Yang-Baxter Equation from SUSY Gauge Theories

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IAS ⟶ KAVLI IPMU

[IGST 2014, Jul/18, DESY]

Happy 10th anniversary, IGST!

mostly based on

M.Y. 1307.1128

D. Xie + M.Y. 1207.0811

M.Y. 1203.5784

Y. Terashima + M.Y. 1203.579

earlier works by

V. V. Bazhanov - S. M. Sergeev 1006.0651, 1106.5874

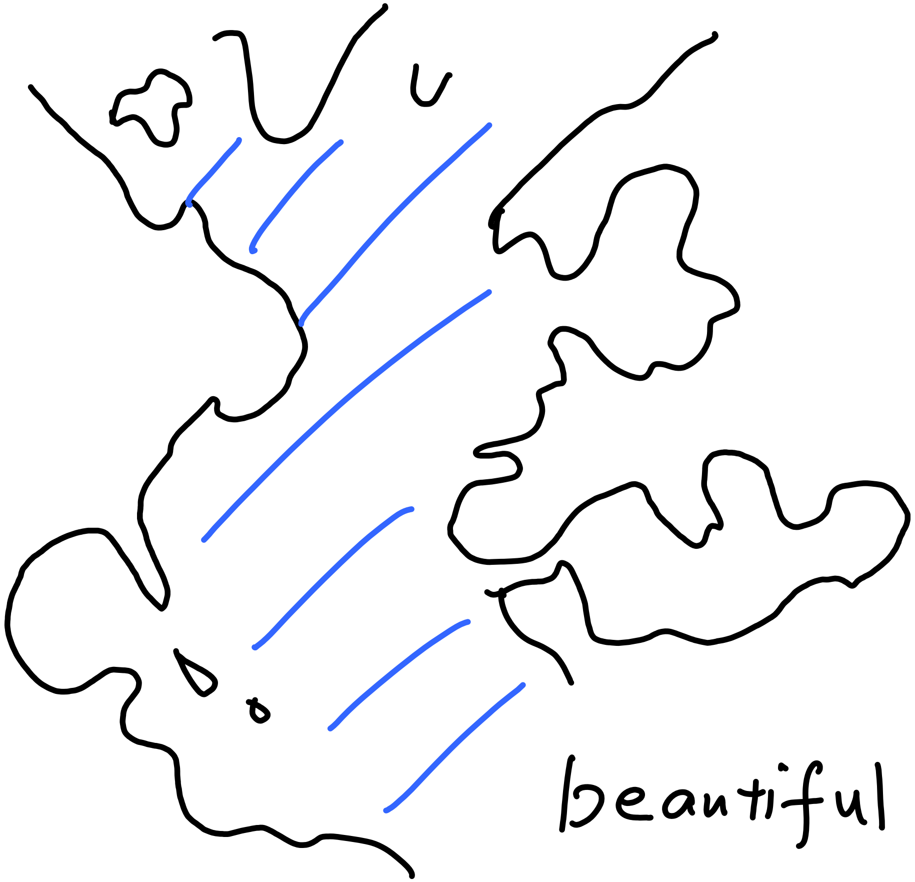
V. P. Spiridonov 1011.3798

yesterday

T. Bargheer, Y. Huang & F. Loebbert + M.Y. 1407.4449

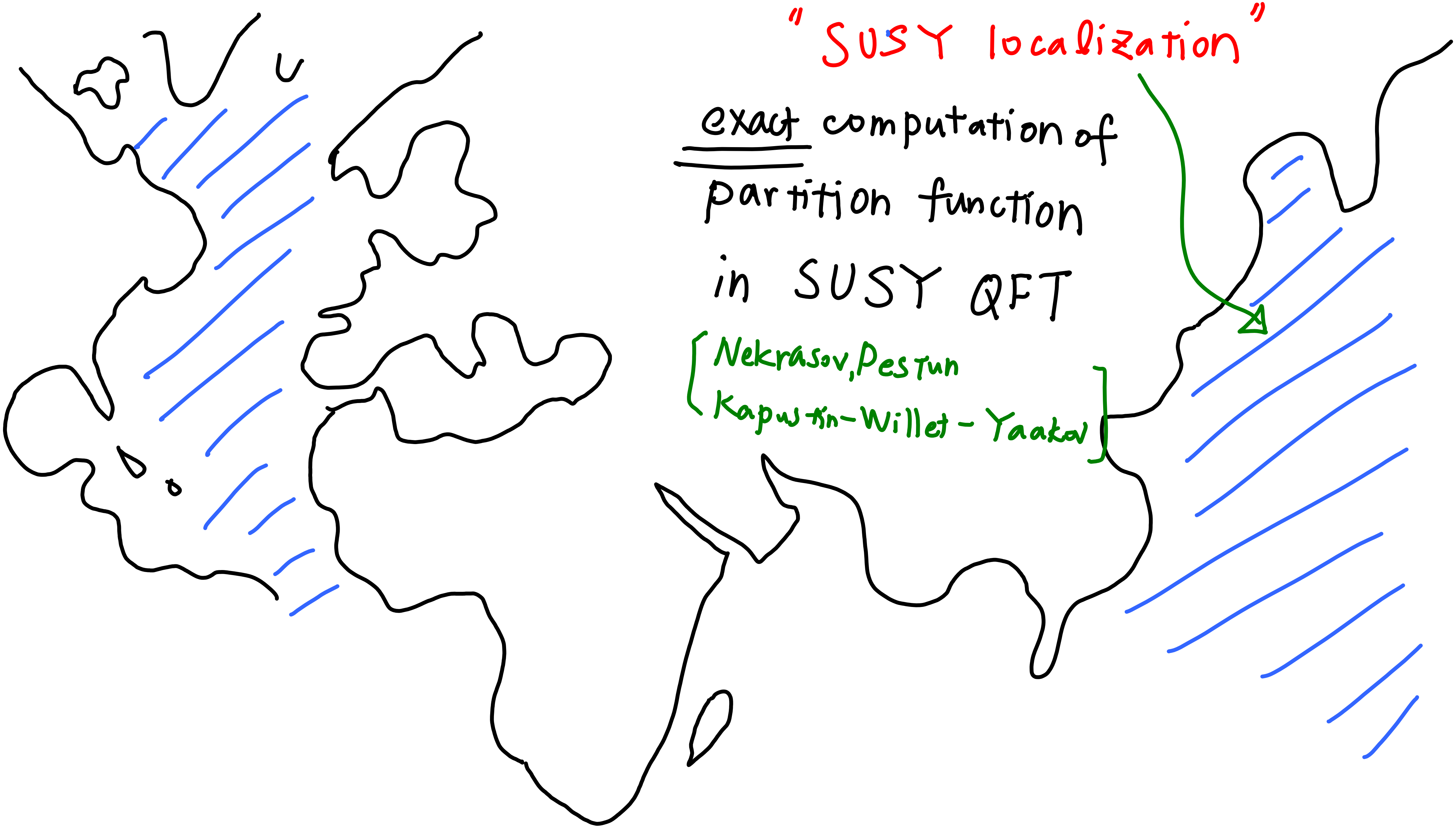
Introduction

ocean of integrability



beautiful developments in understanding 4d $\mathcal{N}=4$, 3d ABJM, ... & their gravity duals

"SUSY localization"



