6 - 1) F_{p,q} + \delta_{M,0} \right] e^{i\rho(a)}$$

$$\hat{I}_{\text{chiral}} = \sum_{\rho \in R} \left[t^{3Q} e^{i\rho(a)} - t^{6-3Q} e^{-i\rho(a)} \right] F_{p,q}$$

$$\hat{I} = \hat{I}_{vect} + \hat{I}_{chiral}$$

$$\hat{I}_{vect} = \sum_{\rho \in Adj \,\leftarrow\, \text{repr.}} \left[(t^{6}-1) F_{\rho,q,M} + \delta_{M,0} \right] e^{i\rho(a)}$$

$$\hat{I}_{chiral} = \sum_{\rho \in R} \left[t^{\overset{\uparrow}{3Q}} e^{i\rho(a)} - t^{6-3Q} e^{-i\rho(a)} \right] F_{\rho,q,M}$$

R-charge

weight repr.

$$\left(M \equiv \rho(\underset{\uparrow\ \text{holonomy}}{m}) \mod p, \quad 0 \le M \le p-1 \right)$$

$$\hat{I}_{\text{rect}} = \sum_{\rho \in \text{Adj}} \left[(t^6-1) F_{p,q,M} + \delta_{M,0} \right] e^{i\rho(a)}$$

$$\hat{I}_{\text{chiral}} = \sum_{\rho \in R} \left[t^{3Q} e^{i\rho(a)} - t^{6-3Q} e^{-i\rho(a)} \right] F_{p,q,M}$$

$$\left(F_{p,1,M} = \frac{1}{1-t^6} \left(\frac{(t^3 y)^M}{1-(t^3 y)^p} + \frac{(t^3 y^{-1})^{p-M}}{1-(t^3 y^{-1})^p} \right) \right)$$

4d index → 3d index

$$I[S^1 \times S^3/\mathbb{Z}_p] \xrightarrow[p\to\infty]{} I[S^1 \times S^2]$$

$$\left(M \equiv \rho(m) \mod p, \quad 0 \leq M \leq p-1 \right)$$

$$\hat{I}_{\text{rect}} = \sum_{\rho \in \text{Adj}} \left[(t^6-1) F_{p,1,M} + \delta_{M,0} \right] e^{i\rho(a)}$$

$$\hat{I}_{\text{chiral}} = \sum_{\rho \in R} \left[t^{3Q} e^{i\rho(a)} - t^{6-3Q} e^{-i\rho(a)} \right] F_{p,1,M}$$

$$\left(F_{p,1,M} = \frac{1}{1-t^6} \left(\frac{(t^3 y)^M}{1-(t^3 y)^p} + \frac{(t^3 y^{-1})^{p-M}}{1-(t^3 y^{-1})^p} \right) \right)$$

$$\left(M \equiv p(m) \ \mathrm{mod}\, p, \quad 0 \leq M \leq p-1 \right)$$

$$\left(F_{p,1,M} = \frac{1}{1-t^6} \left[\frac{(t^3 y)^M}{1-(t^3 y)^p} + \frac{(t^3 y^{-1})^{p-M}}{1-(t^3 y^{-1})^p} \right] \right)$$

$$\Big\downarrow \ p \to \infty, \ y \to 1$$

$$F = \frac{1}{1-t^6} t^{3M} \qquad\qquad F \sim \frac{1}{1-t^6} t^{3(p-M)}$$

$$M \sim 0 \qquad\qquad M \sim p$$

$$\hat{I}_{\text{vect}} \to \sum_{\rho \in \text{Adj}} \left[-t^{3|\rho(m)|} + \delta_{\rho(m),0} \right] e^{i\rho(a)}$$

$$\hat{I}_{\text{hyper}} \to \sum_{\rho \in R} \frac{t^{3Q} e^{i\rho(a)} - t^{6-3Q} e^{-i\rho(a)}}{1-t^6} t^{3|\rho(m)|}$$

precisely reproduces 3d index (Imamura Yokoyama)

