3-manifold Cookbook from Quiver Mutations

Masahito Yamazaki (Princeton)

"Exact Results in SUSY gauge theories"

Jan/12/2013, Rikkyo Univ.

Based on

Y. Terashima + M.Y. to appear [hep-th]

also

Y. Terashima + M.Y. 1103 [hep-th]
1106

K. Nagao + Y. Terashima + M.Y. 1112 [math.AG]

D. Xie + M.Y. in progress

2004

traditions and interactions in physics/math
greater Tokyo area

e.g. "Toda Seminar"

Dec/18/2004 @ Rikkyo Univ.

"Discretization of PDEs
reproducing conservation/dissipation"

"practicalities on research on
prime factorization"

Congratulations on the new Research Center for Mathematical Physics!

Today:

a new class of 3d $\mathcal{N}=2$ theories $T[(Q,\mathfrak{m})]$

from a pair $\begin{cases} Q : \text{quiver} \\ \mathfrak{m} : \text{mutation sequence} \end{cases}$

$$\boxed{Z^{3d\ \mathcal{N}=2}_{T[(Q,\mathfrak{m})]} = Z^{\text{cluster}}_{(Q,\mathfrak{m})}}$$

$$\boxed{\begin{array}{l}\text{6d (2,0) theory on } M \text{ (3-mfd)} \\ \rightsquigarrow \text{3d } \mathcal{N}=2 \text{ theory } T[M]\end{array}}$$

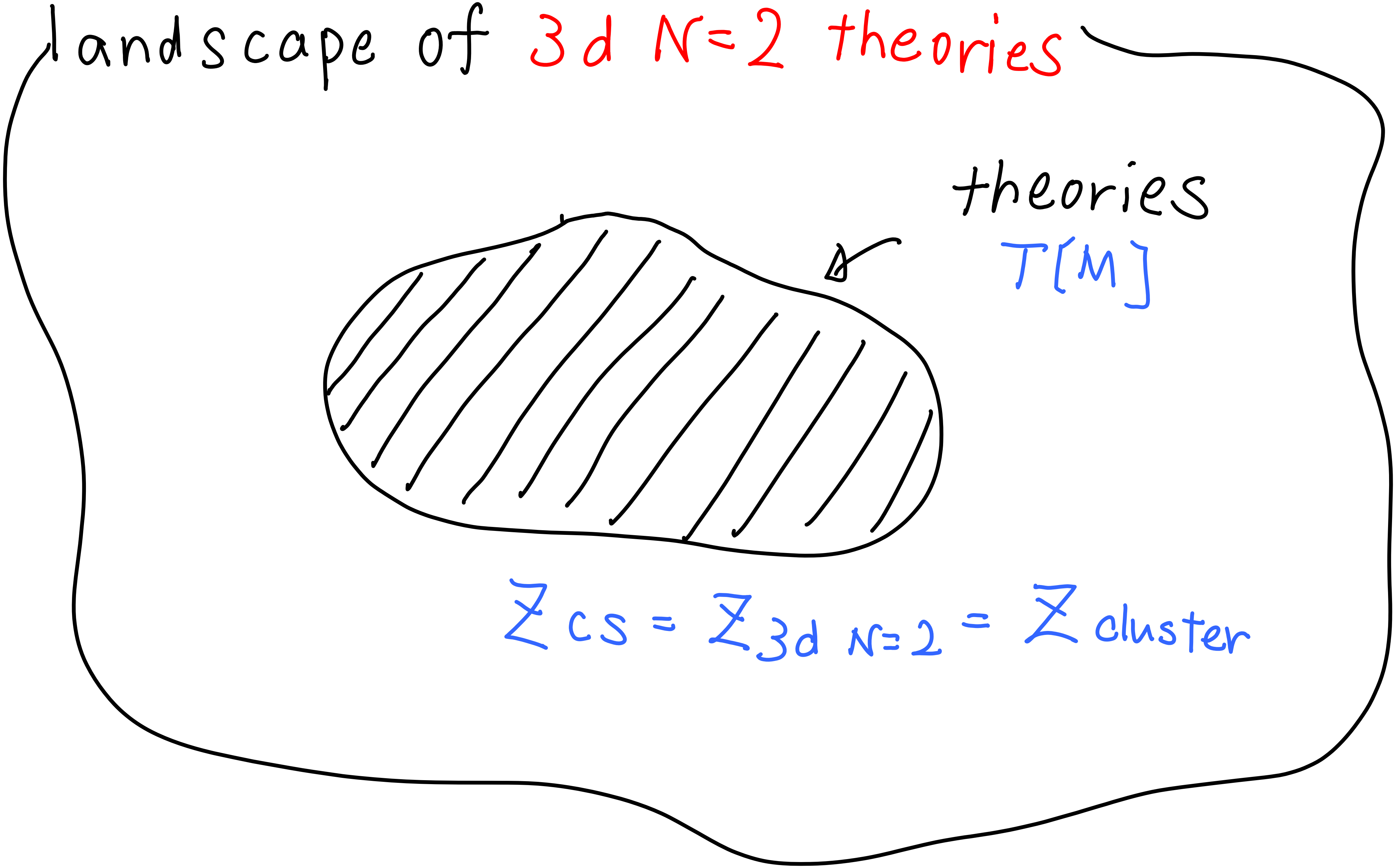
$$Z_{\text{3d } SL(2) \text{ CS}}[M] = Z^{\text{3d } \mathcal{N}=2}_{T[M]}[S^3_b]$$

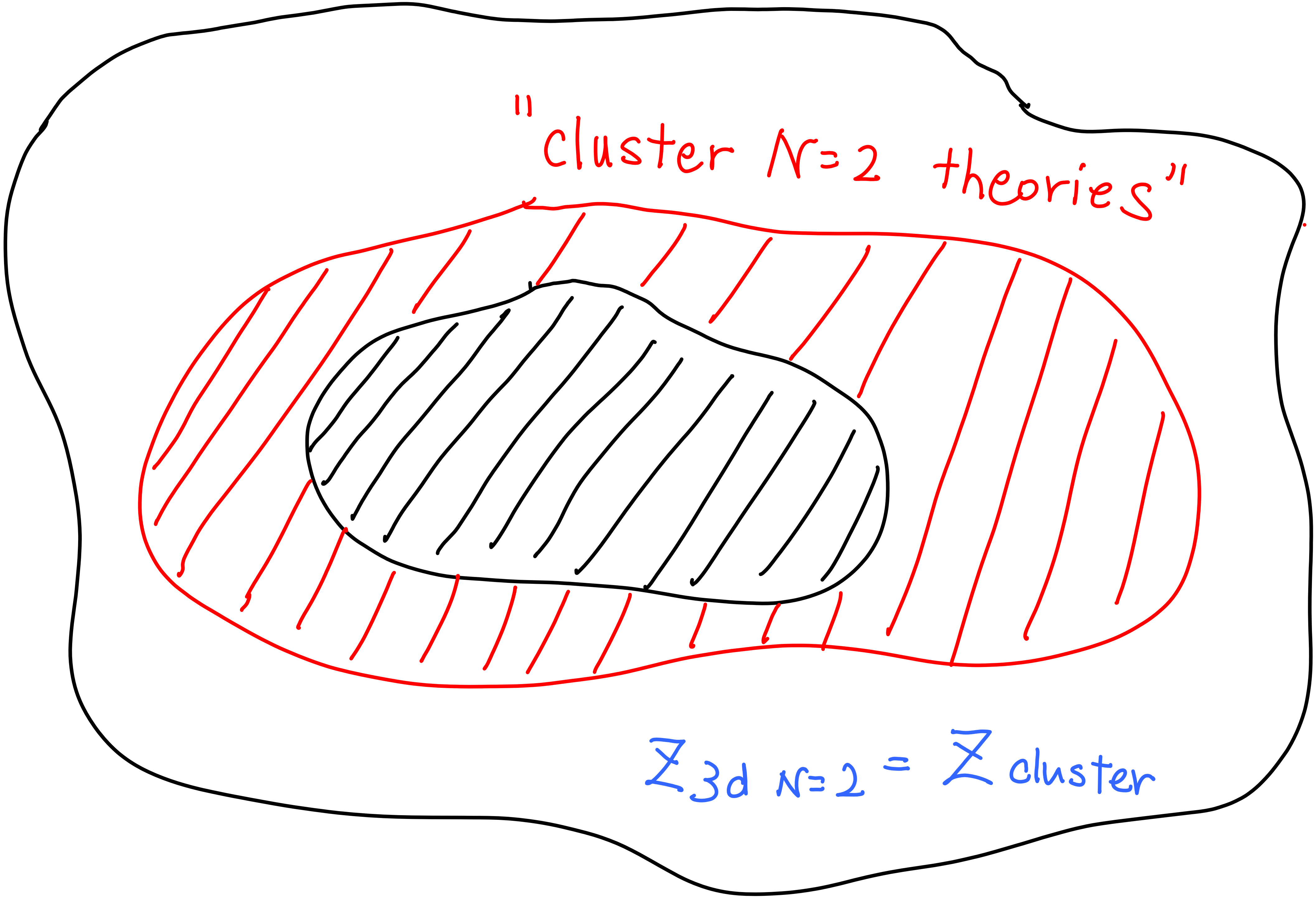
"quantum hyperbolic geometry"

