Integrability in Scattering Amplitudes

M. Yamazaki (Kavli IPMU & IAS)

2014/Sep/23, KIAS
"Autumn Symposium on String/M-theory"

Integrability/Yangian Symmetry in $\underbrace{\text{Scattering}}_{\text{(Deformed)}}$ Amplitudes

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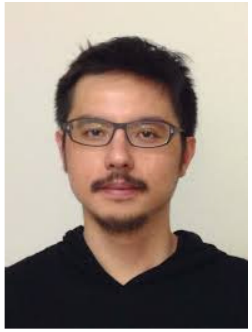
thanks to numerous discussions w/ IAS postdocs

T. Bargheer, S. He, Y. Huang, F. Loebbert

DESY Perimeter NTU Humbolt U.

part of which appeared in

Bargheer, Huang, Loebbert + M.Y

"Integrable Amplitude Deformations for $N=4$ super Yang-Mills and ABJM theory"

1407.4449 [hep-th] ← v2 soon

See also

Ferro-Łukowski-Standacher 1407.6736

as well as earlier works on 4d $\mathcal{N}=4$:

Zwiebel 1111

Ferro-Łukowski-Meneghelli-Plefka-Standacher 1212, 1308

Chicherin-Derkachov-Kirschner 1309

Frassek-Kanning-Ko-Standacher 1312

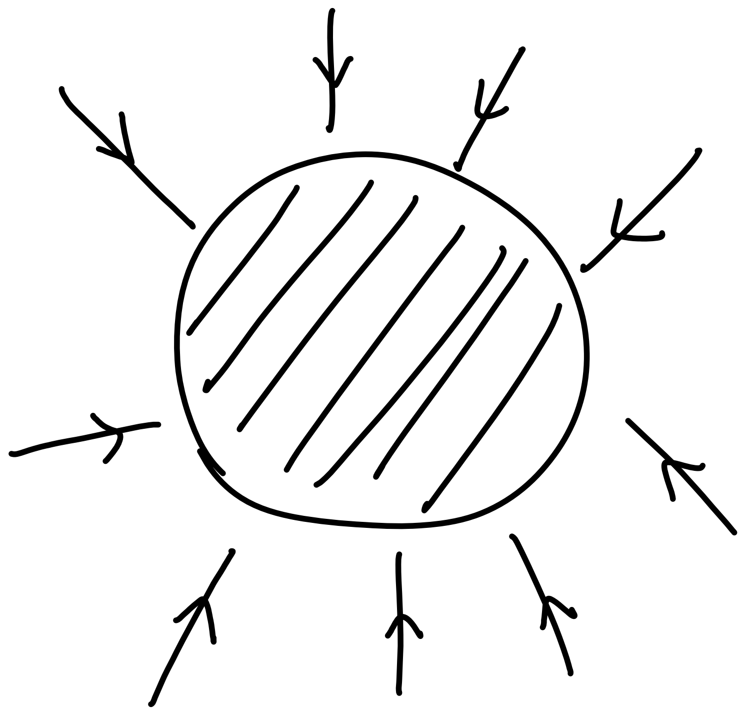
Beisert-Broedel-Rosso 1401

Kanning-Łukowski-Standacher 1403

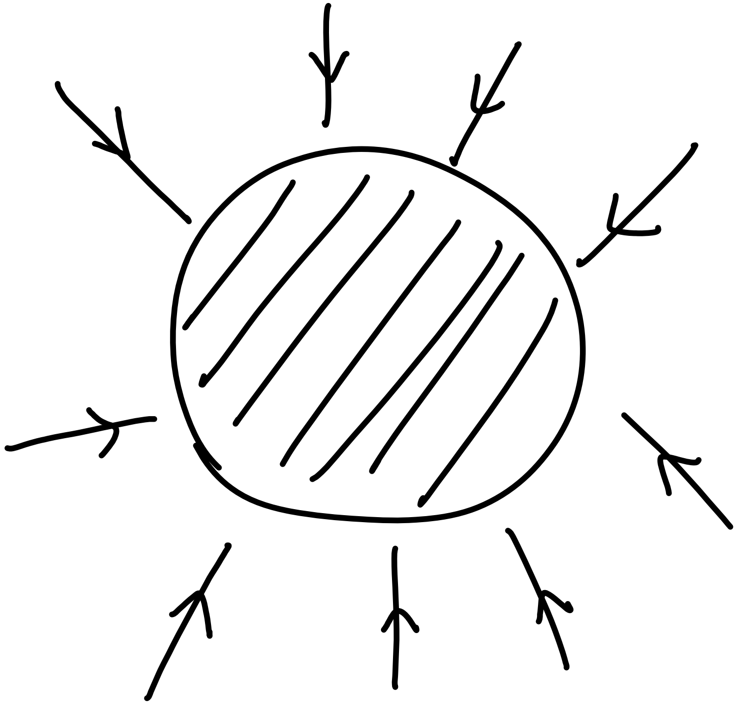
Broedel-Leeuw-Rosso 1403, 1406

Introduction

Scattering amplitudes (on-shell methods)