Sum over holonomies

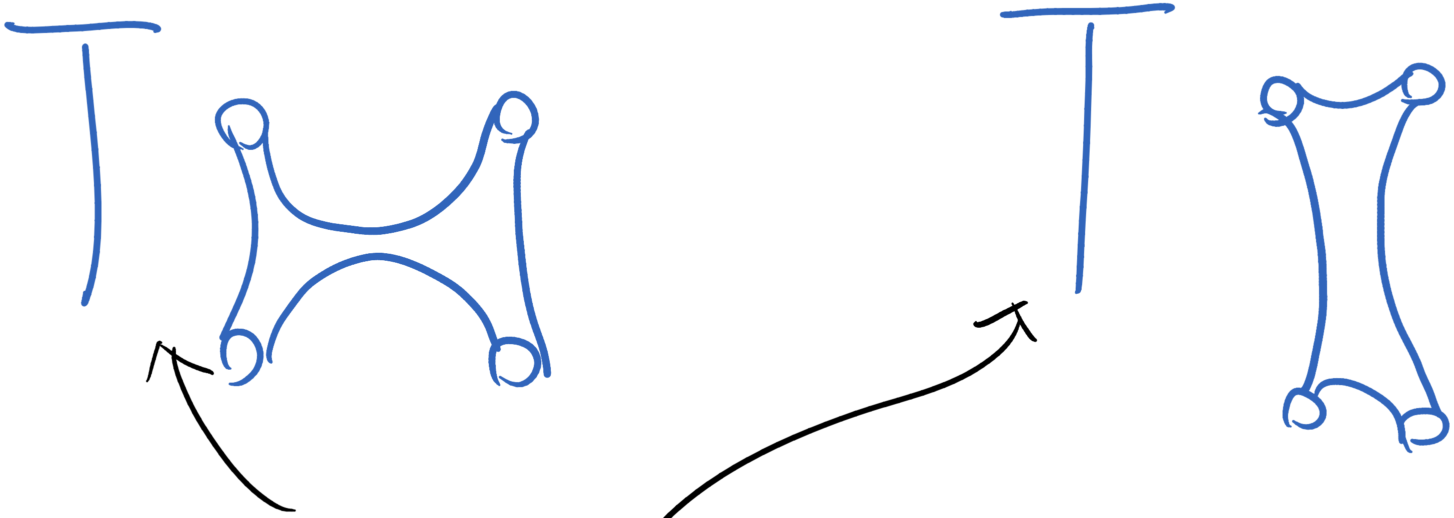
→ sum over monopole charges

Application to M5-brane theories

M5 on Σ_1

pants decomposition