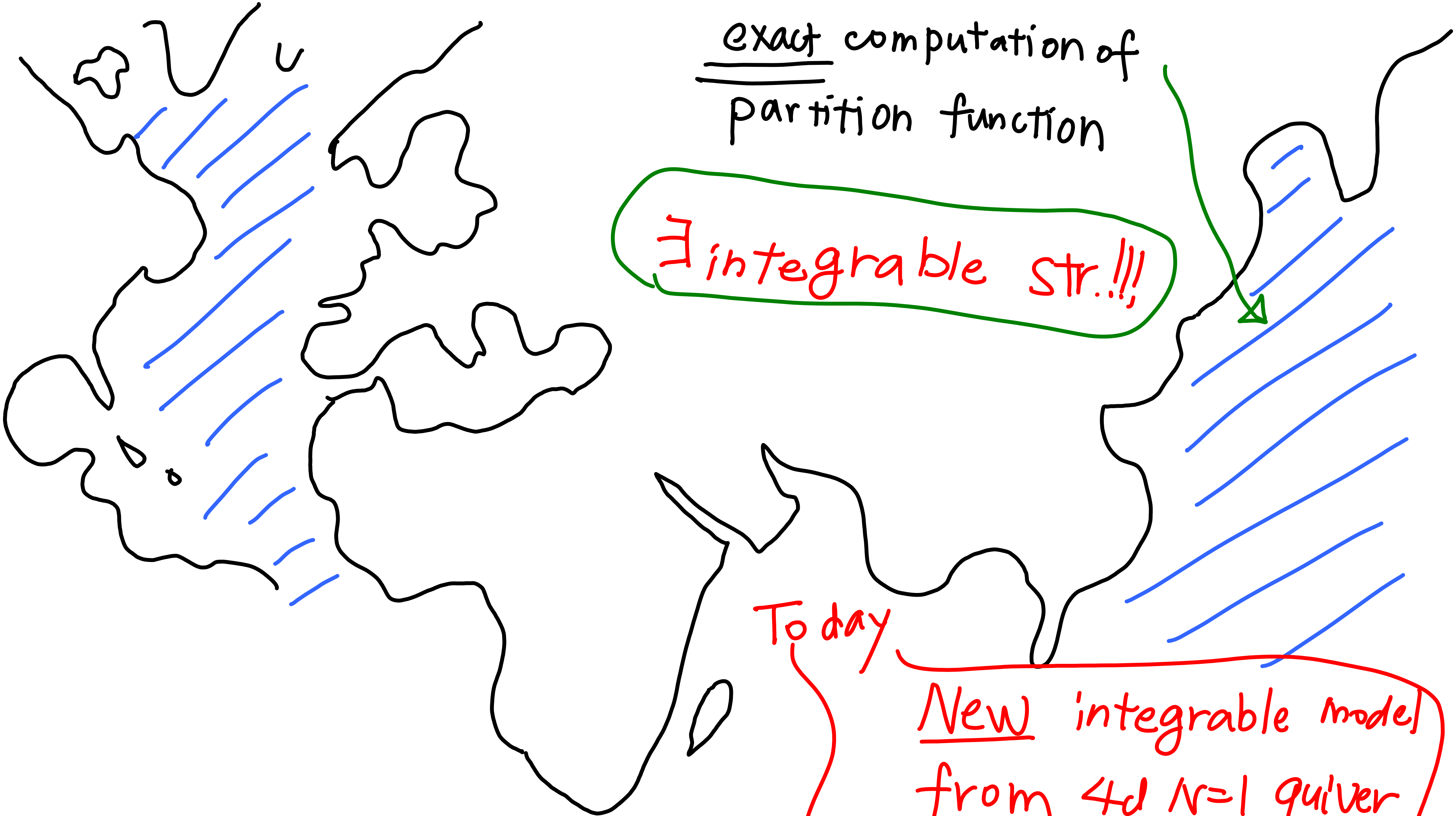
<u>exact</u> computation of partition function in SUSY QFT

[Nekrasov, Pestun
Kapustin-Willett-Yaakov]

"SUSY localization"

exact computation of partition function

∃ integrable str.!!!

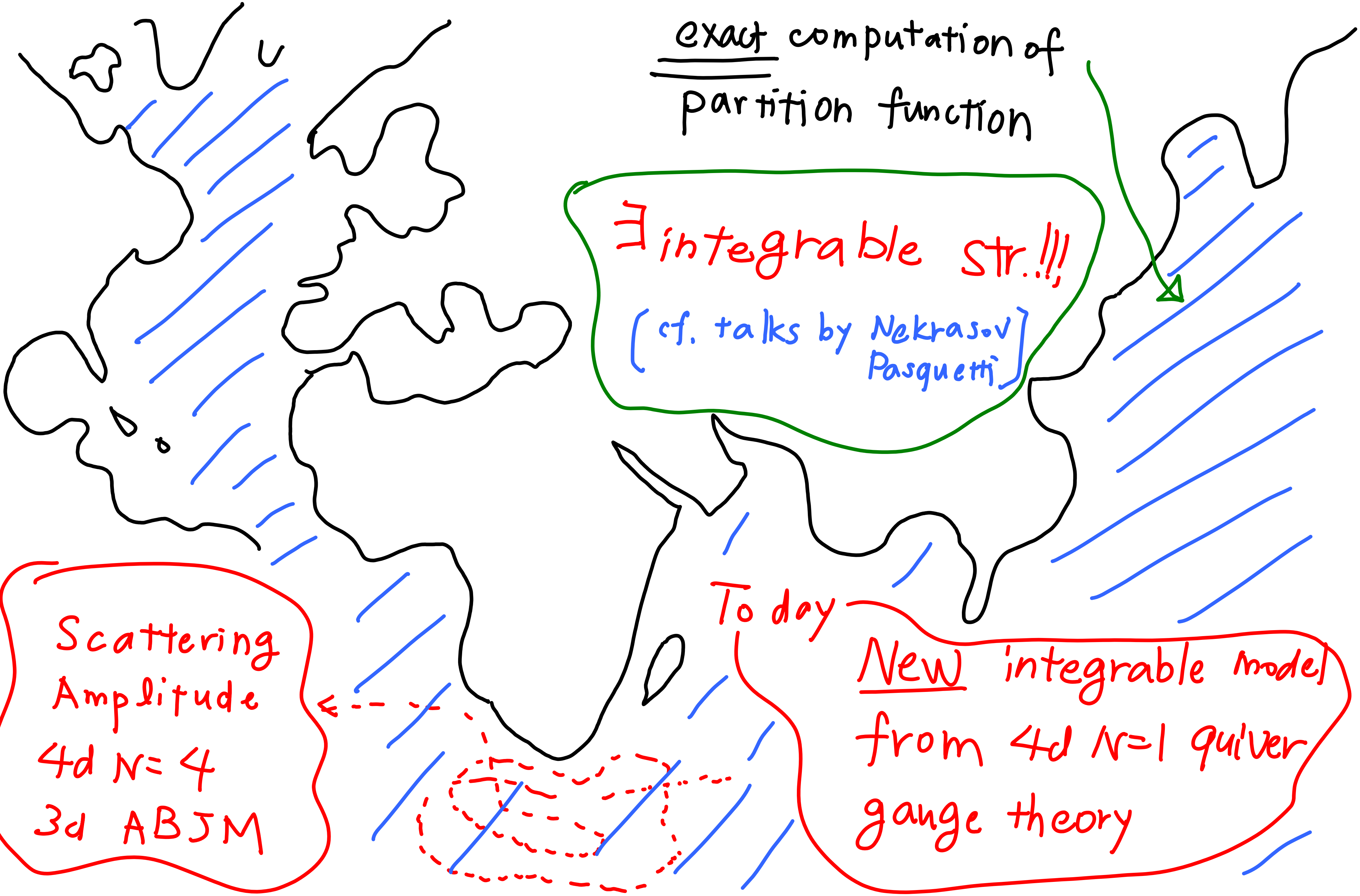


Today

New integrable model from 4d $\mathcal{N}=1$ quiver gauge theory

"SUSY localization"

exact computation of partition function



$\exists$ integrable str.!!!

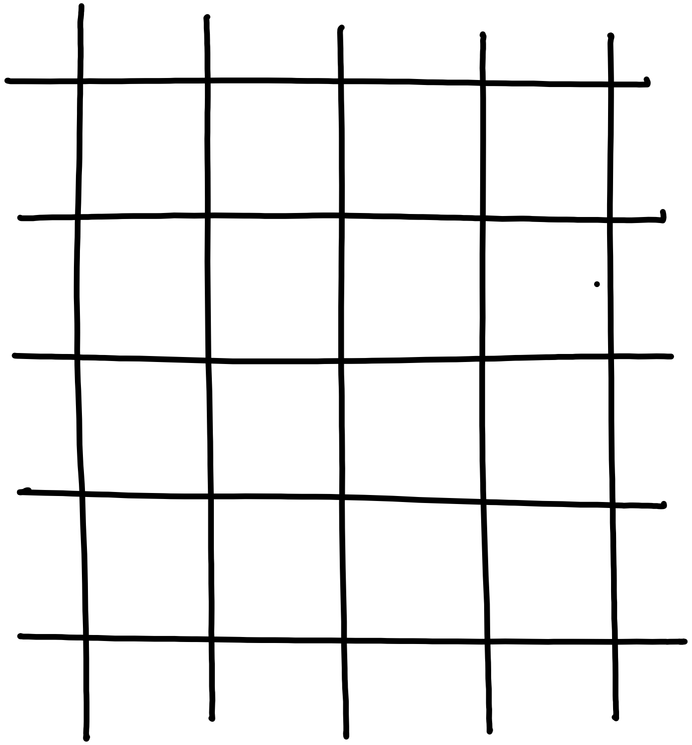
(cf. talks by Nekrasov, Pasquetti)