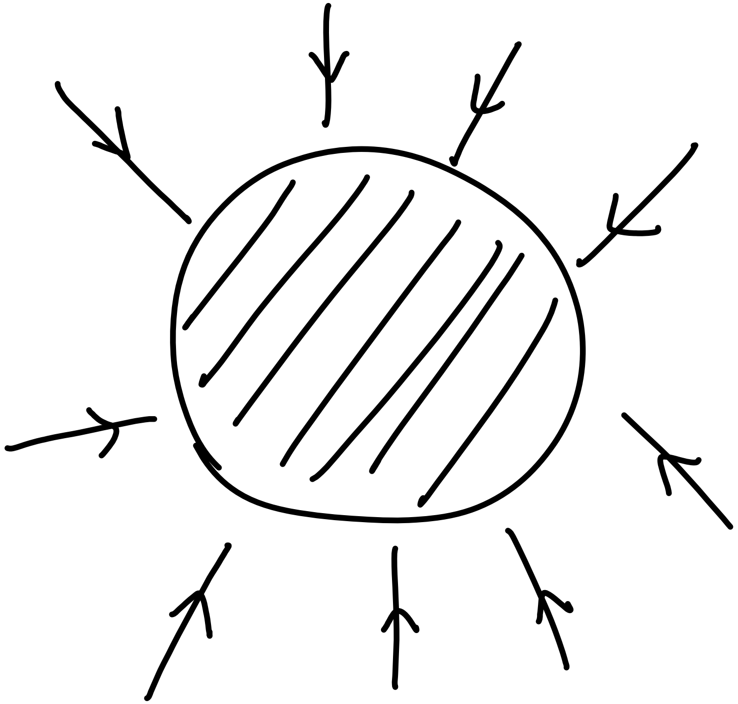


Scattering amplitudes (on-shell methods)



- recent revival of the S-matrix theory, often in SUSY theories
- a number of exciting results, both conceptually and technically

Scattering amplitudes (on-shell methods)



- recent revival of the S-matrix theory, often in SUSY theories
- a number of exciting results, both conceputally and technically
- now is probably the appropriate time to revisit the whole story

Today: planar scattering amplitude of 3d ABJM and 4d $\mathcal{N}=4$ theory

Today: planar scattering amplitude of
3d ABJM and 4d $\mathcal{N}=4$ theory

constrained by

superconformal sym.

ABJM $\quad OSp(6|2,2)$

$$\cup$$

$$\underbrace{Sp(2,2)}_{\substack{\| \\ SO(3,2)_{\text{conformal}}}} \times SO(6)_{R\text{-sym}}$$

$\mathcal{N}=4 \quad PSU(2,2|4)$

$$\cup$$

$$\underbrace{SU(2,2)}_{\substack{\| \\ SO(4,2)_{\text{conformal}}}} \times SU(4)_{R}\text{-sym.}$$

Today: planar scattering amplitude of

3d ABJM and 4d $\mathcal{N}=4$ theory

constrained by

super conformal sym.

$OSp(6|2,2)$

$PSU(2,2|4)$

as well as

DUAL superconformal sym.

[For $\mathcal{N}=4$ Drummond-Henn-Korchemsky - Sokatchev '08]

Today: planar scattering amplitude of

3d ABJM and 4d $\mathcal{N}=4$ theory

constrained by

super conformal sym.

$OSp(6|2,2)$

$PSU(2,2|4)$

$\{J^{(0)}\}$

as well as

DUAL superconformal sym.

$\{J^{(1)}\}$

Yangian

[Drummond-Henn-Plefka '09]

$\{J^{(n)}\}_{n\geq 0}$

level n generators

$Y[OSp(6|2,2)]$

$Y[PSU(2,2|4)]$

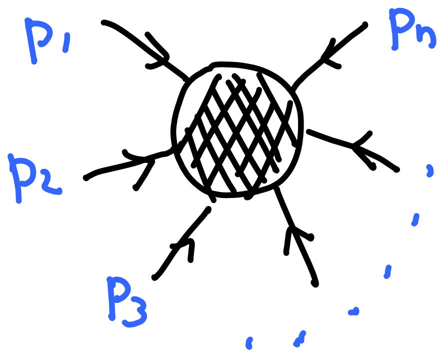
Today we use Yangian symmetry
as a guideline

Q: Is there a deformation of
amplitude consistent w/ Yangian?

Q: Can we make the integrable structure
more manifest?

Yangian Symmetry in Helicity Spinor / Twistor

helicity spinor



n- particle scattering

$A(p_1, \quad , p_n)$ $\quad p_i$: momentum of the i-th particle

spinor-helicity variable for a massless particle

<u>D=4</u>

$$p_\mu \,,\; p^2 = 0 \;\longrightarrow\; p_{\alpha\dot\alpha} = p_\mu (\sigma^\mu)_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$$

$SO(2,2) \simeq SL(2,R) \times SL(2,R)$ $\qquad \det p = 0$

<u>D=3</u>

$$p_\mu \,,\; p^2 = 0 \;\longrightarrow\; p_{\alpha\beta} = p_\mu (\sigma^\mu)_{\alpha\beta} = \lambda_\alpha \lambda_\beta$$

$SO(2,1) \simeq SL(2,R)$ $\qquad \det p = 0$

$$A(p_1 \ldots p_n) \longrightarrow \begin{cases} A(\lambda_1 \ldots \lambda_n, \tilde{\lambda}_1 \cdots \tilde{\lambda}_n) & D=4 \\ A(\lambda_1, \ldots, \lambda_n) & D=3 \end{cases}$$

$$\xrightarrow[\text{SUSY}]{} \begin{cases} A(\lambda_1 \ldots \lambda_n, \tilde{\lambda}_1 \cdots \tilde{\lambda}_n, \eta_1, \cdots, \eta_n) & D=4 \\ A(\lambda_1, \ldots, \lambda_n, \eta_1 \cdots, \eta_n) & D=3 \end{cases}$$