link complement $S^3 \setminus L$

[Terashima-Y, Dimofte-Gukov-Gaiotto, Cecotti-Cordova-Vafa, ⋯]

earlier works by [Dimofte-Lotte-Gukov, Drukker-Gaiotto-Gomis, Hosomichi-Lee-Park, ⋯]


landscape of 3d $\mathcal{N}=2$ theories
theories
$T[M]$
$Z_{CS} = Z_{3d\ \mathcal{N}=2} = Z_{cluster}$


"cluster $\mathcal{N}=2$ theories"
$Z_{3d\,\mathcal{N}=2} = Z_{\text{cluster}}$

Plan

1. Quivers and Mutations Z_{cluster}
2. 3-manifolds $Z_{3\text{-mfd}}$
3. 3d $\mathcal{N}=2$ theories $Z_{3d\ \mathcal{N}=2}$

- Summary

1. Quivers and Mutations

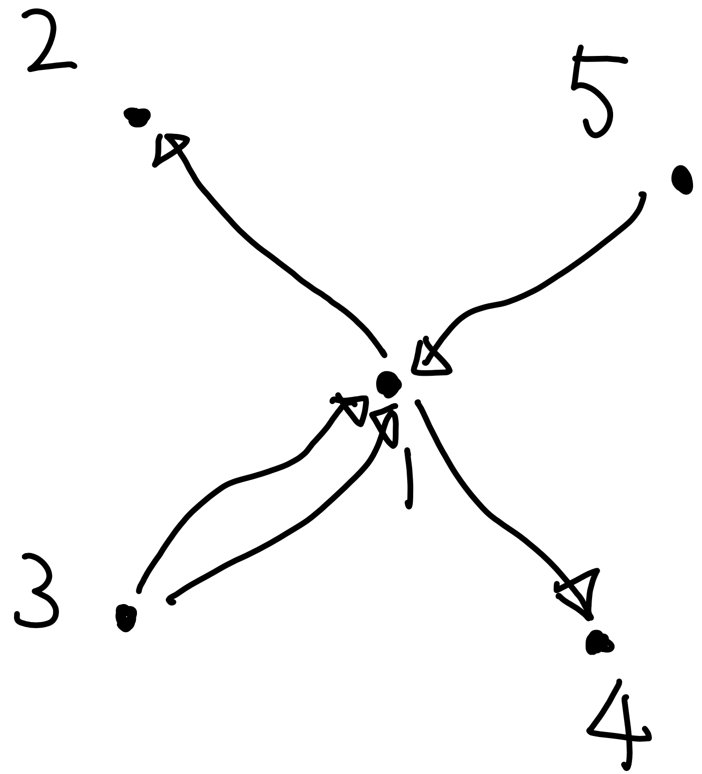
[Fomin Zelevinsky, ...
quantization by
Fock Goncharov ...]

Quiver : oriented graph Q

described by an antisymmetric matrix

$$Q_{i,j} = \#\{i \to j\} - \#\{j \to i\}$$

$i, j \in I = \{\text{vertices of } Q\}$



e.g. $Q_{12} = +1$

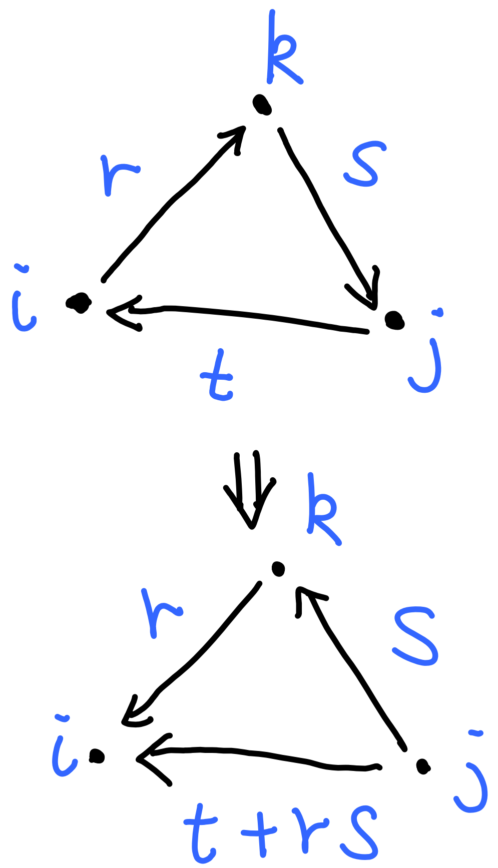
$Q_{15} = -1$

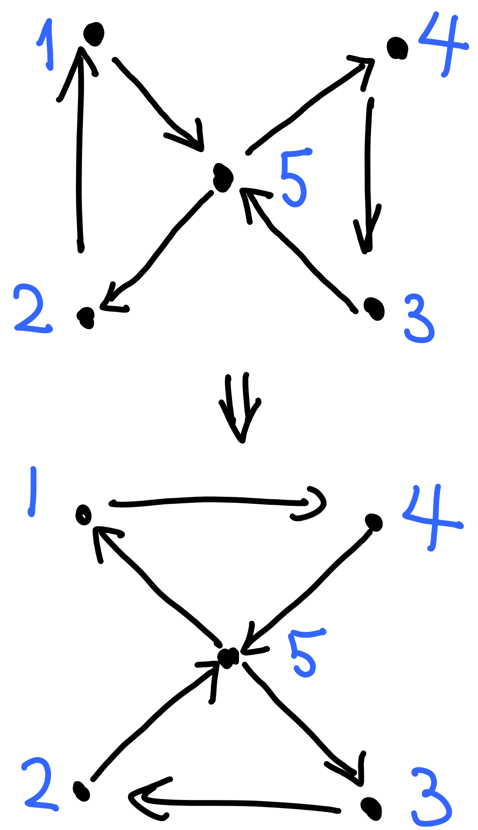
$Q_{13} = -2$

$\mu_k Q$: mutation of quiver Q at vertex k

$$(\mu_k Q)_{ij} := \begin{cases} -Q_{ij} & (i=k \text{ or } j=k) \\ Q_{ij} + [Q_{ik}]_+ [Q_{kj}]_+ - [Q_{jk}]_+ [Q_{ki}]_+ & (i,j \neq k) \end{cases}$$

$([x]_+ := \max(x, 0))$





Q : quiver

$\rightsquigarrow \mathcal{A}_Q$: space generated by y_i $(i \in I)$

with relation

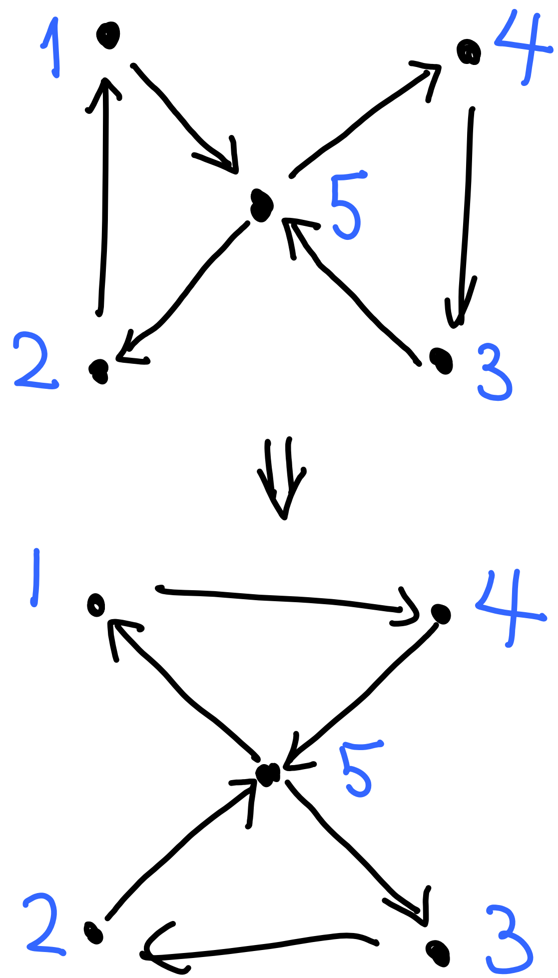
$$y_j y_i = q^{2Q_{ij}} y_i y_j$$

$$q = e^{i\hbar} = e^{i\pi b^2}$$

$$\left(\begin{array}{lll} \text{or} & [Y_i, Y_j] = 2Q_{ij} & \rightsquigarrow \quad [x_i, p_j] = i\hbar\delta_{ij} \\ & \text{for } y_i = e^{Y_i} & \quad [x_i, x_j] = [p_i, p_j] = 0 \end{array} \right)$$