Today

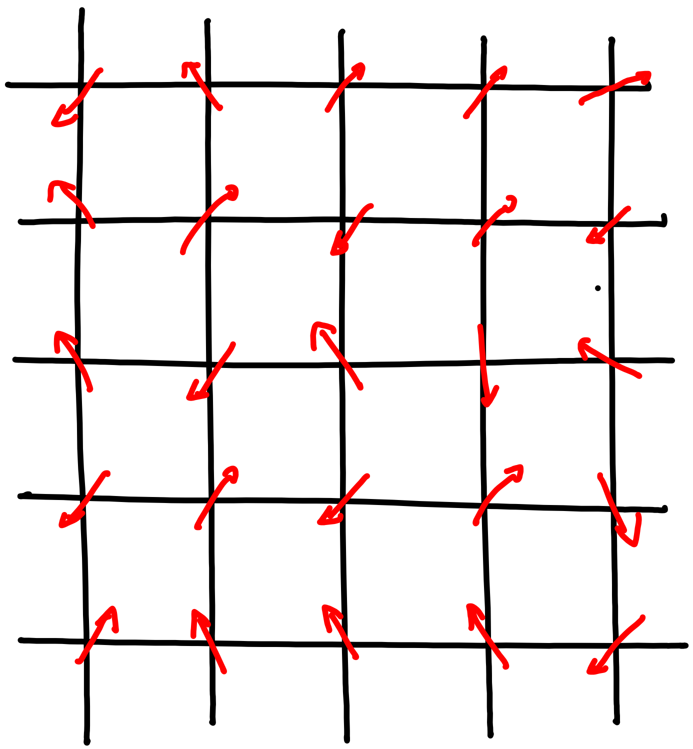
New integrable model from 4d $\mathcal{N}=1$ quiver gauge theory

Scattering Amplitude
4d $\mathcal{N}=4$
3d ABJM

Gauge / YBE



stat-mech



- spin $\vec{S}_v$ at vertex $v \in V$
- energy

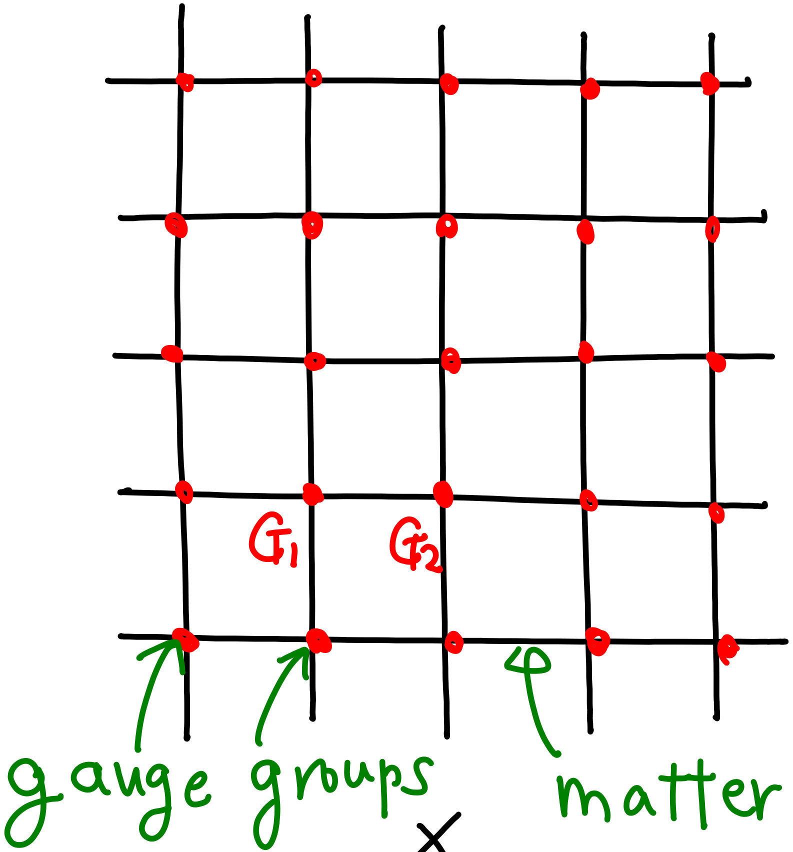
$$\mathcal{E} = \sum_{v \in V} \mathcal{E}_v(\vec{S}_v) + \sum_{e \in E} \mathcal{E}_e(\vec{s}_v, \vec{s}_{v'})$$

edge e between v & v'

- Partition function

$$Z = \sum_{\{\vec{S}_v\}} e^{-\beta \mathcal{E}(\{s_v\})}$$

(quiver) gauge theory



$\times$

$\mathbb{R}^{d,1}$

- gauge group G_v at vertex $v \in V$

  $A_\mu^v(x)$

- matter charged under $G_v \times G_{v'}$ for an edge e between v, v'

  $\Phi_e(x)$

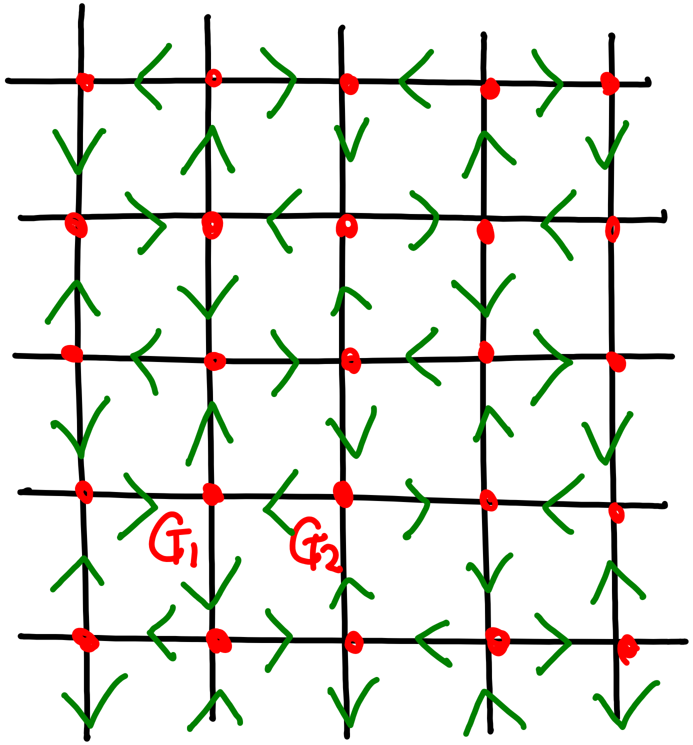
- Lagrangian

$$\mathcal{L} = \sum_{v \in V} \mathcal{L}_v(A_\mu^v) + \sum_{e \in E} \mathcal{L}(\Phi_e, A_\mu^v, A_\mu^{v'})$$

- partition function

$$Z = \int \prod_{v \in V} \mathcal{D}A_\mu^v \prod_{e \in E} \mathcal{D}\Phi_e \, e^{i\mathcal{L}}$$

4d $\mathcal{N}=1$ quiver gauge theory



$\times$

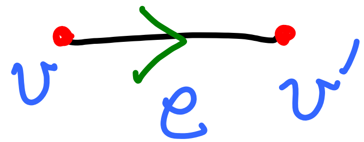
$\mathbb{R}^{3,1}$

For concreteness we choose

- $G_v = SU(N_v)$ vector multiplet at vertex $v \in V$

  mostly $N \equiv N_v$

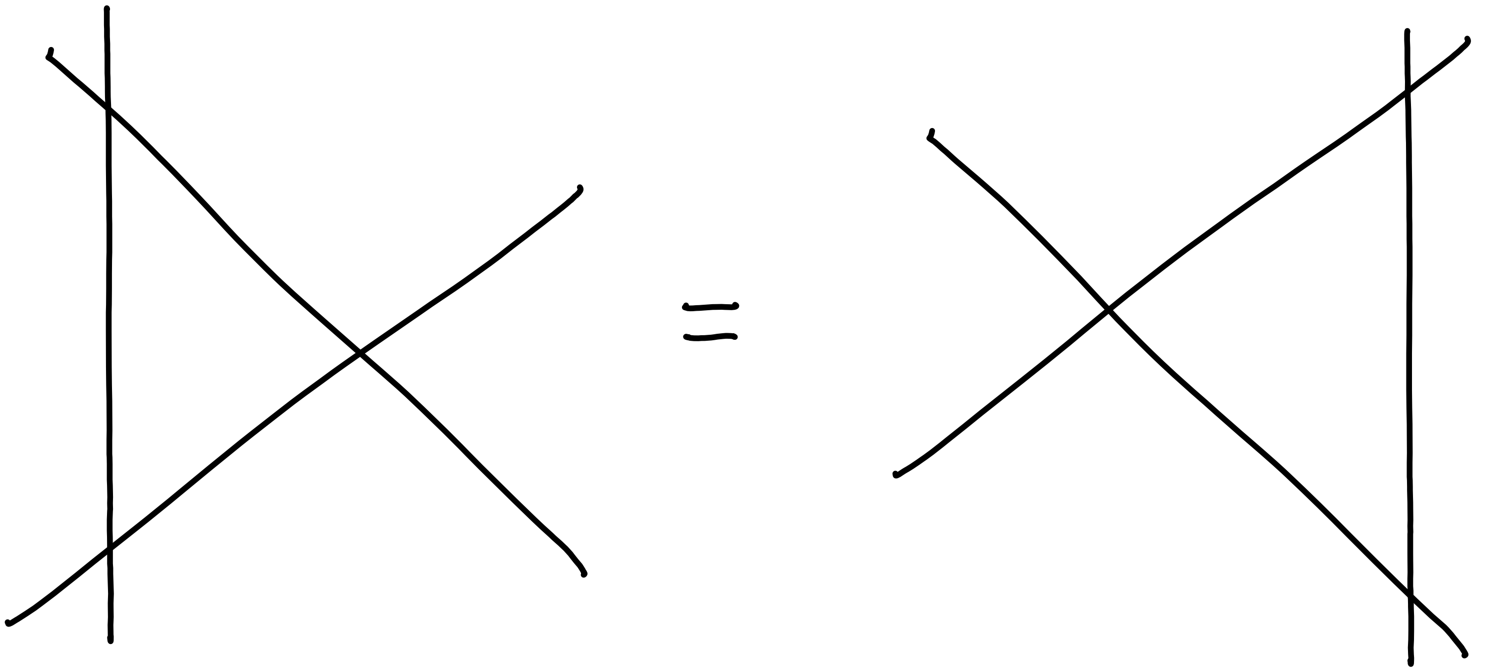
- bifundamental chiral multiplet $(N_v, \overline{N_{v'}})$ under $SU(N_v) \times SU(N_{v'})$