η keeps track of the helicity in the multiplet

e.g. $\mathcal{N}=4$

$$\Phi(\lambda, \tilde{\lambda}, \eta) = G^+(\lambda, \tilde{\lambda}) + \eta^A \psi_A(\lambda, \tilde{\lambda}) + \frac{1}{2}\eta^A \eta^B \phi_{AB}(\lambda, \tilde{\lambda})$$

$$+ \frac{1}{6}\varepsilon_{ABCD}\eta^A \eta^B \eta^C \overline{\psi}_D(\lambda, \tilde{\lambda}) + \frac{1}{24}\varepsilon_{ABCD}\eta^A \eta^B \eta^C \eta^D G^-(\lambda, \tilde{\lambda})$$

Superconformal sym. (level 0)

$\underline{D=4}$

$(\lambda_\alpha, \tilde\lambda_{\dot\alpha}, \eta_A)$

$$L^a{}_b = \lambda^a \partial_b - \frac{1}{2}\delta^a_b \lambda^c \partial_c$$

$$D = \frac{1}{2}\partial_c \lambda_c + \frac{1}{2}\tilde\lambda^{\dot c}\tilde\partial_{\dot c}$$

$$R^A{}_B = \eta^A \partial_B - \frac{1}{4}\delta^A_B \eta^C \partial_C$$

$$Q^{aB} = \lambda^a \eta_B$$

$$S_a{}^B = \partial_a \partial_B$$

$$P^{ab} = \lambda^a \tilde\lambda^{\dot b}$$

$$K^{ab} = \partial_a \tilde\partial_{\dot b}$$

twistor transform linearizes (Fourier transform) superconformal sym.

$$\lambda_\alpha \leftrightarrow \frac{\partial}{\partial \tilde\mu_\alpha}$$

$$Z^A = (\tilde\mu^\alpha, \tilde\lambda^{\dot\alpha}, \eta^A) \quad \text{twistor}$$

$$J^{(0)A}{}_B = \sum_{i=1}^{n} J_i^{(0)A}{}_B$$

act only i-th particle

$$= \sum_{i=1}^{n} Z_i^A \frac{\partial}{\partial Z_i^B} - (\text{supertrace})$$

"oscillator construction" simple

superconformal sym. (level 0)

$D=3$ there is no $\tilde{\lambda}_{\dot\alpha}$, only $\lambda_\alpha, \eta_\alpha$

hence no room for Fourier transform

$$L^a{}_b = \lambda^a \partial_b - \frac{1}{2}\delta^a_b \lambda^c \partial_c$$
$$P^{ab} = \lambda^a \lambda^b$$
$$K_{ab} = \partial_a \partial_b$$
$$D = \frac{1}{2}\lambda^a \partial_a + \frac{1}{2}$$

$$R^{AB} = \eta^A \eta^B$$
$$R^A{}_B = \eta^A \partial_B - \frac{1}{2}\delta^A_B$$
$$R_{AB} = \partial_A \partial_B$$

$$Q^{aA} = \lambda^a \eta^A$$
$$Q^a{}_A = \lambda^a \partial_A$$
$$S_a{}^A = \eta^A \partial_a$$
$$S_{aA} = \partial_a \partial_A$$

in terms of $\Lambda = (\lambda, \eta)$ level-0 generators

are $\quad \Lambda \frac{\partial}{\partial \Lambda}, \quad \Lambda\Lambda, \quad \frac{\partial^2}{\partial\Lambda\partial\Lambda}$

Yangian

"level n" generator

start w/ $\mathfrak{g}$ $\quad [J_a, J_b] = i f_{ab}{}^c J_c$

then $Y[\mathfrak{g}]$ is generated by $\{J_a^{(n)}\}_{n \geq 0}$

s.t. $J_a^{(0)} = J_a$

$$[J_a^{(0)}, J_b^{(0)}] = i f_{ab}{}^c J_c^{(0)}$$

$$[J_a^{(0)}, J_b^{(1)}] = i f_{ab}{}^c J_c^{(1)}$$

satisfying the Serre relation

$$[J_a^{(1)}, [J_b^{(1)}, J_c^{(0)}]] + (\text{cyclic}) = \frac{1}{4} f_{ag}{}^d f_{bh}{}^e f_{ck}{}^f f^{ghk} J_{\{d}^{(0)} J_e^{(0)} J_{f\}}^{(0)}$$

For our purposes we can concentrate on $J_a^{(0)}, J_a^{(1)}$ from which we can generate $J_a^{(n)}$

conformal $\qquad$ dual conformal

level 1 = dual conformal

$J_a^{(0)}$ and $J_a^{(0)}$ are a priori independent. However for our purposes we can use

evaluation representation of the Yangian

$$\begin{cases} J_a^{(0)} = J^a = \sum_{i=1}^{n} J_i^a & \text{acts on a single particle} \\ J_a^{(1)}(\vec{u}) = \sum_{i<j} J_i^a J_j^a + \sum_i u_i J_i^a \end{cases}$$

acts on a pair "evaluation parameter"

these $J_a^{(0)}, J_a^{(1)}$ and the resulting $J_a^{(n)}$ satisfy the defining relation of the Yangian for $N=4$ / ABJM

Yangian Symmetry of Scattering Amplitudes

$$\frac{\text{on-shell}}{\text{amplitude}} = \text{① basic building block} + \text{② gluing (BCFW recursion)}$$

① basic building block

($\mathcal{N}=4$ 3 pt
ABJM 4 pt)