has standard repr. on Hilbert space $\mathcal{H}_Q$

mutation at vertex k of quiver Q $\rightsquigarrow$ $\hat{\mu}_k : \mathcal{A}_Q \to \mathcal{A}_{\mu_k Q}$



$$
\begin{cases}
y_1' = y_1(1+q y_5) \\
y_2' = y_2(1+q y_5^{-1})^{-1} \\
y_3' = y_3(1+q y_5) \\
y_4' = y_4(1+q y_5^{-1})^{-1} \\
y_5' = y_5^{-1}
\end{cases}
$$

quiver Q + a chain of mutations

$$m = (m_1, m_2, \cdots, m_L)$$

$\rightsquigarrow$ "cluster partition function"

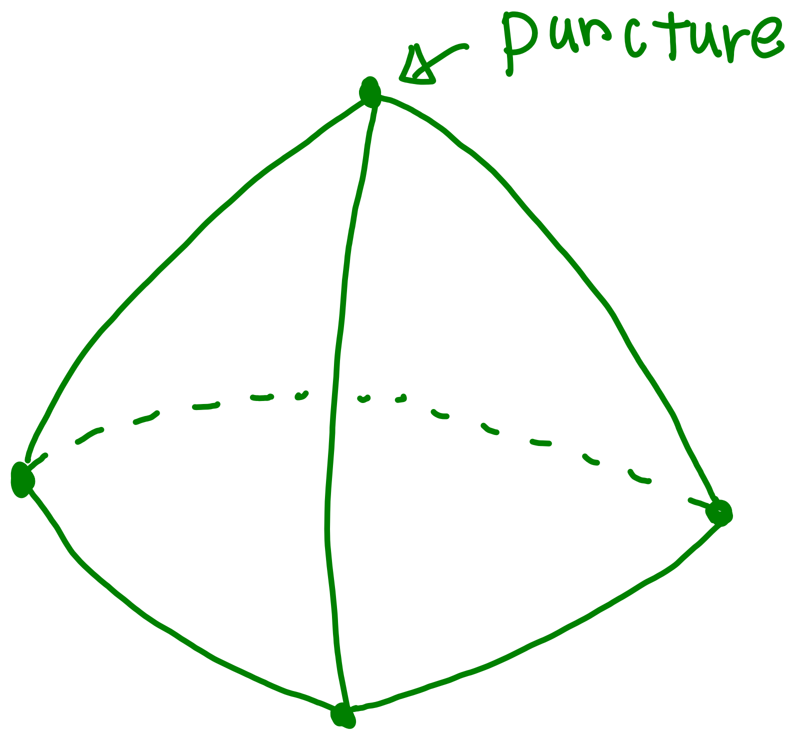
$$Z_{(Q,m)} := \langle \mathrm{in} | \hat{\mu}_{m_1} \hat{\mu}_{m_2} \cdots \hat{\mu}_{m_L} | \mathrm{out} \rangle$$

$$|\mathrm{in}\rangle \in \mathcal{H}_Q$$

$$|\mathrm{out}\rangle \in \mathcal{H}_{\mu_L \cdots \mu_1 Q}$$

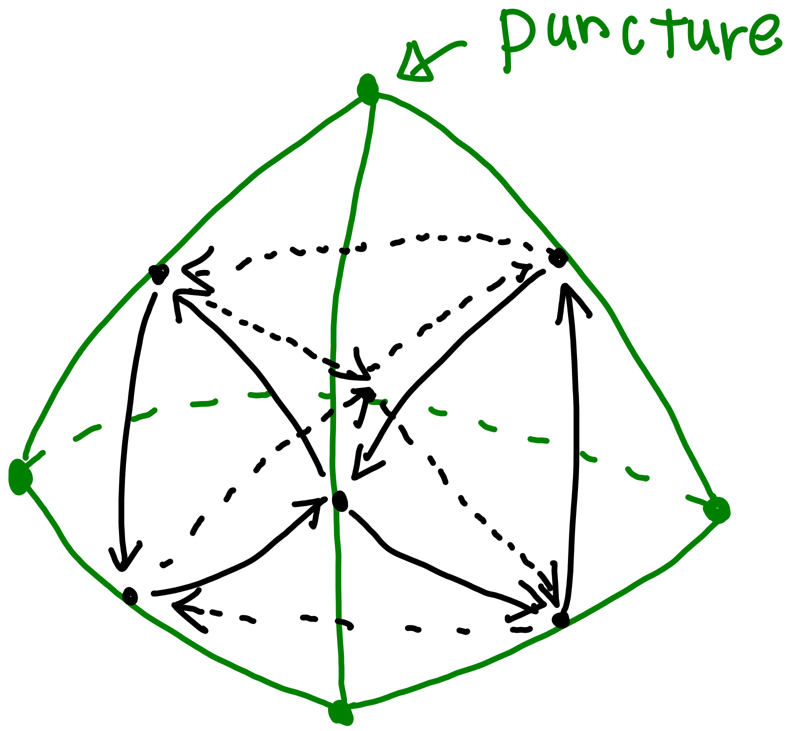
2. 3-manifolds

Σ : 2d punctured Riemann surface
w/ an ideal triangulation ($\chi<0$)

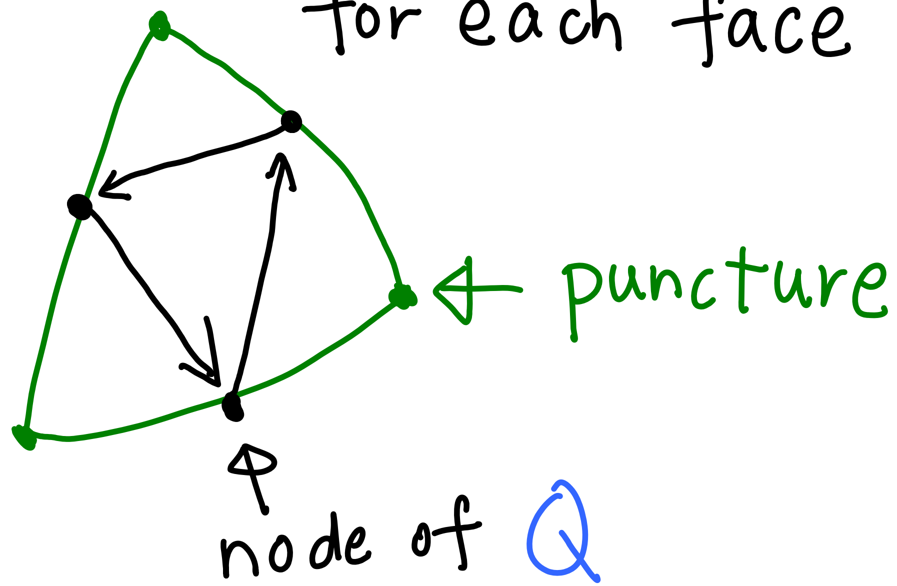


Σ : 2d punctured Riemann surface
w/ an ideal triangulation ($\chi < 0$)