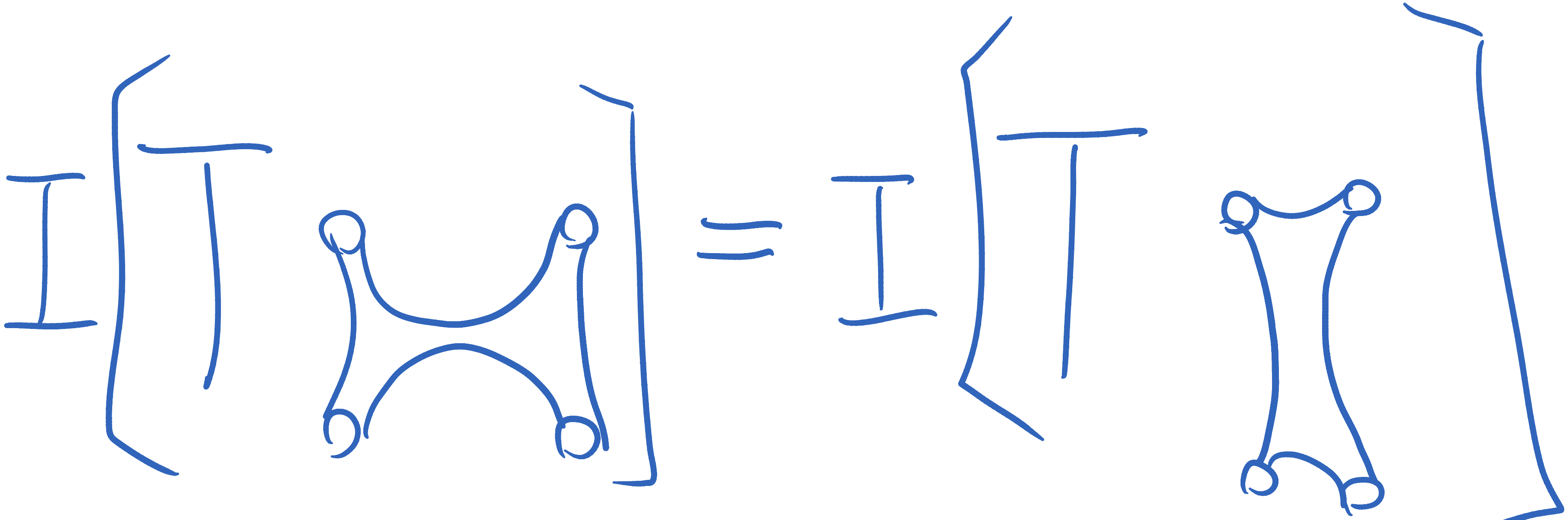
M5 on Σ,



4d $\mathcal{N}=2$ SCFT

Gaiotto

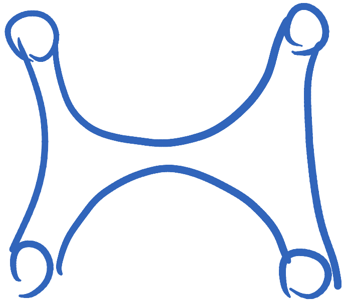
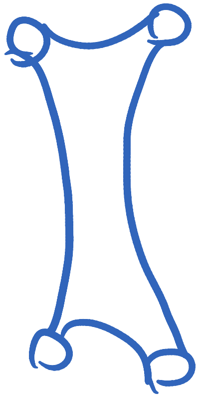
M5 on Σ,