② + gluing (BCFW recursion)

[Britto-Cachazo-Feng-Witten '04]

Let's first discuss $\mathcal{N}=4$

building block

$\mathcal{N}=4$ the basic building block is the 3-pt

$p_1+p_2+p_3=0$, $p_i=\lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}$

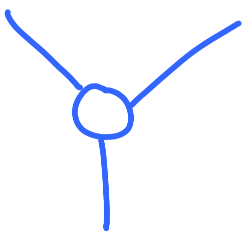
$$p_i\cdot p_j=\frac{p_k^2-p_i^2-p_j^2}{2}=0$$

$$= (\lambda_i\lambda_j)(\tilde{\lambda}_i\tilde{\lambda}_j) = \langle ij\rangle[\tilde{i}\tilde{j}]$$

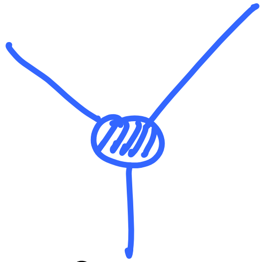
$$\Downarrow$$

$$\langle ij\rangle=0 \text{ or } [\tilde{i}\tilde{j}]=0$$

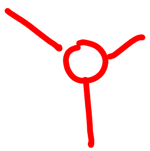
two possibilities



λ_i all parallel



$\tilde{\lambda}_i$ all parallel

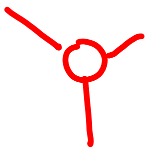
A [] = ?

A [] = ?

building block

$\mathcal{N}=4$

$$\left.\begin{array}{l}\text{Lorentz invariance}\\ \text{(Super) momentum conservation}\\ \text{cyclicity}\\ \text{dimensional analysis}\end{array}\right\}\text{ gives}$$

A [] $= \dfrac{\delta^4(P)\,\delta^4(\tilde{Q})}{[12][23][31]}$

A [] $= \dfrac{\delta^4(P)\,\delta^4(Q)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$

$\langle ij\rangle \equiv \lambda_i \lambda_j \qquad [ij] = \tilde{\lambda}_i \tilde{\lambda}_j$

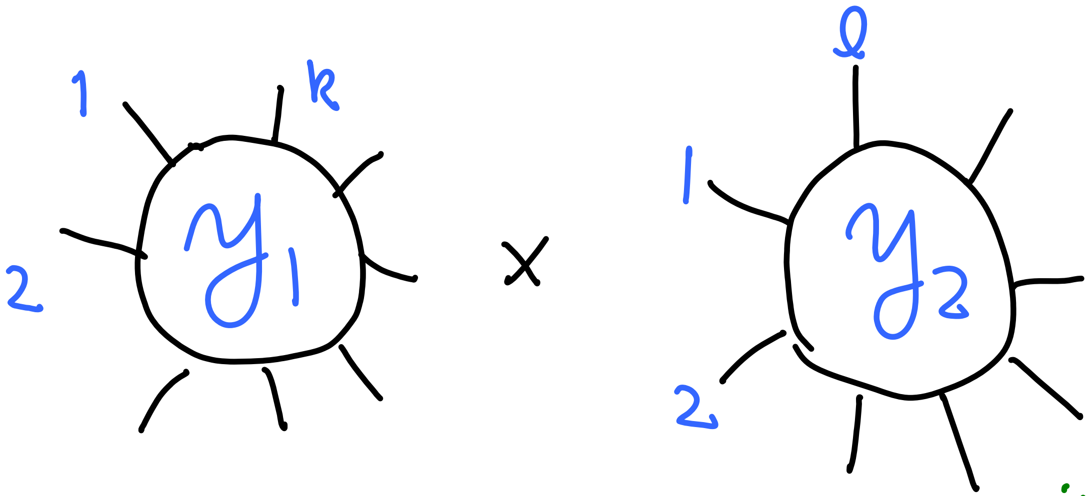
$P \equiv \sum \lambda_i \tilde{\lambda}_i \qquad Q \equiv \sum \lambda_i \tilde{\eta}_i \qquad \tilde{Q} \equiv \widetilde{[12]\eta_3} + \widetilde{[23]\eta_1} + \widetilde{[31]\eta_2}$