$\rightsquigarrow$ quiver Q

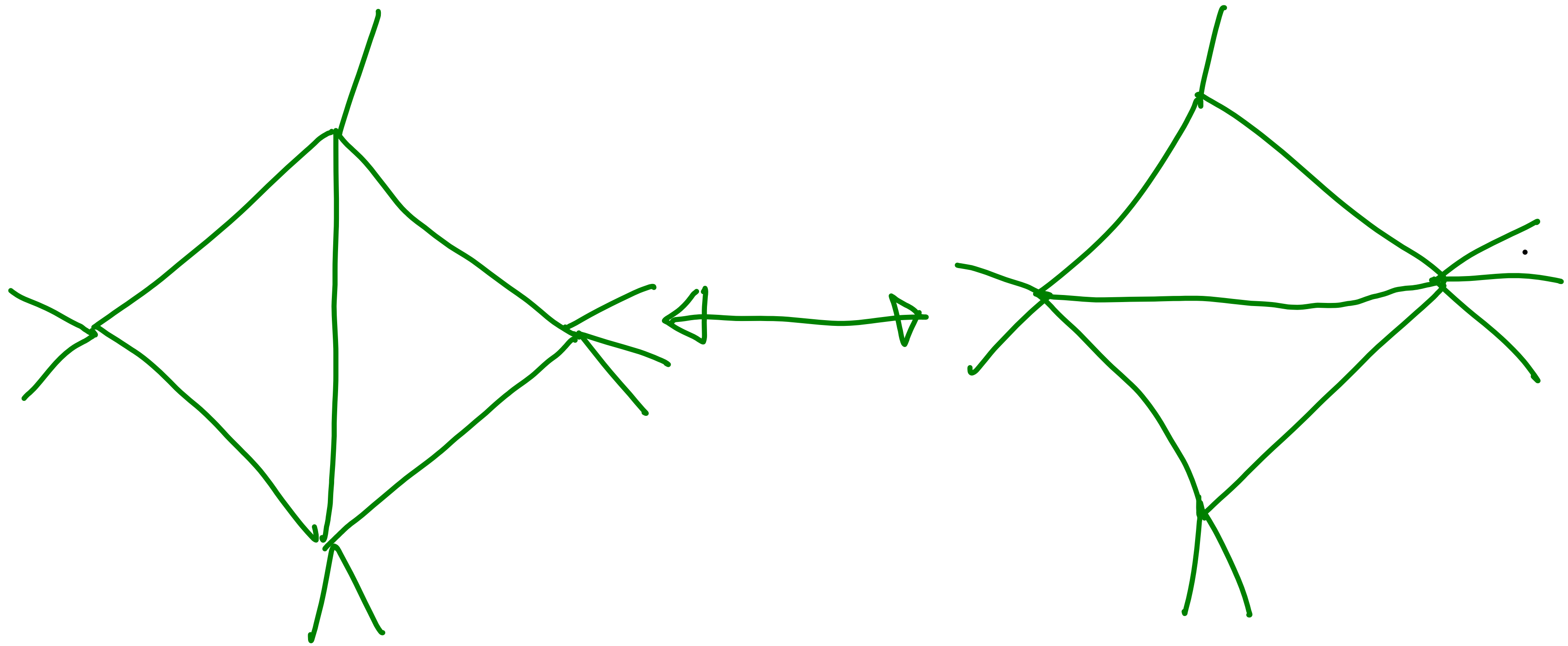


associate a quiver
for each face



change of triangulation

$\rightsquigarrow$ a chain of flips on triangulation

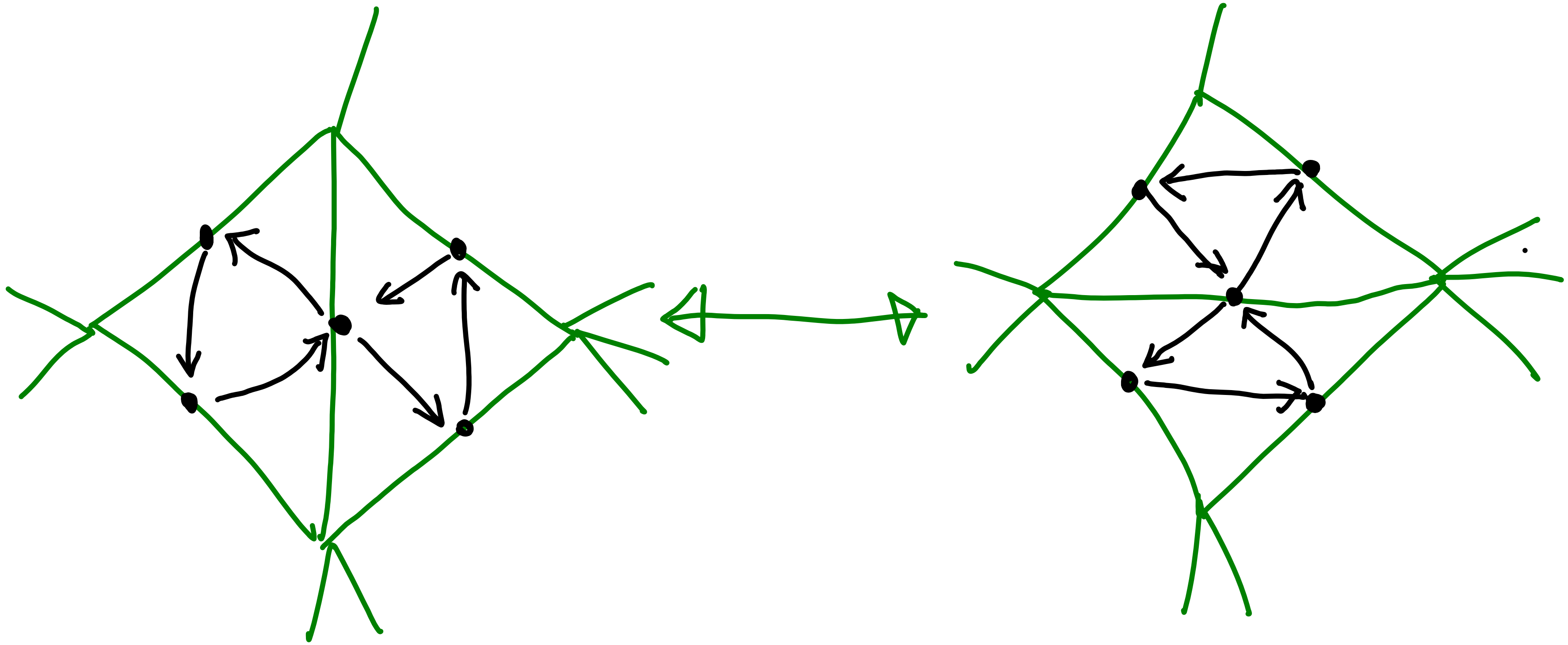


change of triangulation

$\rightsquigarrow$ a chain of flips on triangulation

$\qquad\qquad\qquad\quad\ \ \ \parallel$

a chain of mutations on quiver



$\mathcal{A}_Q$, $\mathcal{H}_Q$ $\rightsquigarrow$ quantum Teichmüller space $\mathcal{A}_\Sigma, \mathcal{H}_\Sigma$