chiral

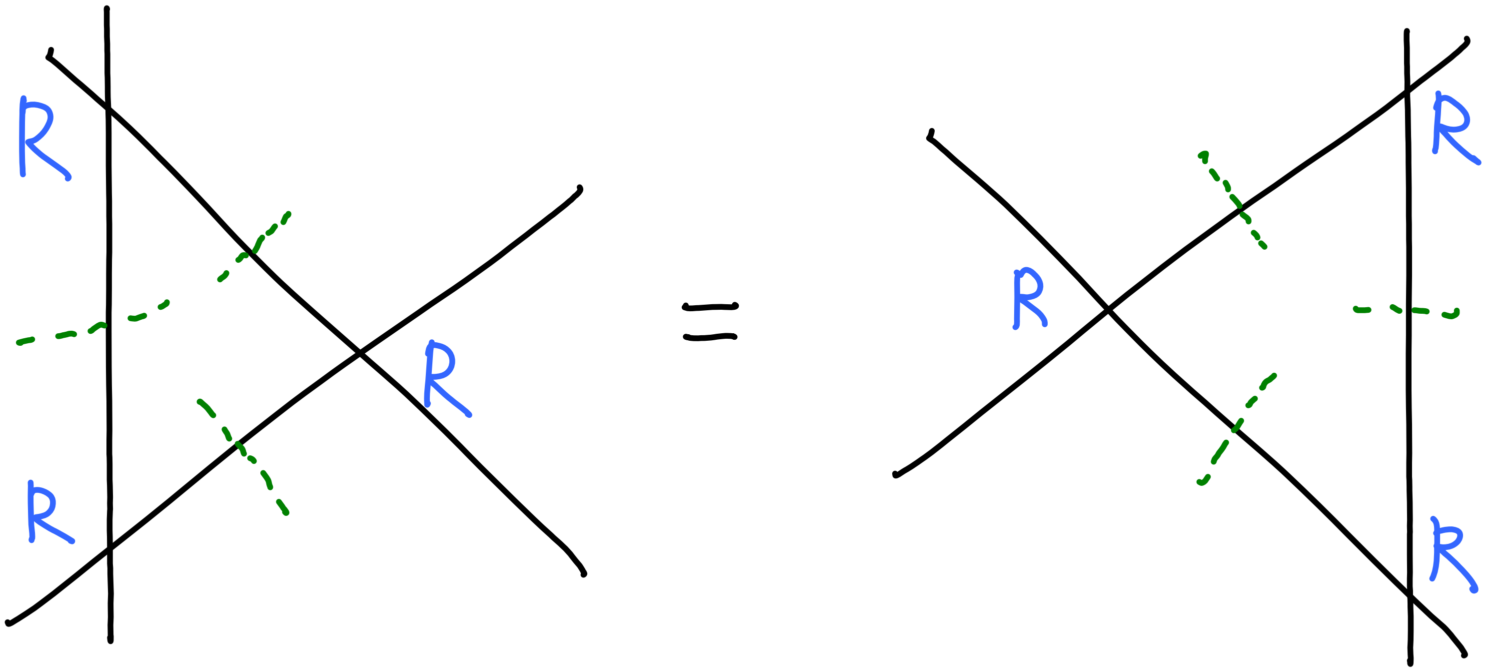
integrability ?

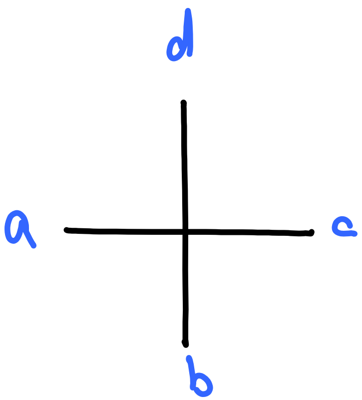
YBE



integrability ?

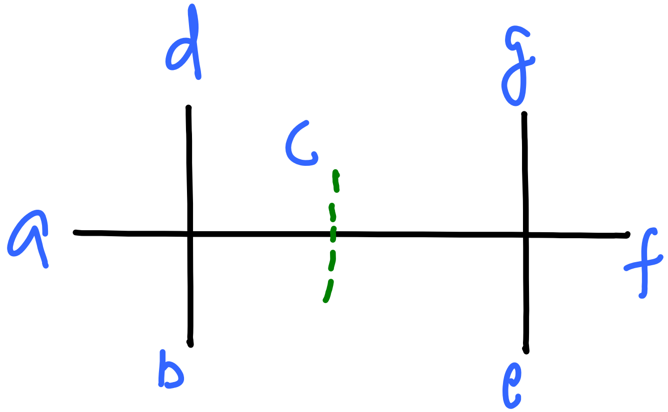
YBE





$= R_{a}{}^{d}{}_{b}{}^{c}$

$R : V_1 \otimes V_2 \to V_3 \otimes V_4$



$$= \sum_{c} R_{a}{}^{d}{}_{b}{}^{c}\, R_{c}{}^{g}{}_{e}{}^{f}$$

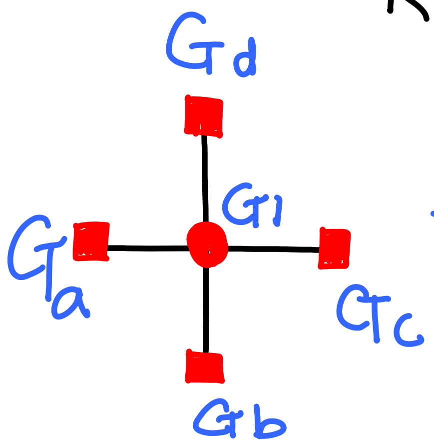
$$= (R \circ R)_{a}{}^{d}{}_{b}{}_{e}{}^{g}{}_{f}$$

lift to gauge theory?

spin $\rightsquigarrow$ gauge group (vect mult.)

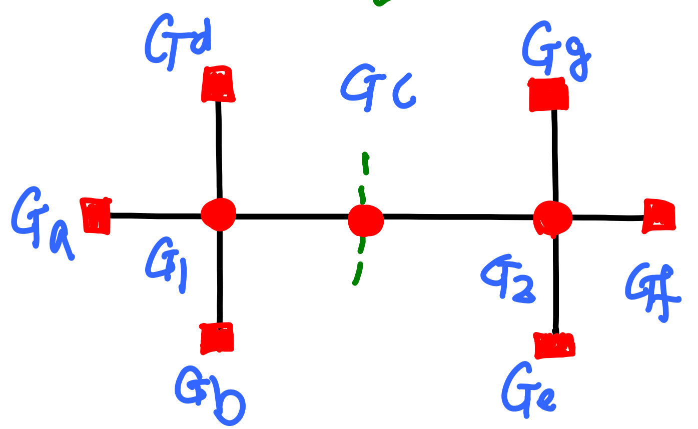
V $\qquad$ G

"the theory for the R-matrix"



$= T[R]$

gluing = gauging



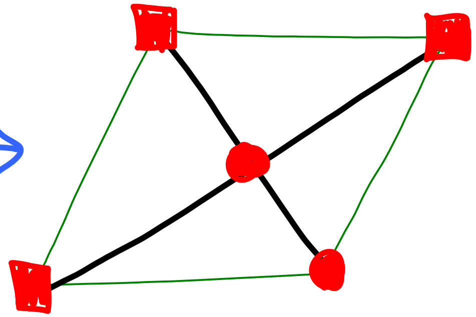
$= T[R \circ R]$

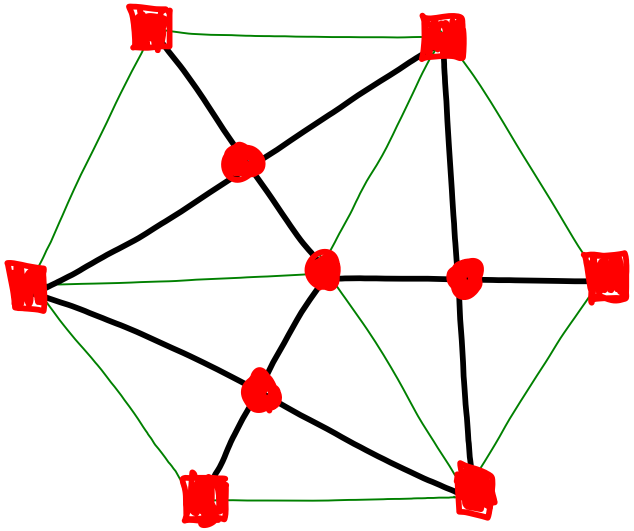
● = gauge sym; dynamical; sum in Z

■ = flavor sym; non-dynamical; fixed parameter in Z

YBE as a duality?