next gluing

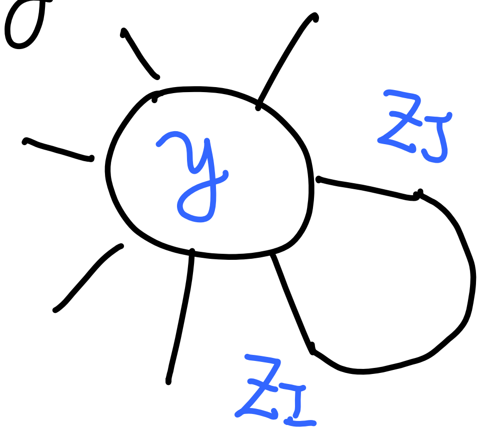
any on-shell graph can be obtained by

1. product

$Y_1 \times Y_2$



2. gluing



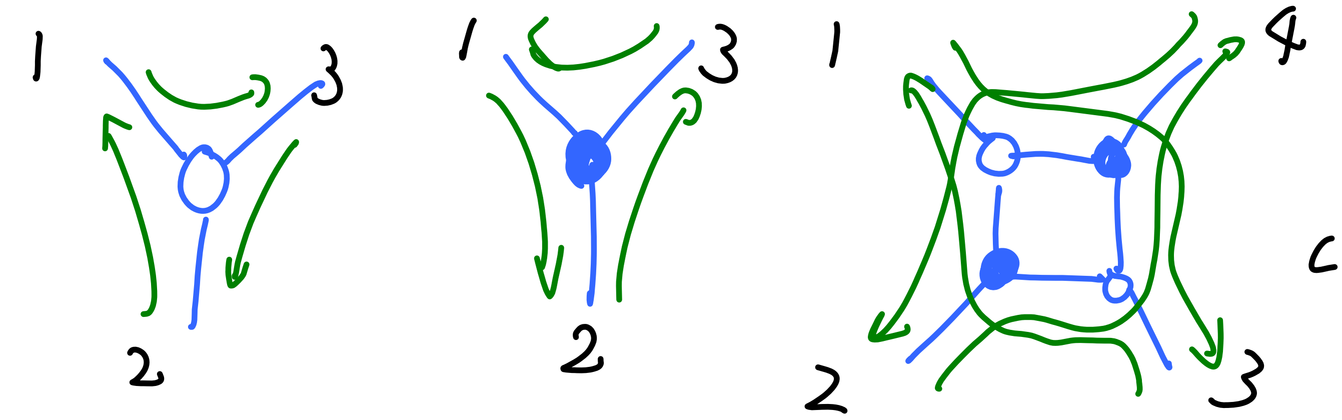
inverse momentum

$$\int d^{3|4} Z_I \; Y(\cdots Z_I, Z_{\bar{J}})$$

preserves Yangian symmetry!

✓ Yangian

to be systematic, use (decorated) permutation



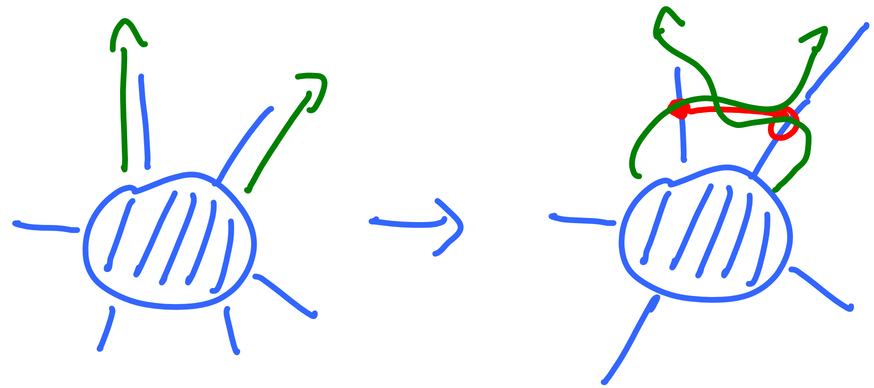
$$\begin{array}{ccc}1&2&3\\ \downarrow&\downarrow&\downarrow\\ 3&1&2\end{array}\qquad \begin{array}{ccc}1&2&3\\ \downarrow&\downarrow&\downarrow\\ 2&3&1\end{array}\qquad \begin{array}{cccc}1&2&3&4\\ \downarrow&\downarrow&\downarrow&\downarrow\\ 3&4&1&2\end{array}$$

$\Big\updownarrow$ 1:1

cell of $Gr_{\geq 0}$

[Arkani-Hamed
Bourjaily
Cachazo
Goncharov
Postnikov
Trnka '12]

any permutation can be obtained by adjacent transposition / BCFW bridge



each BCFW shift $_{i,j}$ mixes the column of c_i

as $c_j \to c_j + \alpha c_i$, giving a parametrization

$C(\vec{\alpha})$ for a cell of $(Gr_{k,n})_{\geq 0}$

+ helicity

$$\hat{y}_\sigma = \int \prod_{j=1}^{n_F - 1} \frac{d\alpha_j}{\alpha_j} \, \delta^{4k|4k}\left(C(\vec{\alpha}) \cdot Z \right)$$

permutation
||
on-shell graph

face variable

start from C_{VAC} eg

$$C_{VAC} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{matrix} k=2 \\ n=4 \end{matrix}$$

and applying $c_j \to c_j + \alpha c_i$

for each BCFW

$$\hat{y}_\sigma = \int \prod_{j=1}^{n_F-1} \frac{d\alpha_j}{\alpha_j}\, \delta^{4k|4k}\left(C(\vec{\alpha}) \cdot Z \right)$$

inside the delta function, shift in C can be traded for a shift in Z:

$$c_i Z_i + c_j Z_j \to c_i Z_i + (c_j + \alpha c_i) Z_j = c_i (Z_i + \alpha Z_j) + c_j Z_j$$

hence we arrive at

$$\hat{y}_\sigma = R_{\sigma_\ell} R_{\sigma_{\ell-1}} \cdots R_{\sigma_1}\, \delta^{4k|4k}\left(C_{VAC} \cdot Z \right)$$

with

$$R_{(ij)} f(Z) \equiv \int \frac{d\alpha}{\alpha} \left[f(Z) \Big|_{Z_i \to Z_i + \alpha Z_j} \right]$$

$$R_{(ij)} f(Z) = \int \frac{d\alpha}{\alpha} \left[f(Z) \Big|_{Z_i \to Z_i + \alpha Z_j} \right]$$

It turns out this R-matrix satisfies the Yang-Baxter equation. We can then make a dictionally w/ the integrable model literature, i.e. QISM/ABE.