$m \rightsquigarrow$ a sequence of flips

repr. by $\varphi \in \dot{M}CG(\Sigma)$

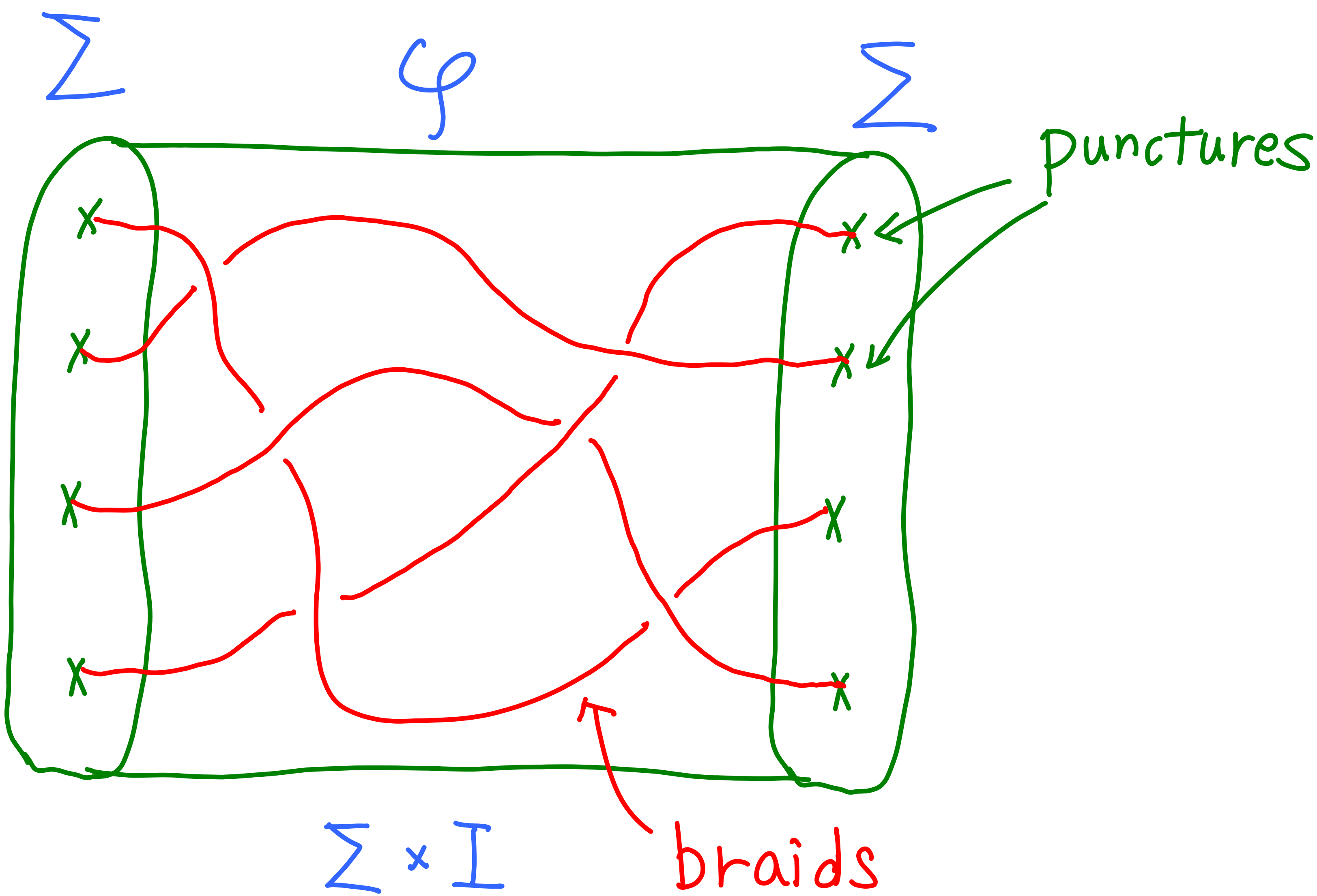
$\hat{\varphi} \in \mathrm{Aut}(\mathcal{H}_\Sigma)$

$Z_{(Q,m)} \rightsquigarrow Z(\Sigma, \varphi) = \langle in | \hat{\varphi} | out \rangle$

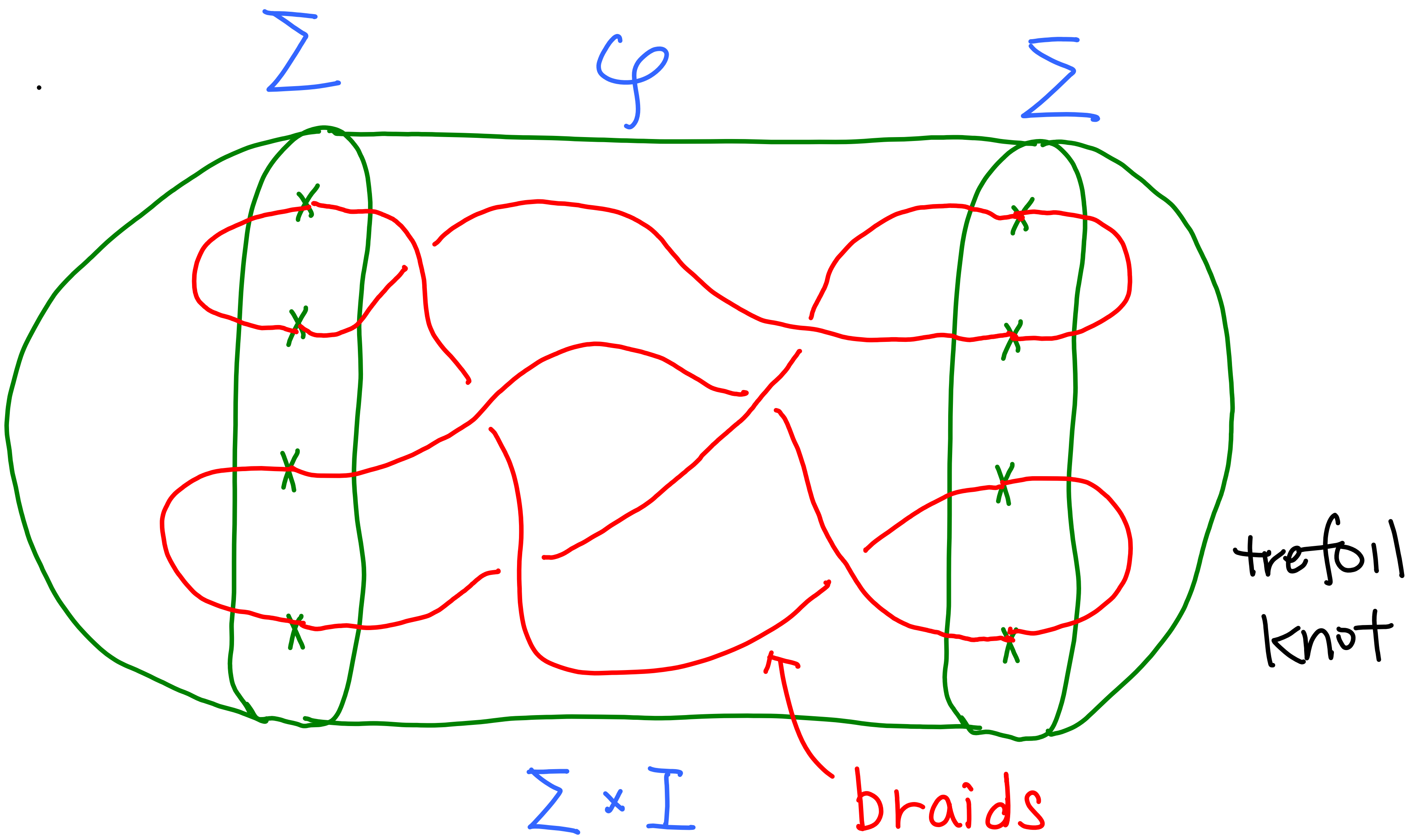
$|in\rangle, |out\rangle \in \mathcal{H}_\Sigma$

- (Σ, φ) defines a 3-mfd M

(mapping cylinder)



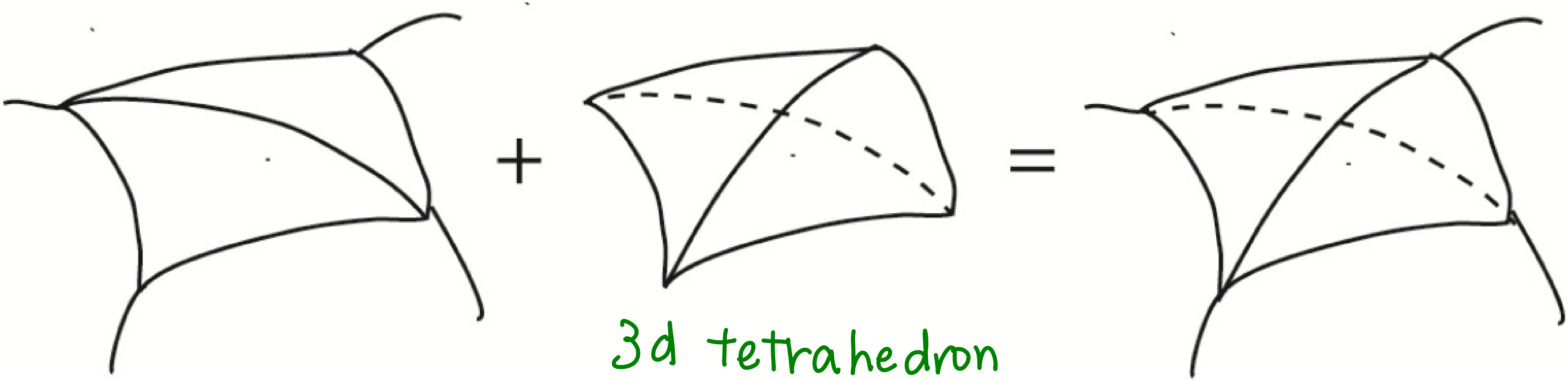
We obtain an arbitrary link in S^3



(2n-plat representation)