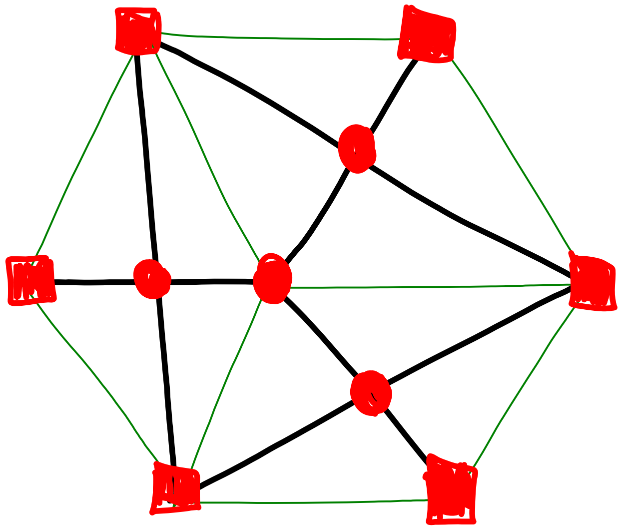
$R \rightarrow$





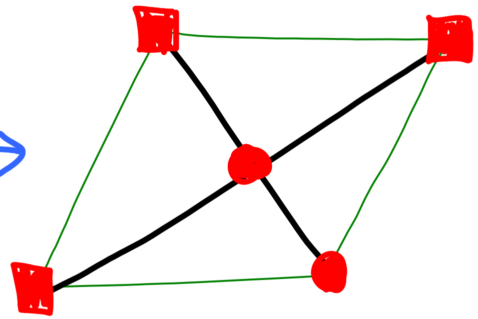
$\overset{??}{=}$

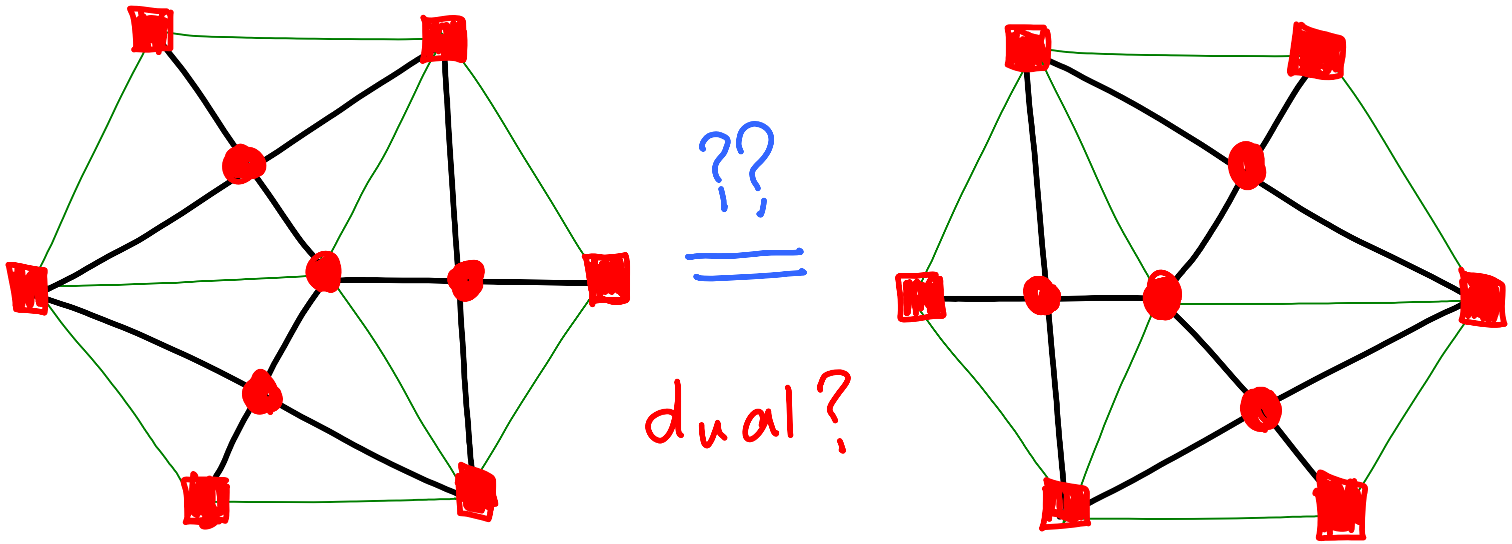
dual?



YBE as a duality?

R →





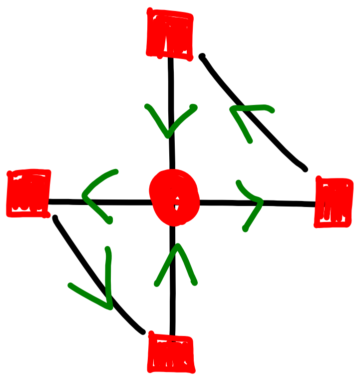
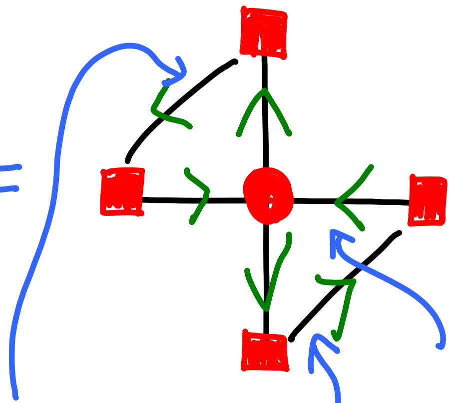
unfortunately there is no such known duality

a bit too naive so far...

the story works with a little modification,

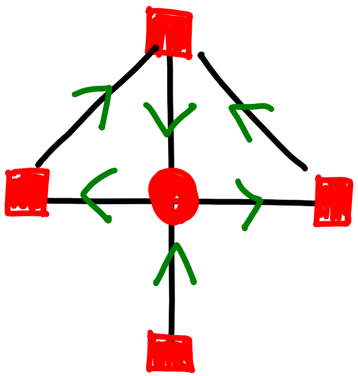
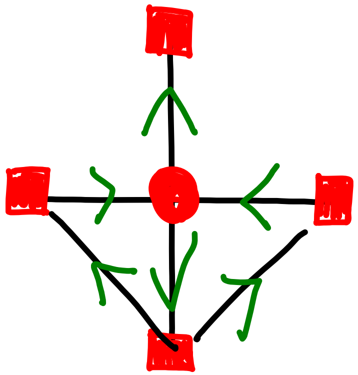
with the help of 4d Seiberg duality

[for $SU(N)$ $N_f = 2N$]