$$\hat{y}_\sigma = R_{\sigma_\ell} R_{\sigma_{\ell-1}} \cdots R_{\sigma_1} \delta^{4k|4k}(C_{vac} \cdot Z)$$

length n spin chain

R-matrix
"
BCFW shift

vacuum
Chicherin et al '13
Broedel et al '14
Kanning et al '14

amplitude	spin chain
n particles	length n (n sites)
Trivial amplitude $\delta^{4k\|4k}(C_{vac}\cdot Z)$	vauum
BCFW shift	R- matrix
BCFW shift preserves Yangian	RLL = LLR relation
Yangian generator $J^{(k)}$	conserved charges $J^{(k-1)}$
deformation	spectral parameter

spectral param.

$$T(u) = \sum_{k=0}^{n} u^{n-k} J^{(k-1)}$$

transfer matrix

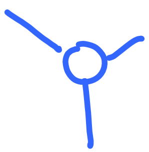
We still need spectral parameters

$$R_{(ij)}(a) f(Z) = \int \frac{d\alpha}{\alpha^{1+a}} \left[f(Z)\Big|_{Z_i \to Z_i + \alpha Z_j} \right]$$

this satisfies YBE

$$R_{ij}(u)\, R_{ik}(u+v)\, R_{jk}(v) = R_{jk}(v)\, R_{ik}(u+v)\, R_{ij}(u)$$

and modifies the 3-pt

$A\Big[$  $\Big]$ $= \dfrac{\delta^4(P)\,\delta^4(\tilde{Q})}{[12]^{1+a_3}[23]^{1+a_1}[31]^{1+a_2}}$

$A\Big[$  $\Big]$ $= \dfrac{\delta^4(P)\,\delta^4(Q)}{\langle 12\rangle^{1+a_3}\langle 23\rangle^{1+a_1}\langle 31\rangle^{1+a_2}}$

$(\sum a_i = 0)$

breaks cyclicity

$\Leftrightarrow$ inhomogeneity of spin chain

[deformation of amplitudes starting with Ferro et al ('12)] how could this be possibly Yangian invariant?

deformations a_i are related linearly to

central charge c_i and evaluation parameters u_i

[Ferro et al ('13)
Beisert et al ('14)]

$$\begin{cases} J_a^{(0)} = J^a = \sum_{i=1}^{n} J_i^a \\ J_a^{(1)}(\vec{u}) = \sum_{i<j} J_i^a J_j^a + \sum_i u_i J_i^a \end{cases}$$

$$GL(2,2|4) \xrightarrow[\text{str}=0]{} SU(2,2|4) \xrightarrow[\substack{\text{central charge} \\ c=0}]{} PSU(2,2|4)$$

$$\mathcal{C} = \sum_i \mathcal{C}_i = \sum_i \left(-Z_i^c \frac{\partial}{\partial Z_i^c} \right)$$

This completes the dictionary

on-shell diagram $\longleftrightarrow$ integrable model

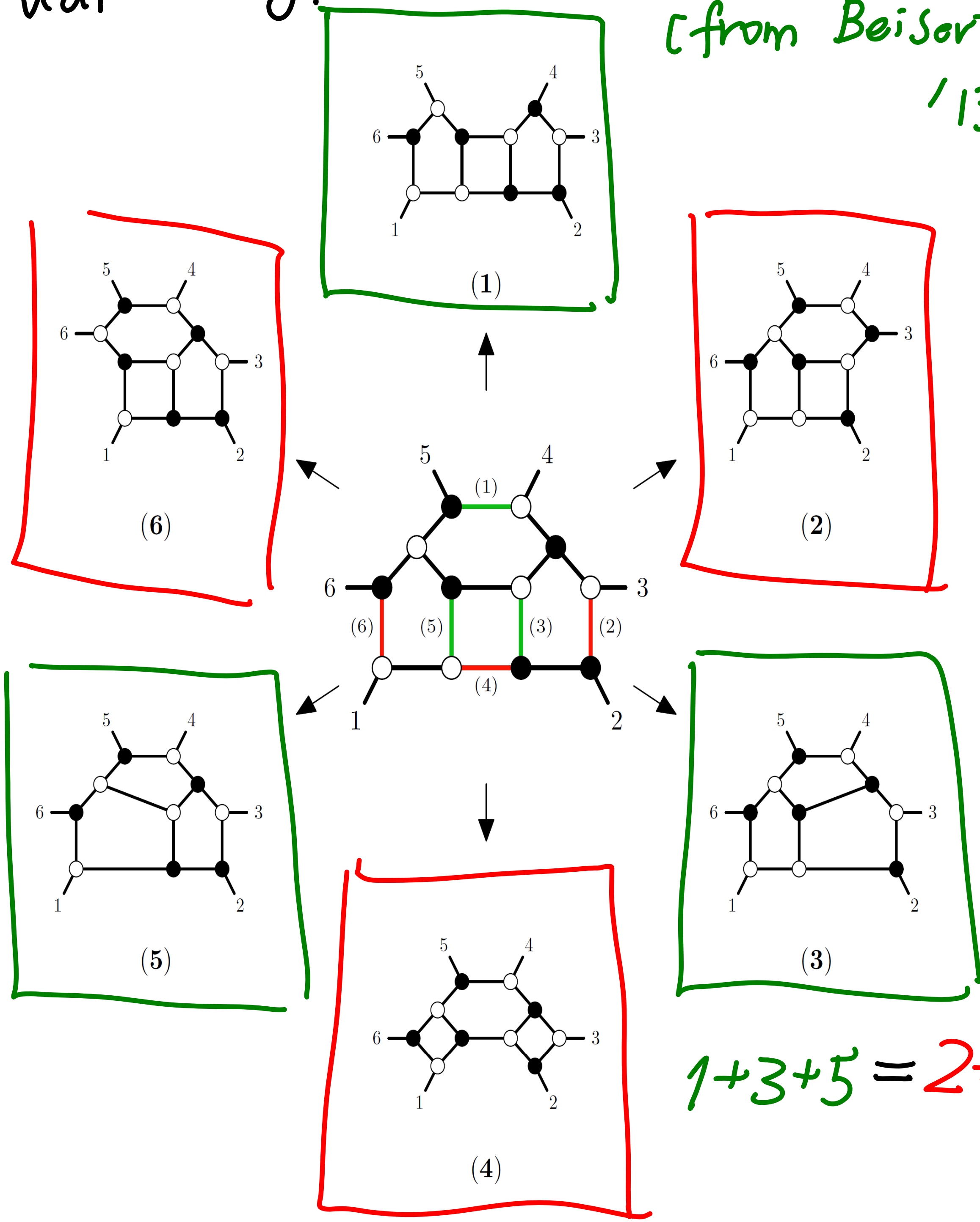
making Yangian sym. manifest.