3-manifold M has

canonical 3d ideal triangulation



2d triangulation + 3d tetrahedron = another 2d triangulation

flip of 2d triangulation

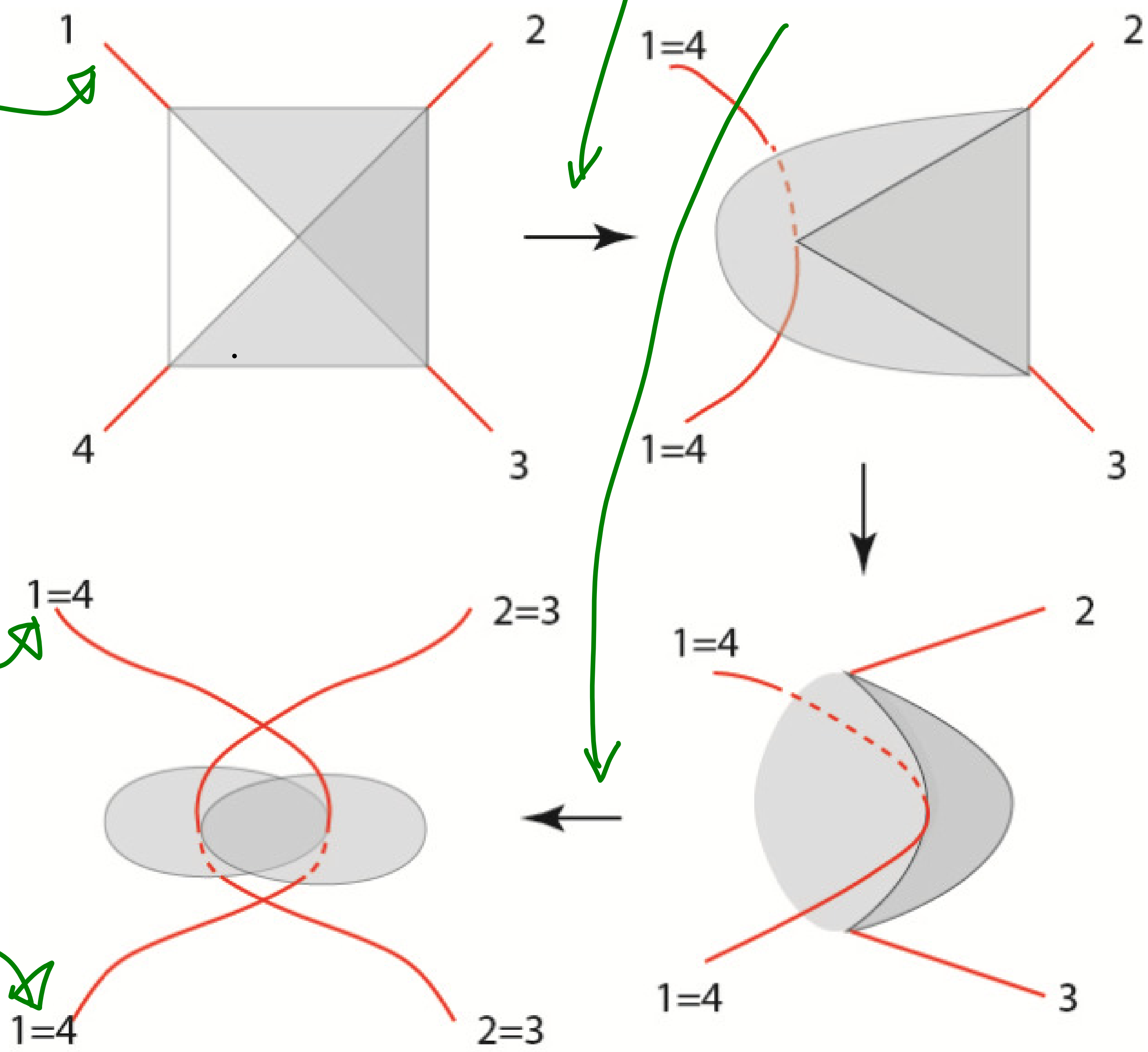
= 3d tetrahedron

[Sakuma-Weeks, Terashima-Y]

identify 2 faces

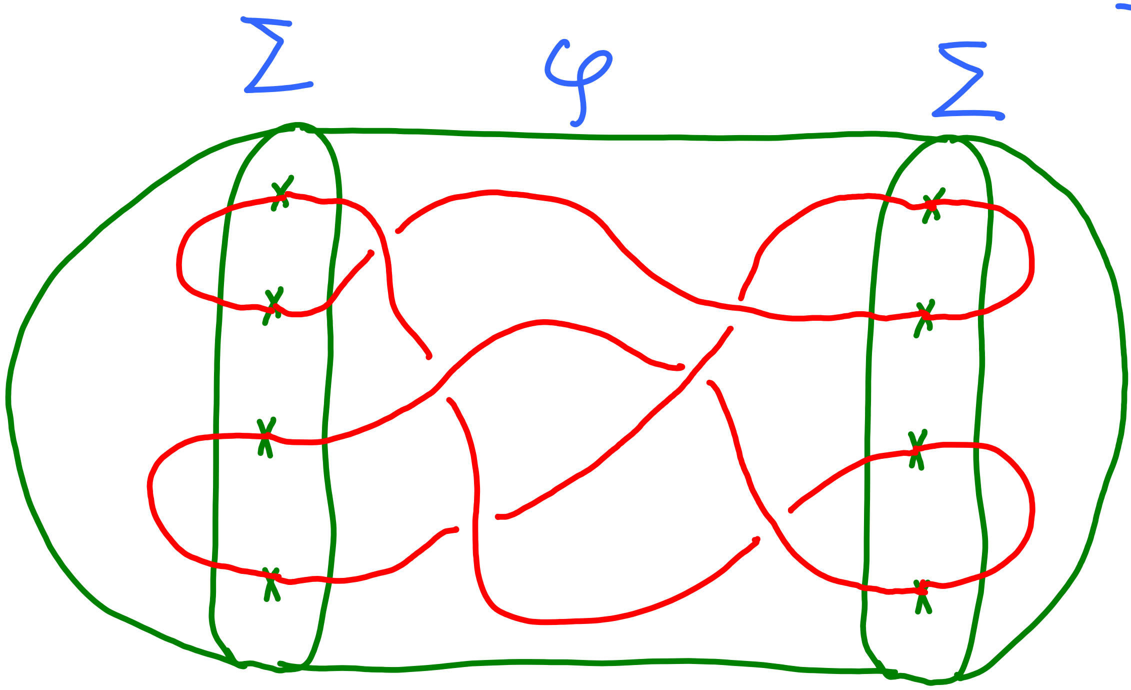
Puncture = braid

braids identified



$$Z_{(Q,m)} = Z_{(\Sigma,\varphi)} = \langle \text{in} | \hat{\varphi} | \text{out} \rangle$$

$$= Z_{SL(2)\,CS}$$



$$\underbrace{}_{\langle \text{in} |} \quad \underbrace{}_{\hat{\varphi}}^{\Sigma \times I} \quad | \text{out} \rangle \; .$$