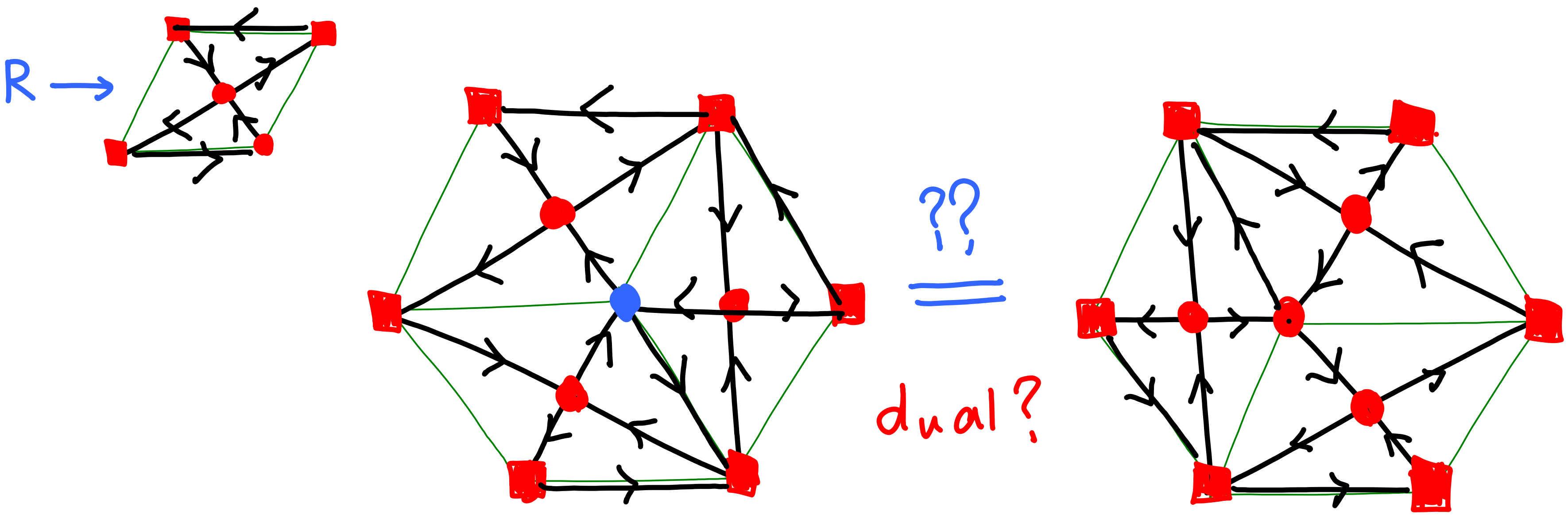
 $=$  $= T[R]$

meson M

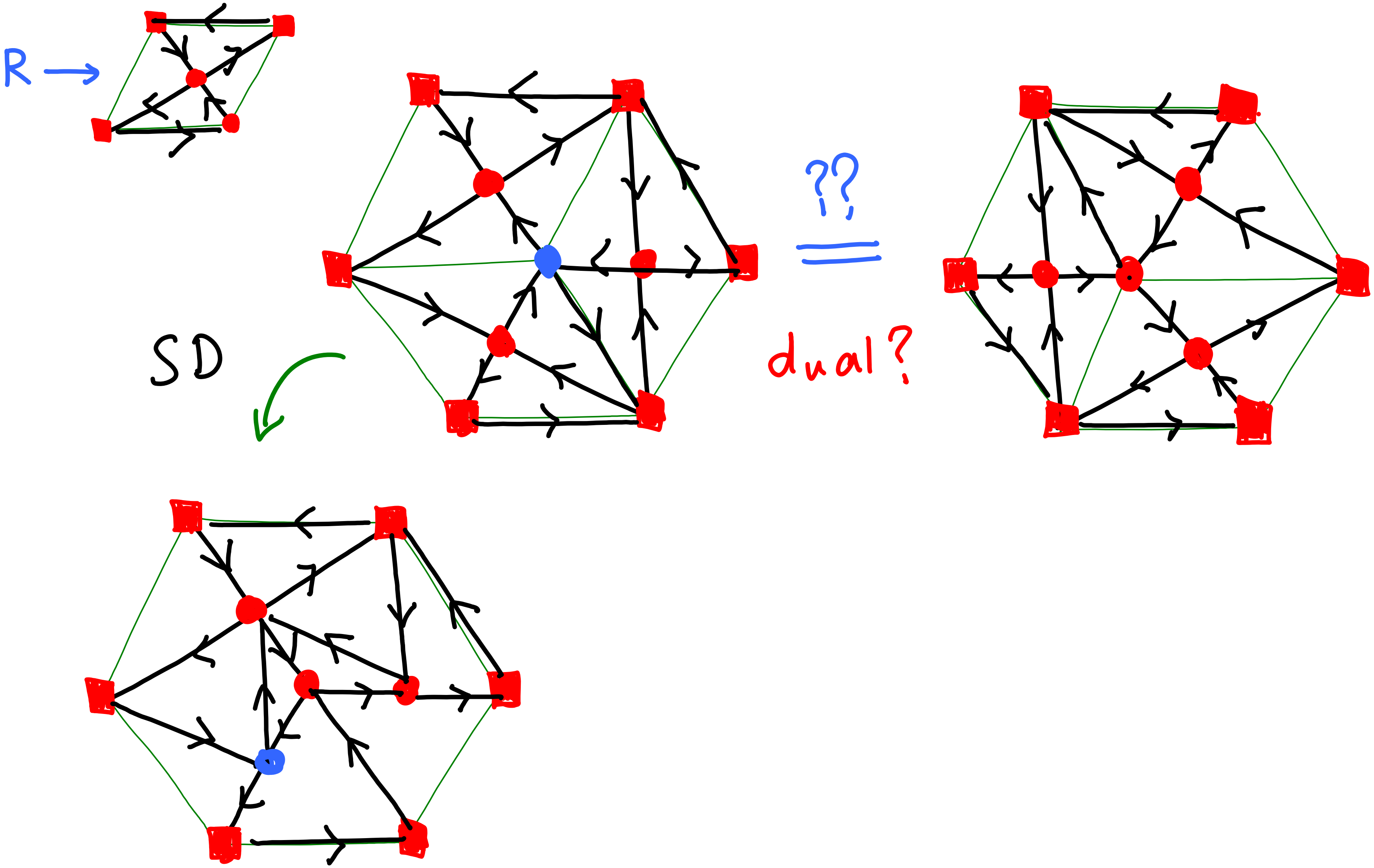
closed loop: superpotential $W = Mq\tilde{q}$

 $=$ 

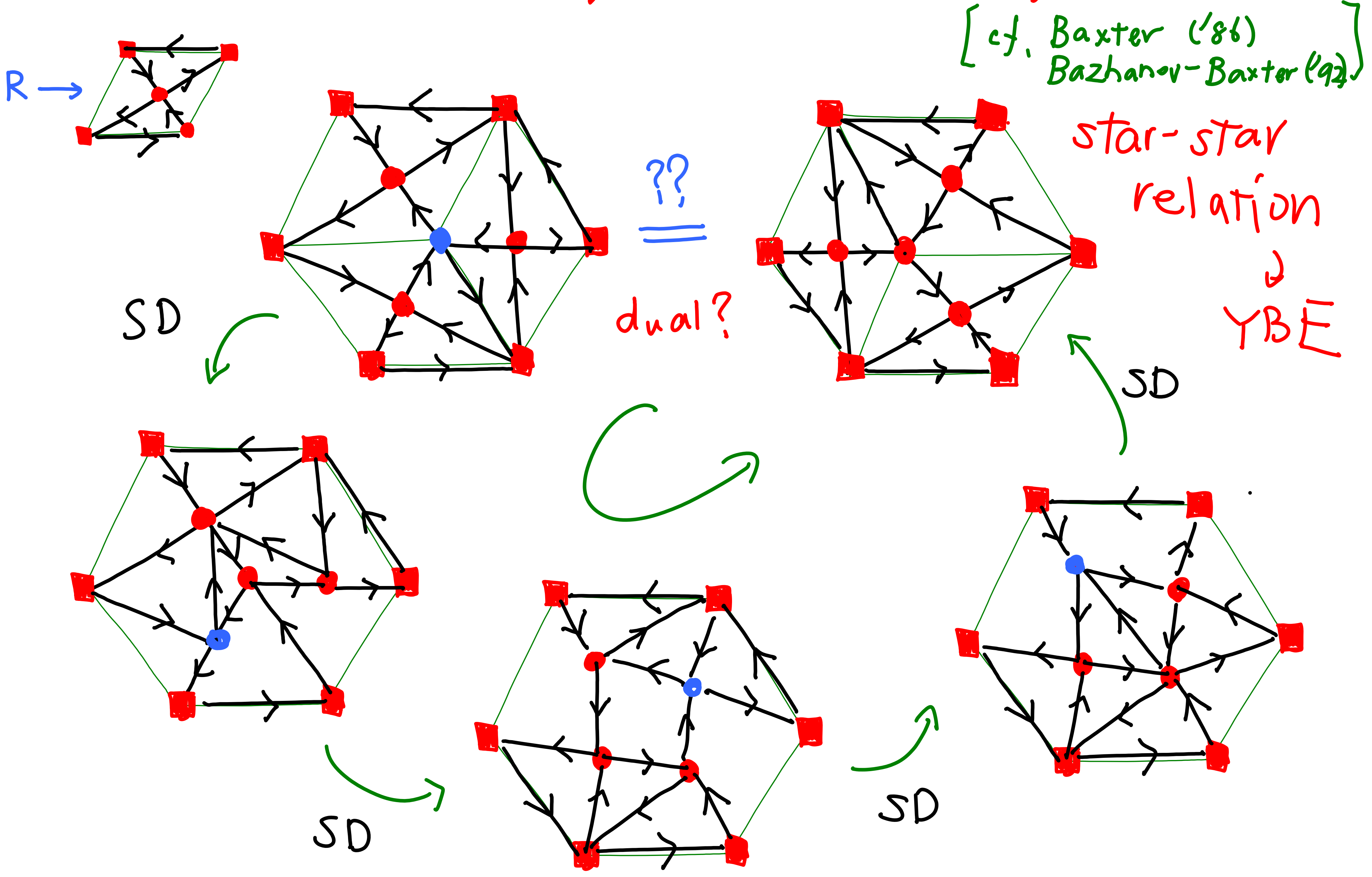
YBE ⇝ duality?



YBE $\rightsquigarrow$ duality?