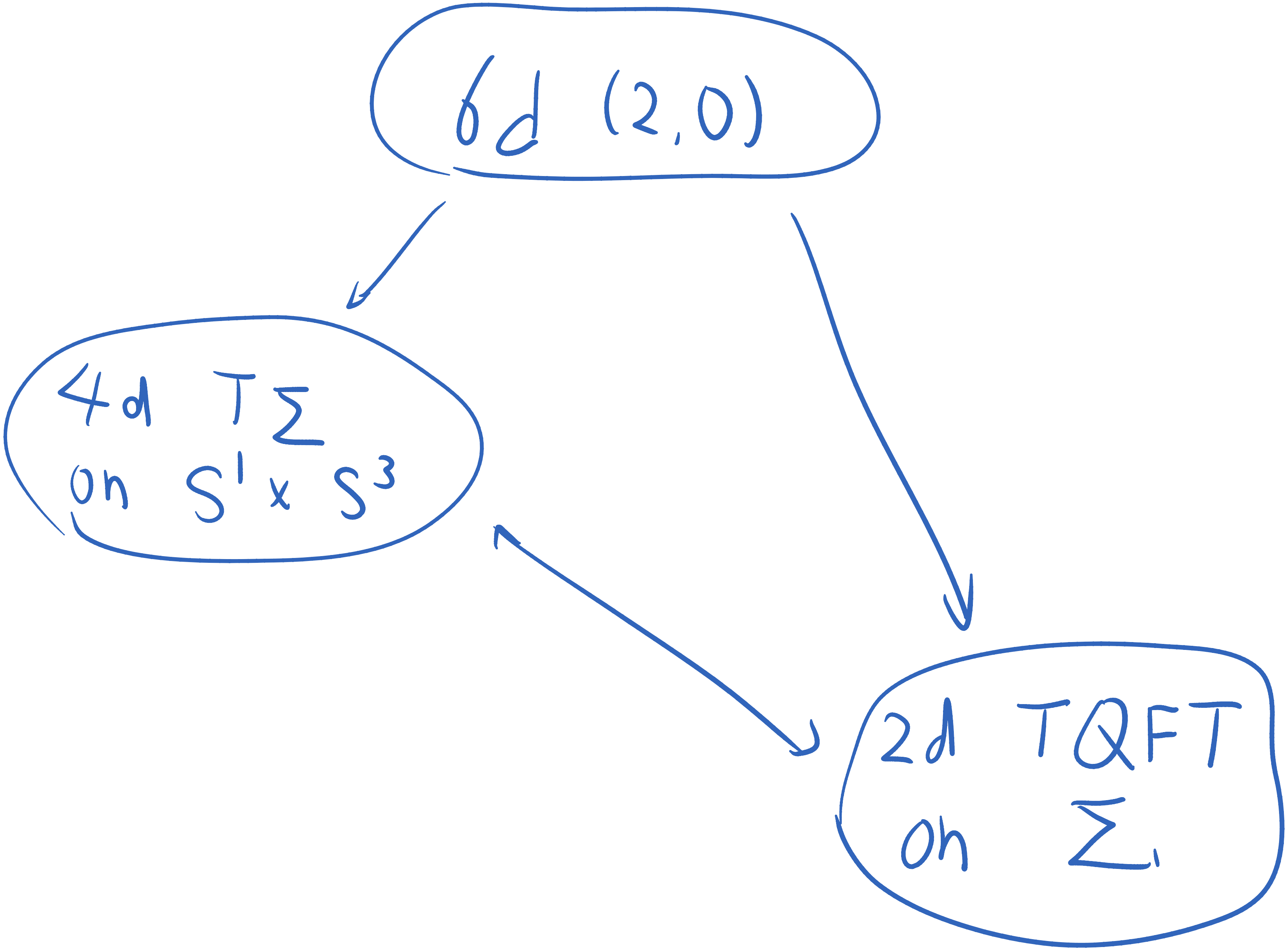
defines 2d TQFT
(proven)