$\mathcal{M}^{\text{flat}}_{SL(2)}$

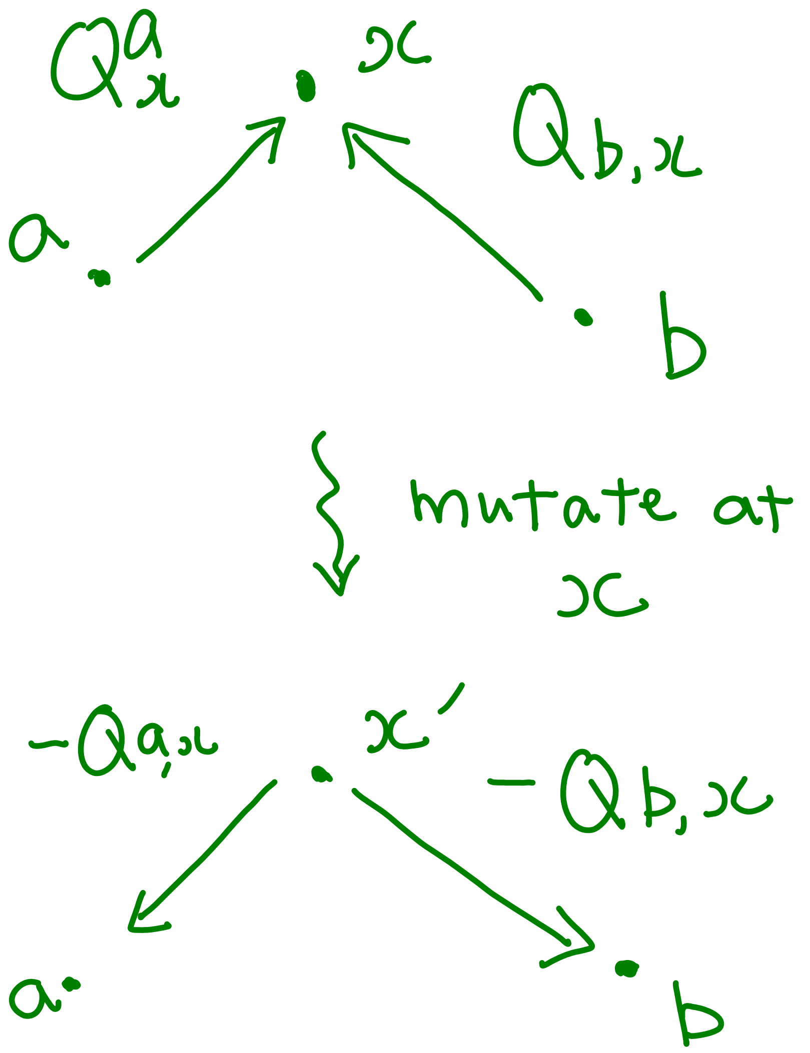
$\left[\mathcal{H}_\Sigma \sim \mathcal{M}^{\text{flat}}_{SL(2)} \right]$

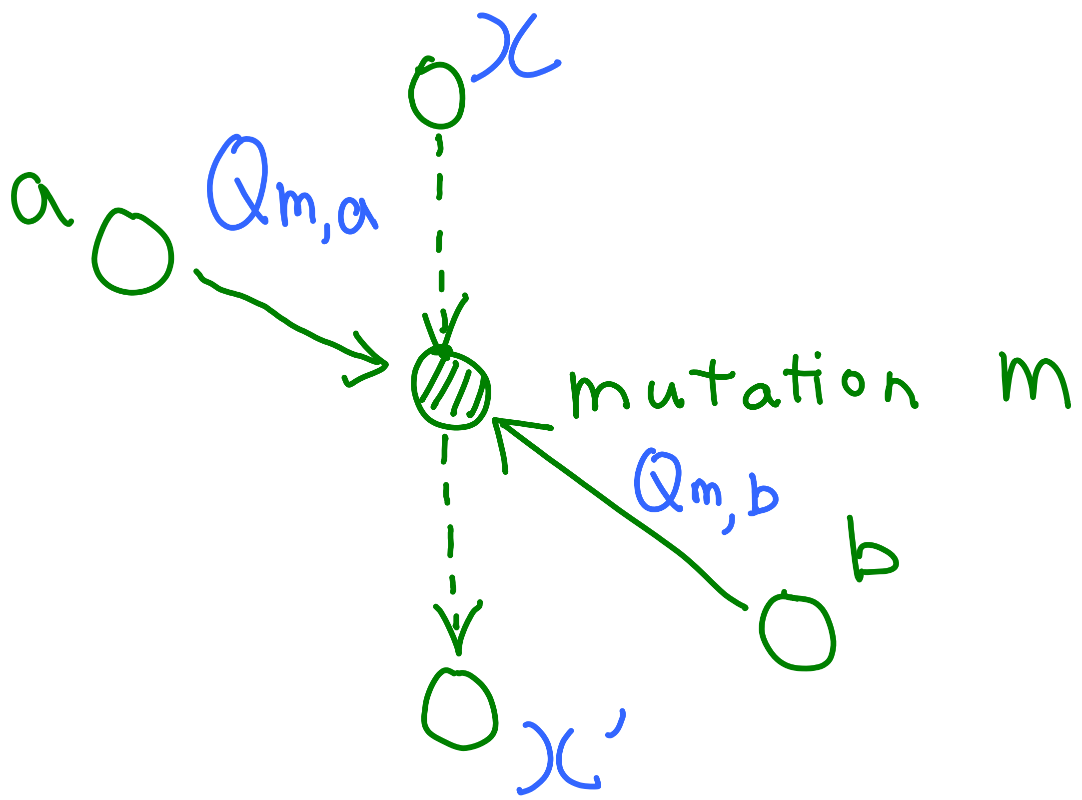
3. 3d Gauge Theories

We can construct 3d $\mathcal{N}=2$ theories $T[(Q,m)]$

s.t. $Z_{T[(Q,m)]}[S^3_b] = Z_{(Q,m)}$

Useful to represent (Q, m) by **mutation networks**





We can evaluate $Z^{\text{cluster}}_{(Q,m)}$ explicitly [Kashaev–Nakanishi, Terashima–Y]

$$Z^{\text{cluster}}_{(Q,m)} = \int \prod_{w \in W} dx_w \prod_{m \in B} \left[S_b\left(\frac{Z(m)}{2\pi b} + \frac{i(b+b^{-1})}{2} \right) \times \exp\left(-\frac{i}{8\pi b^2} Z(m)^2 + \frac{1}{4\pi b^2} Z(m) Z'(m) \right) \right]$$

$S_b(x)$: quantum dilogarithm function

$$Z(m) := 2\left[-X_m - X'_m + \sum_a [Q_{m,a}]_+ x_a \right]$$

$$Z'(m) := 2\left[+X_m + X'_m - \sum_a [-Q_{m,a}]_+ x_a \right]$$

(cf. cluster y-variable)

S_b^3 Partition Function

$$b^2(x_1^2 + x_2^2) + b^{-2}(x_3^2 + x_4^2) = 1$$

[Kapustin Willett Yaakov, Jafferis, Hama Hosomichi Lee, ...]

consider Abelian gauge/global symmetries

$G = \bigotimes_{i \in I} U(1)_i$ ($N=2$ vector mult. V_a)

$V_a \ni (A_\mu^i, \sigma_i)$

A_μ^i: vector mult.

σ_i: scalar

$G \to G_{\text{gauge}} \; (i \in I_{\text{gauge}})$, $G_{\text{flavor}} \; (i \in I_{\text{flavor}})$

σ_i : real mass

+ Chern-Simons terms

$$\frac{1}{4\pi} \sum_{i,j} k_{ij} \int A_i \wedge dA_j$$

+ FI param

ζ_i

matters Φ_a ($\mathcal{N}=2$ hyper mult.)

with $\begin{cases} \text{charges } Q_a^i \text{ under } U(1)_i \\ \text{R-charge } r_a \end{cases}$

+ Superpotential

$$W \ni \prod_I \mathcal{O}_I$$

$$\rightsquigarrow \sum (\text{flavor charge})_I = 0$$

$$\sum (\text{R-charge})_I = 2$$

$$Z[S_b^3] = \int \Big(\prod_{i \in I_{\mathrm{gauge}}} d\sigma_i \Big) \times Z_{\mathrm{cl}} \times Z_{\text{1-loop}}$$

$$Z_{\mathrm{cl}} = \exp\Big(-\sqrt{-1}\,\pi \sum_{i \in I} \zeta_i \sigma_i - \sqrt{-1}\,\pi \sum_{i,j \in I} k_{ij} \sigma_i \sigma_j \Big)$$

$$Z_{\text{1-loop}} = \prod_a S_b\Big(\frac{iQ}{2}(1 - g_a) - \sum_i Q_a^i \sigma_i \Big)$$

Very similar to Z^{cluster}

Recipe

recall

$$Z^{\text{cluster}}_{(Q,m)} = \int \prod_{w\in W} dx_w \prod_{m\in B} \left[S_b\left(\frac{Z(m)}{2\pi b} + \frac{i(b+b^{-1})}{2}\right) \times \exp\left(-\frac{i}{8\pi b^2} Z(m)^2 + \frac{1}{4\pi b^2} Z(m) Z'(m)\right)\right]$$

1. ◍ = mutation of quiver

(= tetrahedron)

= $\mathcal{N}=2$ hyper mult.