YBE ⇝ "YB duality" = (Seiberg duality)^4 !!!



back to integrable models

$$T[R_{12}\circ R_{13}\circ R_{23}] \overset{\text{dual}}{\longleftrightarrow} T[R_{23}\circ R_{13}\circ R_{12}] \quad \text{YB duality}$$

$$\rightsquigarrow \quad Z\left[T[R_{12}\circ R_{13}\circ R_{23}]\right] = Z\left[T[R_{23}\circ R_{13}\circ R_{12}]\right]$$

back to integrable models

$$T[R_{12}\circ R_{13}\circ R_{23}] \overset{\text{dual}}{\longleftrightarrow} T[R_{23}\circ R_{13}\circ R_{12}]$$ YB duality

gauging $\rightsquigarrow$

$$Z\big[T[R_{12}\circ R_{13}\circ R_{23}]\big] = Z\big[T[R_{23}\circ R_{13}\circ R_{12}]\big]$$

$$\| \qquad\qquad\qquad\qquad\qquad \|$$

$$Z\big[T[R_{12}]\big]\circ Z\big[T[R_{13}]\big]\circ Z\big[T[R_{23}]\big] \qquad Z\big[T[R_{23}]\big]\circ Z\big[T[R_{13}]\big]\circ Z\big[T[R_{12}]\big]$$ YBE

sum

$$Z = \int \prod_{v\in V} \mathcal{D}A_\mu^v \prod_{e\in E} \mathcal{D}\Phi_e \; e^{i\mathcal{L}} \xrightarrow[\text{localization}]{} Z = \int \prod_{v\in V} d\sigma_v \; e^{i\mathcal{E}}$$

"spin"

$$\mathcal{L} = \sum_{v\in V} \mathcal{L}_v(A_\mu^v) + \sum_{e\in E} \mathcal{L}(\Phi_e, A_\mu^v, A_\mu^{v'})$$

$$\begin{pmatrix} A_\mu(x) \to \sigma \\ \Phi_e \to 0 \end{pmatrix}$$

$$\mathcal{E} = \sum_{v\in V} \mathcal{E}(\sigma_v) + \sum_{e\in E} \mathcal{E}(\sigma_v, \sigma_{v'})$$

$$Z = Z[S^1 \times S^3/\mathbb{Z}_r] \quad \text{[Benini-Nishioka-Y ('11)]}$$

$$= Z_{N,r}[p, q; \vec{R}_i]$$

N: rk of $G = SU(N)$

r: order of $S^3/\mathbb{Z}_r$

p, q: fugacity "quantum parameter"

$\vec{R}_i$: (R-charge) = (spectral parameter)

"spin" $\vec{S}_v$ = holonomy of the gauge field along $S^1 \times S^3/\mathbb{Z}_r$

$$= \vec{z}_v \in U(1)^{N-1} \ \& \ \vec{m}_v \in (\mathbb{Z}_r)^{N-1}$$

The resulting model is one of the most general solutions to YBE known in the literature

$$Z = \sum_{s_v\ (v\in V)} \left(\prod_{v\in V} \mathbb{S}^v(s) \right) \left(\prod_{e\in E} \mathbb{W}^e(R;s) \right)$$

$$= \sum_{m_v\ (v\in V)} \int_{|z_{v,m}|=1} \prod_{v\in V} \prod_{i=1}^{N-1} \frac{dz_{v,i}}{2\pi\sqrt{-1}z_{v,i}} \left(\prod_{v\in V} \mathbb{S}^v(z,m) \right) \left(\prod_{e\in E} \mathbb{W}^e(R;z,m) \right) ,$$

$$\mathbb{S}^v(s) = \mathbb{S}^v(s_v) = \left(\prod_{a=0}^{r-1} \frac{1}{n_{v,a}!} \right) \mathcal{S}_0^v(s_v) \left(\Gamma_{r,0}(1;p,q)\right)^{-(N-1)} \prod_{i\neq j} \Gamma_{r,[\![m_i-m_j]\!]} \left(\frac{z_{v,i}}{z_{v,j}};p,q \right)^{-1} ,$$

$$\mathcal{S}_0^v(s_v) = (p\,q)^{-\frac{1}{4r}\sum_{i\neq j}[\![m_{v,i}-m_{v,j}]\!][\![-m_{v,i}+m_{v,j}]\!]} .$$

$$\Gamma_{r,[\![m]\!]}(x;p,q) = \Gamma(xp^{[\![m]\!]};pq,p^r)\,\Gamma(xq^{r-[\![m]\!]};pq,q^r) ,$$

$$\Gamma(x;p,q) = \prod_{n_1,n_2\geq 0} \frac{1-x^{-1}p^{n_1+1}q^{n_2+1}}{1-x\,p^{n_1}q^{n_2}} .$$

$$
\begin{aligned}
\mathbb{W}^e(R;s) &= \mathbb{W}^e_{R_e}(s_{t(e)}, s_{h(e)}) \\
&= \mathcal{W}^e_0(s_{t(e)}, s_{h(e)}) \prod_{1\le i,j\le N} \Gamma_{r,[\![m_{t(e),i}-m_{h(e),j}]\!]}\left((pq)^{\frac{R_e}{2}}\frac{z_{t(e),i}}{z_{h(e),j}};p,q\right) ,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}^e_0(s_{t(e)}, s_{h(e)}) = \prod_{1\le i,j\le N} \Bigg[& (p\,q)^{\frac{1}{4r}[\![m_{t(e),i}-m_{h(e),j}]\!][\![-m_{t(e),i}+m_{h(e),j}]\!](1-R_e)} \\
& \times \left(\frac{p}{q}\right)^{-\frac{1}{12r}[\![m_{t(e),i}-m_{h(e),j}]\!][\![-m_{t(e),i}+m_{h(e),j}]\!](2[\![m_{t(e),i}-m_{h(e),j}]\!]-r)} \\
& \times \left(\frac{z_{t(e),i}}{z_{h(e),j}}\right)^{-\frac{1}{2r}[\![m_{t(e),i}-m_{h(e),j}]\!][\![-m_{t(e),i}+m_{h(e),j}]\!]} \Bigg] .
\end{aligned}
$$

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$$

Comments

The model has parameters $N, r; \underbrace{p, q}_{\text{elliptic}}$

We can consider specializations

$S^1 \times S^3/\mathbb{Z}_r \xrightarrow{r=1} S^1 \times S^3 \xrightarrow{p,q\to 1 \text{ + Higgsing}}$ Faddeev-Volkov (S^3) ← trigonometric

Benini Nishioka

$S^1 \times S^2 \to S^2$

$S^3/\mathbb{Z}_r \to S^2$

Bazhanov Sergeev model → Faddeev-Volkov, Kashiwara Miwa, chiral Potts

root of unity

Kashiwara Miwa → Fateev-Zamolodchikov

chiral Potts → Fateev-Zamolodchikov → Ising

$U_{p,q;r}(sl_N)$?

Feigin, Odesskii??

elliptic

$U_{p,q}(sl_N)$

Sklyanin

Cherednik

Witten, 1990 "gauge theories, vertex models & quantum groups"

There are several obvious areas for further investigation. In terms of statistical mechanics, one compelling question is to understand the origin of the spectral parameter (and the elliptic modulus) in IRF and vertex models; this is essential for explaining the origin of integrability.

✓ spectral parameter = R-charge

Another question, which may or may not be related, is to understand the spin models formulated only rather recently [24] in which the spectral parameter is not an abelian variable but is a point on a Riemann surface of genus greater than one.

If possible, one would also like to understand vertex models and quantum groups directly at physical values of q, without the less than appealing analytic continuation that is used in this paper.

Perhaps related to this, one would like if possible to see the origin of quantum groups and not just quantum Lie algebras.