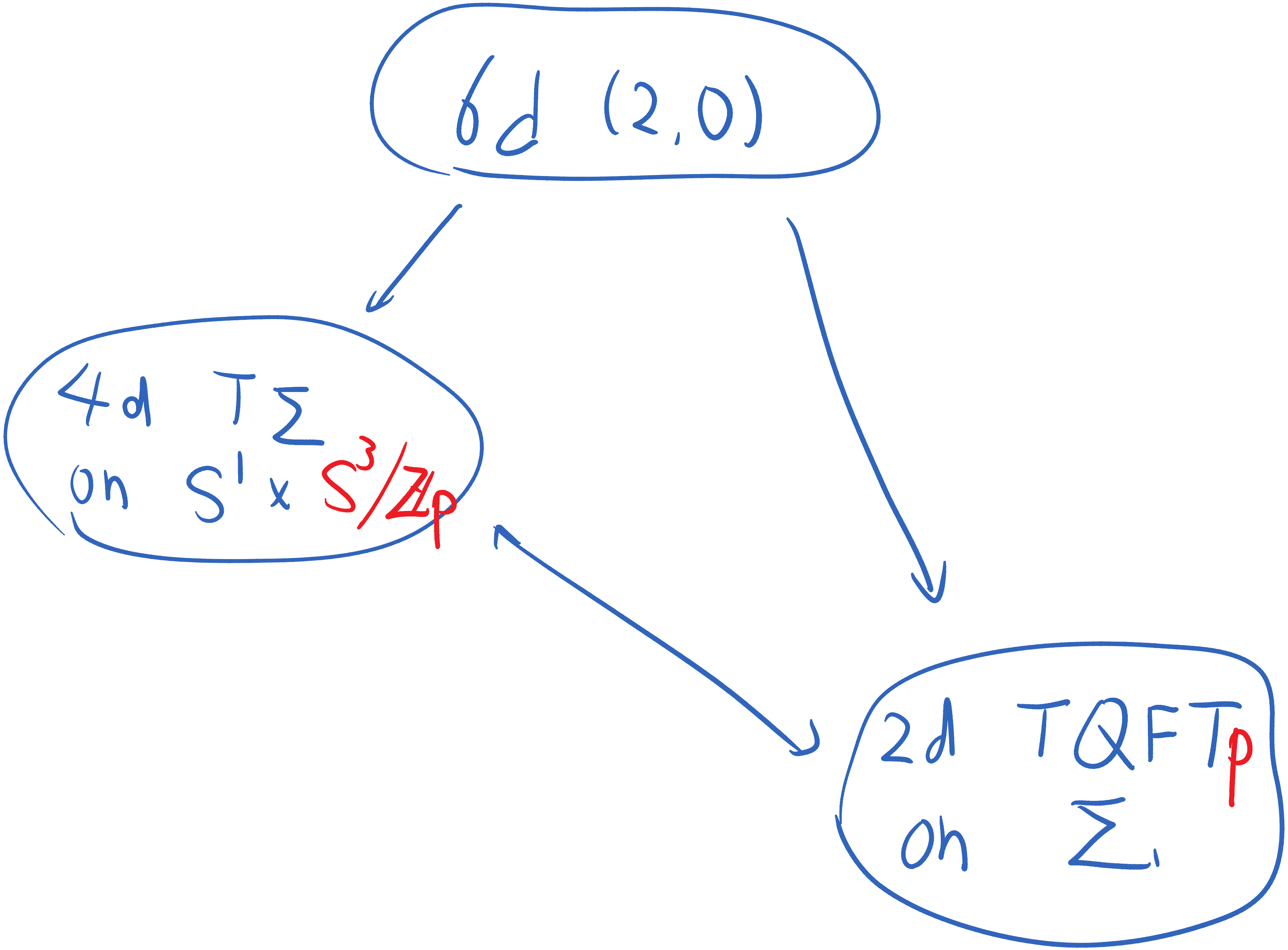
Gadde
Rastelli
Razamat
Yan

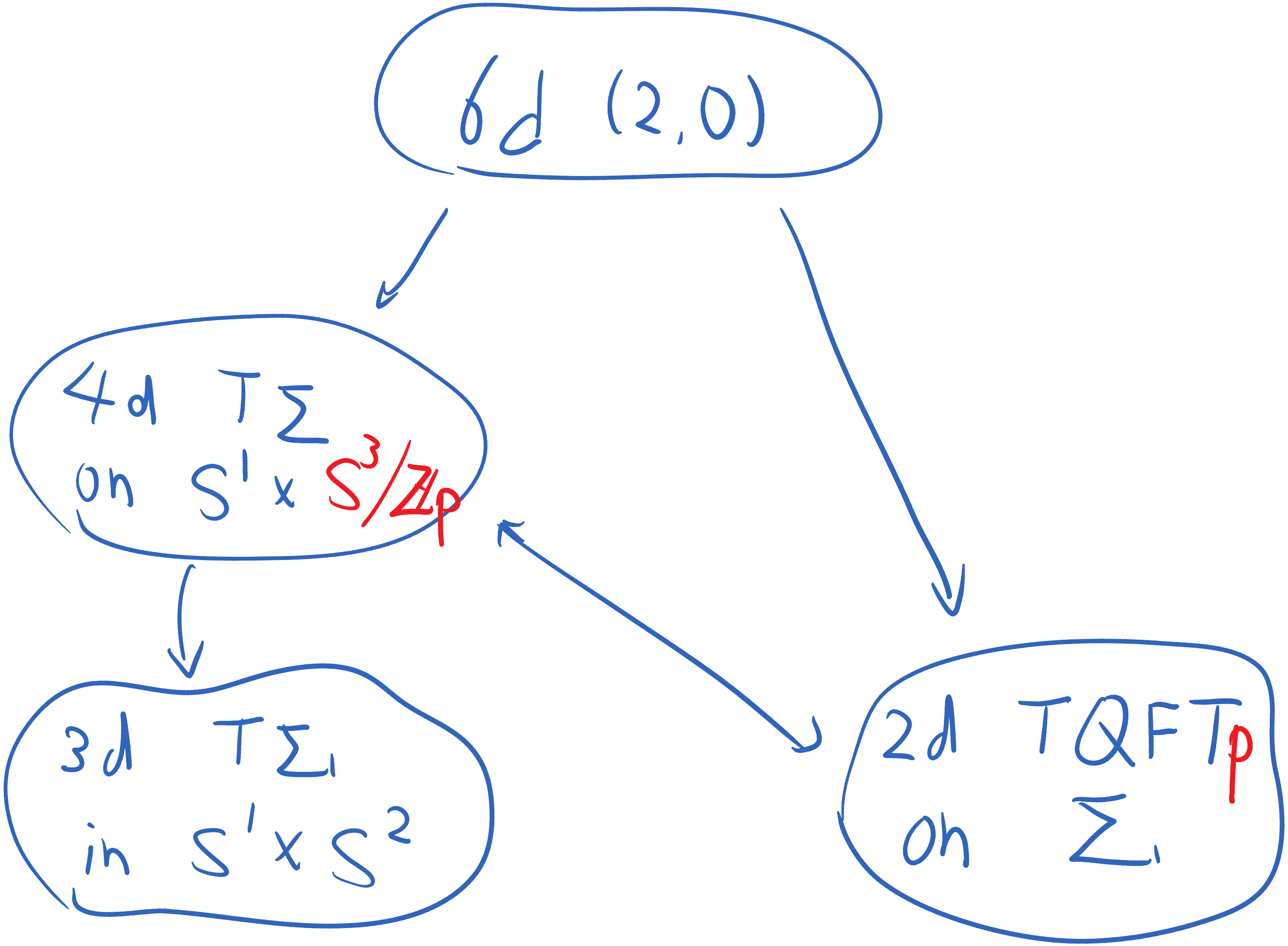
M5 on Σ,

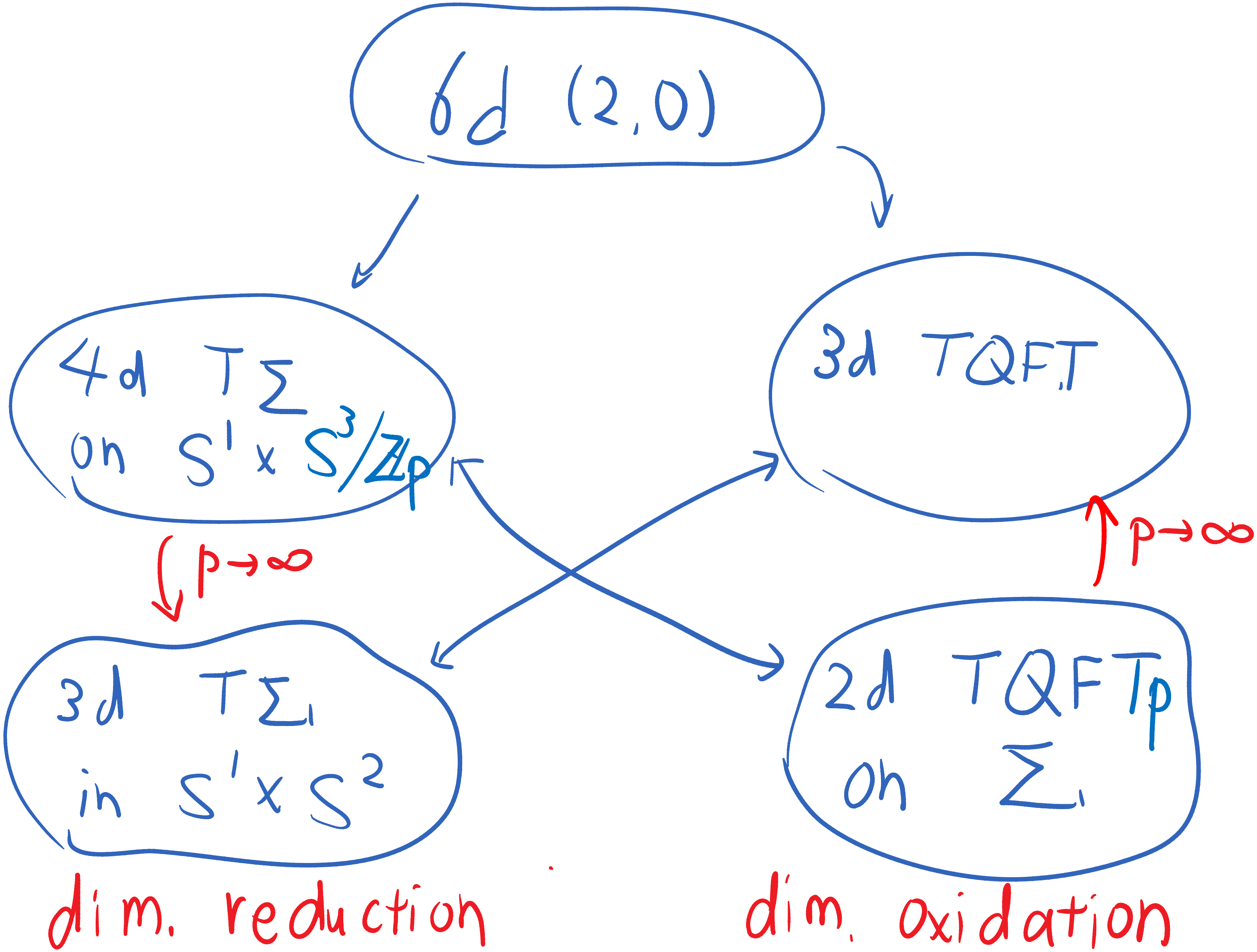
$I_{p,q}\Big[T$  $\Big] = I_{p,q}\Big[T$  $\Big]$

defines 2d TQFT
(not yet proven)


6d (2,0)
4d T_Σ on $S^1 \times S^3$
2d TQFT on Σ


6d (2,0)
4d T_Σ on $S^1 \times S^3/\mathbb{Z}_p$
2d TQFT_p on Σ,


6d (2,0)
4d T_Σ on $S^1 \times S^3/\mathbb{Z}_p$
3d T_{Σ_1} in $S^1 \times S^2$
2d TQFT$_p$ on Σ_1


6d (2,0)
4d TΣ_3 on $S^1 \times S^3/\mathbb{Z}_p$
3d TQFT
$p \to \infty$
$p \to \infty$
3d TΣ_1 in $S^1 \times S^2$
2d TQFT$_p$ on Σ_1
dim. reduction
dim. oxidation

Summary

$$\begin{tikzcd}
 & I_{3d}[S^1 \times S^2] \\
I_{4d}[S^1 \times L(p,q)] \arrow[ur] \arrow[dr] \arrow[dd, equal] & \\
 & Z_{3d}[S^3_b] \\
\sum_m \int [da]\, e^{iB_m(a)} \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \hat{I}(t^n, y^n; e^{ina})\right] &
\end{tikzcd}$$