2. ○ = edge of quiver

$$\text{(= edge of tetrahedron)} \begin{cases} \text{internal} \\ \text{external} \end{cases}$$

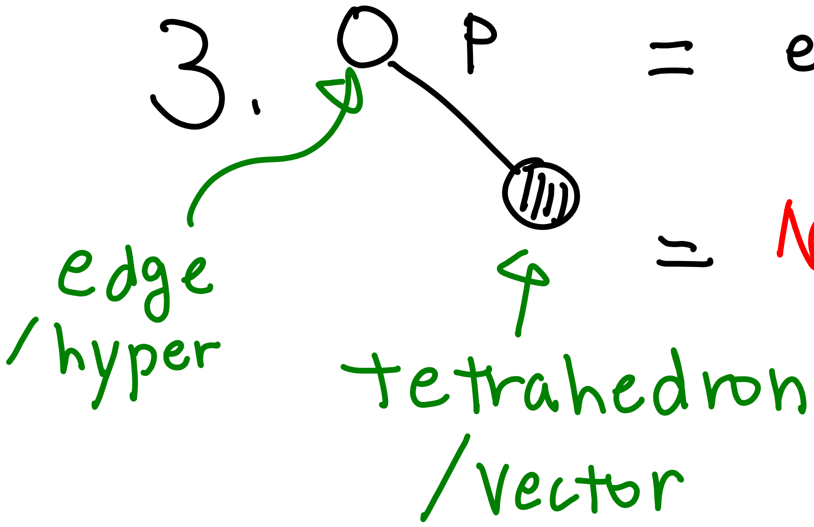
$$= U(1) \text{ symmetry} \begin{cases} \text{gauge} \\ \text{global} \end{cases}$$

[$\mathcal{N}=2$ vector mult.]

(※ many redundancies : electric/magnetic dual)

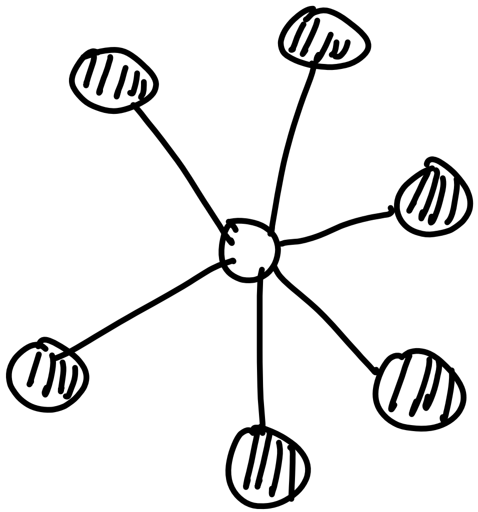
recall

$$\begin{cases} Z(m) := 2\left[X_m + X'_m - \sum_a [Q_{m,a}]_+ x_{m,a}\right] \\ Z'(m) := 2\left[-X_m - X'_m + \sum_a [-Q_{m,a}]_+ x_{m,a}\right] \end{cases}$$

3. 

= edge belongs a tetrahedron

= $\mathcal{N}=2$ hyper mult. has charge p under the $U(1)$ sym.

4. 

= (= gluing of tetrahedra along an edge)

[cf. Nagao-Terashima-Y]

= Superpotential constraint

[※ in general involves monopole operators]

5. change of $|\mathrm{in}\rangle, |\mathrm{out}\rangle$ in $\langle \mathrm{in}|\hat{\varphi}|\mathrm{out}\rangle$
by $Sp(2N,\mathbb{Z})$ [or rather $Mp(2N,\mathbb{Z})$]

$= Sp(2N,\mathbb{Z})$ action on 3d $\mathcal{N}=2$ theories

[Kapustin Strassler, Witten]

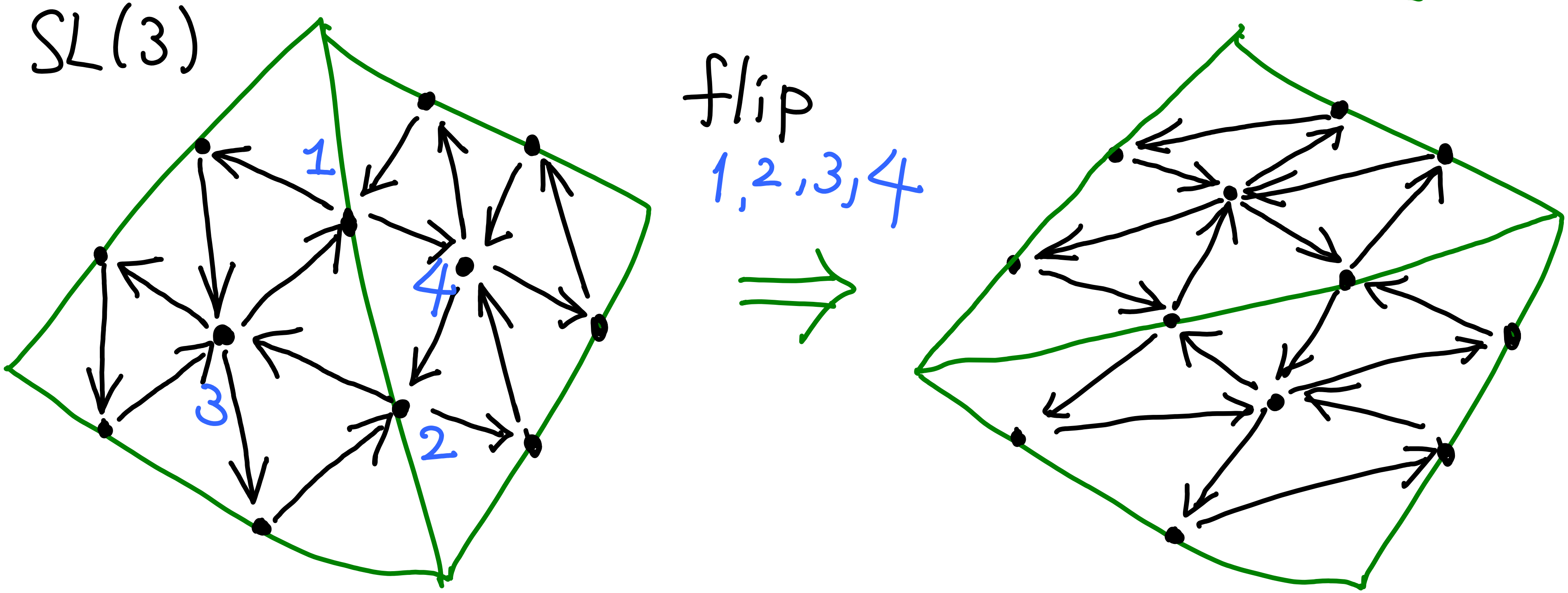
($\leftarrow$ duality group of 4d $U(1)^N$ theories)

※ in general $Sp(2N,\mathbb{Z})$ gives rise to
Chern-Simons terms

※ Our construction is not limited to theories on A_1 (2,0) theory on 3-mfd

e.g. SL(N) Chern-Simons / higher Teichmüller theory [Fock Goncharov]

SL(3)



※ $\frac{1}{6}(N^3-N)$ flips for SL(N)

※ quantum dilogarithm identity [Keller, ...]

$\rightsquigarrow$ new 3d mirror symmetries

[to appear w/ D. Xie]

※ cluster algebras appear in

4d $\mathcal{N}=2$ wall crossing

4d $\mathcal{N}=1$ SCFTs

dimer integrable models

scattering amplitudes

⋮

4d $\mathcal{N}=1$ quiver theory

$I[S^1 \times S^3]$

$= Z$ 2d integrable spin chain

3d $\mathcal{N}=2$ quiver theory

$Z[S^3_b]$

quiver

?

1/2 BPS states inside 4d $\mathcal{N}=2$ theory

KS monodromy

3d $\mathcal{N}=2$ theory on the boundary of 4d $\mathcal{N}=2$ theory

$Z[S^3_b]$

Summary

quiver Q + mutation m

$Z^{\text{cluster}}_{(Q,m)}$

3d $\mathcal{N}=2$ theory $T[(Q,m)]$

$Z_{T[(Q,m)]}[S^3_b]$

3-manifold

$M = S^3 \setminus L$

$Z_{CS}[M]$

Question:

$$Z_{T[(Q,m)]}[S^3_b] = Z^{\text{cluster}}_{(Q,m)}$$

why?