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? We found $U_{p,q,r}(\mathfrak{g})$ See however [Costello ('13)]

Gauge/YBE from 6d

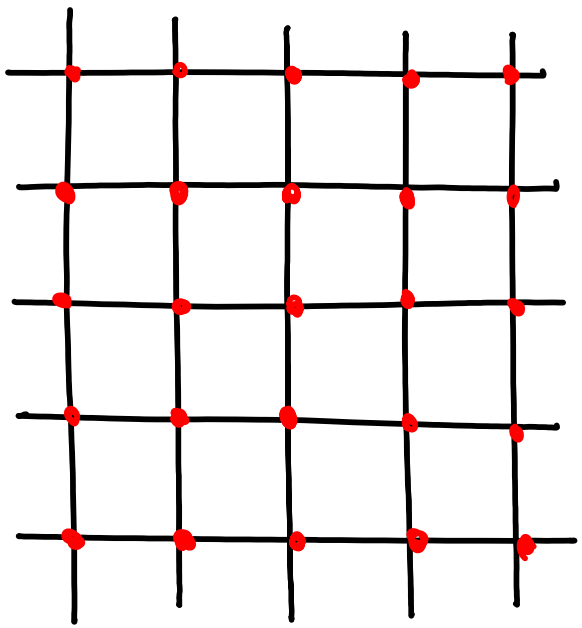
(Similar to AGT)

6d origin: D5/NS5 system

fixed boundary condition [Heckman-Vafa-Xie-MY] ('12)

periodic boundary condition [Imamura, Imamura-Kimura-Isono-Y ('07)]

2d spin system



$\times \ \mathbb{R}^{3,1}$

4d gauge theory

directly gives integrable model ☺

Gauge / YBE

$$I_{S^1\times S^3/\mathbb{Z}_n}\begin{bmatrix}4d\\ \mathcal{N}=1\end{bmatrix} = Z\left[\text{integrable model}\right]$$

$$\not\parallel$$

Gauge / Bethe

[Nekrasov Shatashvili]

Yang-Yang function

$$W_{\text{twisted}}\begin{bmatrix}2d\\ \mathcal{N}=(2,2)\end{bmatrix} = Y\left[\begin{matrix}\text{integrable}\\ \text{model}\end{matrix}\right]$$

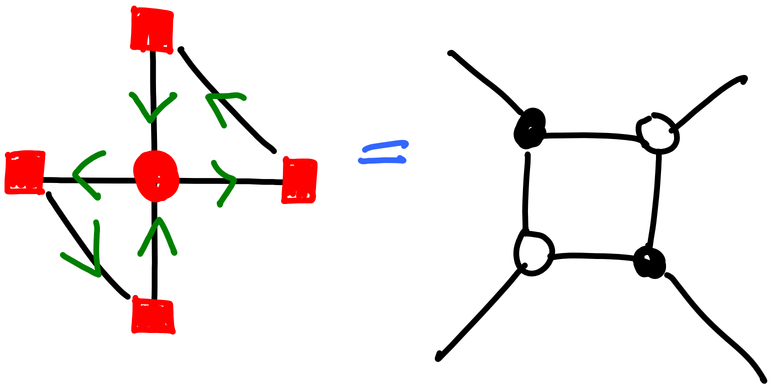
(※ in both cases we have "integrability in theory space")

Scattering Amplitude

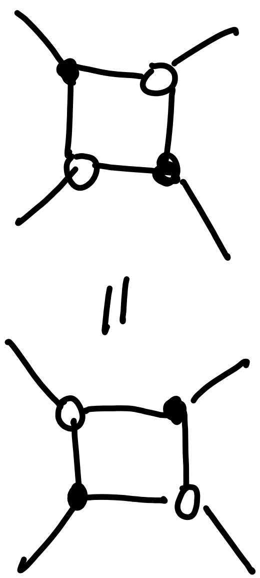
quivers $\longleftrightarrow$ bipartite graph $\rightsquigarrow$ 4d $\mathcal{N}=1$ gauge theory

[Hanany-Kennaway, ...
"brane tilings"
Xie-Yamazaki, Franco ('12)]

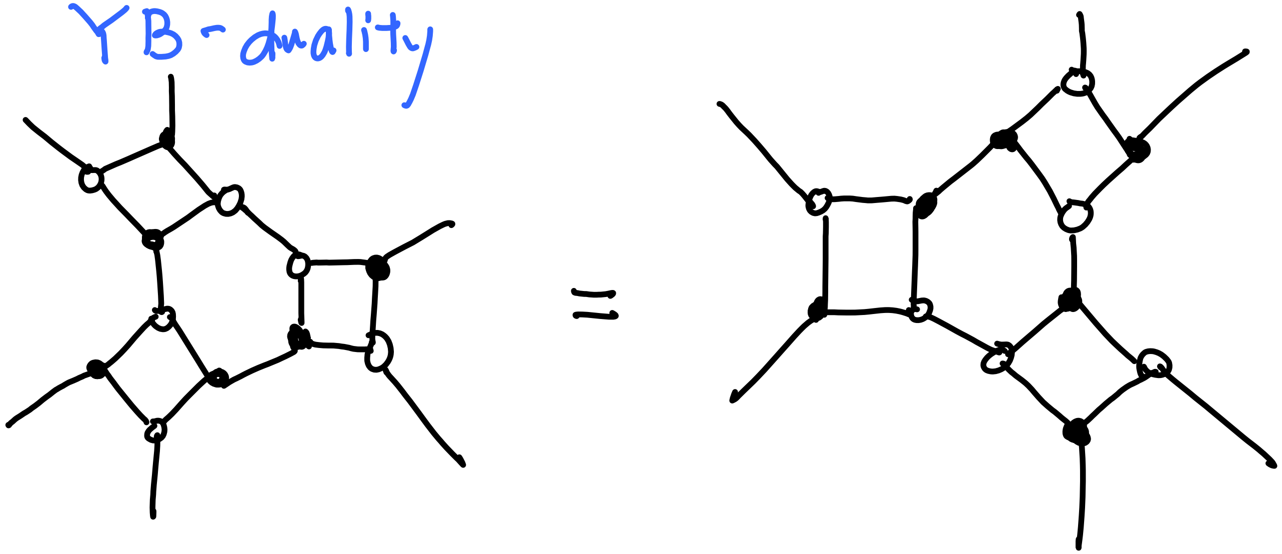
R-matrix:



Seiberg duality



YB-duality



On-shell diagram for 4d $\mathcal{N}=4$

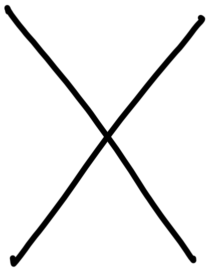
[Arkani-Hamed, Bourjaily, Goncharov, Postnikov, Trnka ('12)]

[also Ferro-Łukowski-Meneghelli-Plefka-Staudacher]

The connection with YBE is simpler for

3d ABJM theory

building block : 4 pt [ABGPT, Huang-Wen('13)]



orthogonal Grassmannian

$$A_4(z) = \frac{\delta^3(P)\,\delta^3(Q)}{\langle 12\rangle^{1+z}\,\langle 3\rangle^{1-z}} = \int \frac{d^{2\times 4}C}{|GL(2)|}\,\frac{\overbrace{\delta^3(C\cdot C^T)}^{OG(2,4)}\,\delta^4(C\cdot\Lambda)}{M_1^{1+z}\;M_2^{1-z}}$$

invariant under Yangion $Y[OSp(6|4)]$

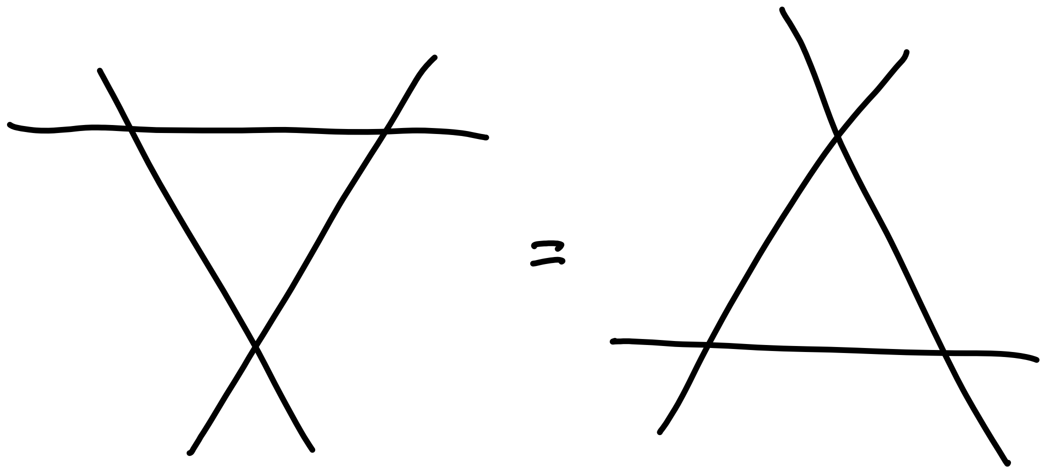
with evaluation parameters $(z, 0, z, 0)$

[Bargheer - Loebbert - Huang - Y]
(yesterday)

R- matrix : 4-pt

$$R(z) = A_4(z)$$

YBE : 6-pt / triangle move $\leftsquigarrow$ uniqueness of top cell of $OG(3,6)$



The Grassmannian integral $R(z)$

is an R-matrix for $Y[OSp(6|4)]$;

satisfy YBE / RLL = LLR relation

Conclusion

- $\text{YB-duality} = (\text{Seiberg duality})^4 \overset{Z}{\rightsquigarrow} \text{YBE}$

  (4d quiver gauge theory) (IM)

  $$I_{S^1 \times S^3/\mathbb{Z}_r}\left[\begin{matrix} 4d \\ N=1 \end{matrix}\right] = Z\left[\text{integrable model}\right]$$

- new, one of the most general solutions to YBE

- mysterious parallel w/ scattering amplitude

Q: We now understand R-matrix/YBE, but can we lift the rest to SUSY gauge theory?

[e.g. RLL = LLR relation as a duality?
BAE?]

Q: Integrable structure in space of QFTs?

New formulation of QFTs?