- ✓ Q: Is there a deformation of amplitude consistent w/ Yangian?
- ✓ Q: Can we make the integrable structure more manifest?

However the actual amplitude is a Sum of on-shell diagram contributions

e.g. 6-pt NMHV

[from Beisert et al '13]



$1+3+5 = 2+4+6$

Such a decomposition is not unique but gives the same answer the uniqueness of answer stems from the Cauchy's residue theorem for the Grassmannian formula, which we deform to be

$$\mathcal{Y}_{n,k}(Z_1, \ldots, Z_n) = \int \frac{d^{k\times n}C}{|GL(k)|} \frac{\delta^{4k|4k}(C\cdot Z)}{M_1^{1+b_1} \cdots M_n^{1+b_n}}$$

M_k : k-th minor of $k\times n$-matrix C

[Bargheer et al, Ferro et al '14]

the choice of contour, however, is not clear after the deformation

Summary up to now:

- The on-shell amplitude can be expressed as an integrable spin chains, making the Yangian (= conformal + dual conformal) manifest

- The spectral parameter of the integrable spin chain gives deformed amplitude, in the evaluation repr. of the Yangian

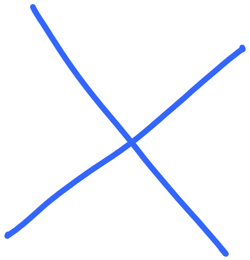
physical meaning?

mathematically, Gr-type formula for $Y[psu(2,2|4)]$ R-matrix

ABJM

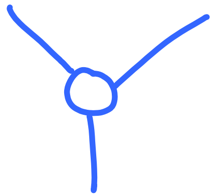
- building block:

  4-pt amplitude

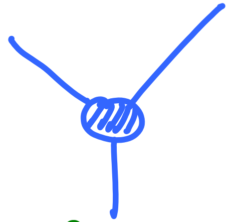


on-shell diagram
looks already as the R-matrix

recall for $\mathcal{N}=4$



λ_i all parallel



$\tilde{\lambda}_i$ all parallel

we have $\tilde{\lambda}=\lambda$ in $D=3$

$$A_4(z)(\bar{\Phi}_1, \Phi_2, \bar{\Phi}_3, \Phi_4) = \frac{\delta^3(P)\,\delta^6(Q)}{\langle 12\rangle^{1+z}\langle 34\rangle^{1-z}}$$

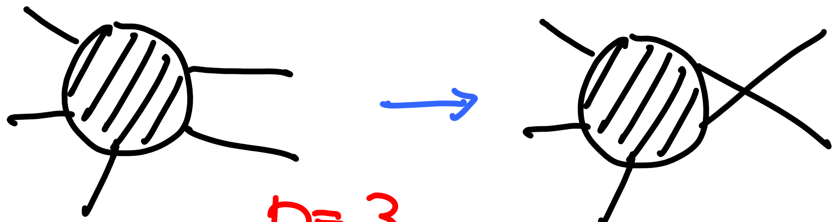
identified as the spectral param.

$$\Phi(\lambda,\eta) = \phi^4(\lambda) + \eta^A\psi_A(\lambda) + \tfrac{1}{2}\varepsilon_{ABC}\eta^A\eta^B\phi^C(\lambda) + \tfrac{1}{6}\varepsilon_{ABC}\eta^A\eta^B\eta^C\psi_4(\lambda)$$

$$\bar{\Phi}(\lambda,\eta) = \bar{\psi}^4(\lambda) + \eta^A\bar{\phi}_A(\lambda) + \tfrac{1}{2}\varepsilon_{ABC}\eta^A\eta^B\bar{\psi}^C(\lambda) + \tfrac{1}{6}\varepsilon_{ABC}\eta^A\eta^B\eta^C\bar{\phi}_4(\lambda)$$

R- matrix

$$(R_{jk}(z) \circ f)(\cdots \lambda_j, \lambda_k \cdots)$$
$$= \int d\lambda_\# \, d\lambda_b \, A_4(z)(\lambda_j, \lambda_k, i\lambda_\#, i\lambda_b) f(\cdots \lambda_\#, \lambda_b \cdots)$$



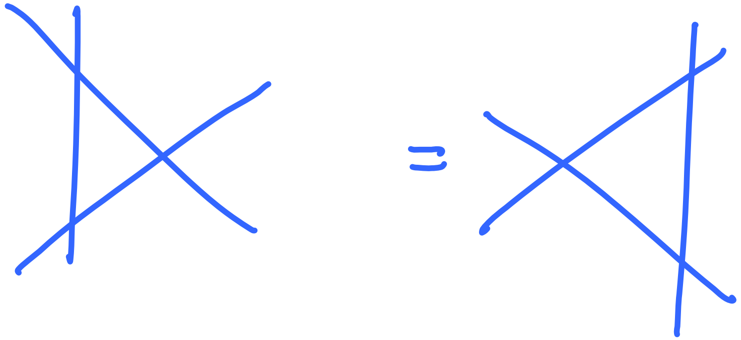
D=3 version of BCFW shift [Gang-Huang-Koh-Lee-Lipstein '10, Huang-Wen '13, ...]

again satisfies YBE (RRR= RRR
RLL= LLR)

[Bargheer et al '14]

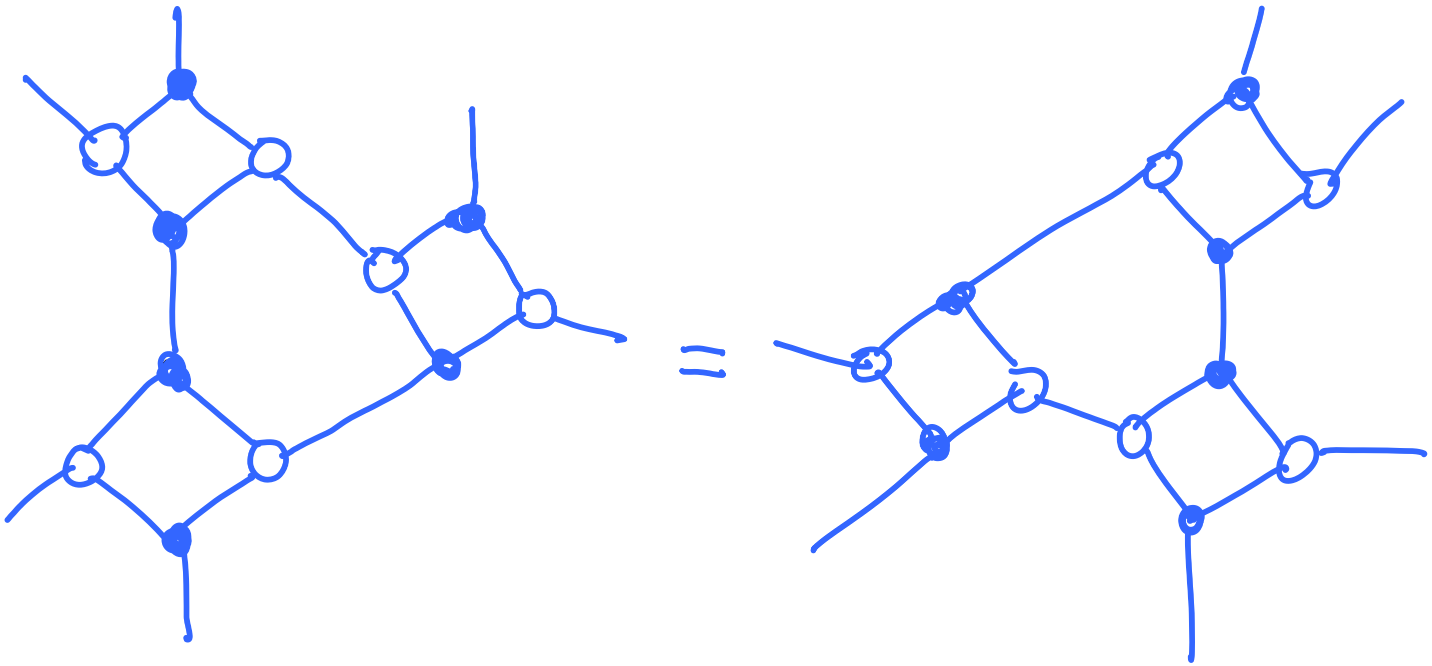
preserves Yangian

interestingly, YBE here is the fundamental move of on-shell diagram



"triangle move"

cf. YBE for $\mathcal{N}=4$



[already in Arkani-Hamed et al '12]

deformed Grassmannian

$$\mathcal{G}_{2k}(\vec{b}) = \int \frac{d^{k\times 2k}C}{|GL(k)|} \frac{\delta^{k(k+1)/2}(C\cdot C^T)\,\delta^{2k|3k}(C\cdot\Lambda)}{\prod_{i=1}^{k} M_i^{1+b_i}}$$

[Bargheer et al]

※ subtlety : $OG(k,2k)_{\geq 0}$ has branches

e.g. 4-pt $OG(2,4)_{\geq 0}$ has 2 branches

$\langle 12\rangle = \pm\langle 34\rangle$

each contribution is separately Yangian inv.

i.e. Yangian is NOT enough for fixing amplitude

Questions

- physical meaning of the deformation?

  IR regulator for loop? Contour in Gr-formula?

  continuous spin? BCFW?

- Yangian in other dim, e.g. $D=2$ scattering
- bonus symmetry?
- amplituhedron?
- similarity w/ 4d $\mathcal{N}=1$ quiver theories?

  ⋮

Lesson?

integrability / Yangian

Severely constrains scattering amplitudes,

but there are more to the latter.

more structure than the spectral problem of planar $N=4$?