Polology of Conformal Blocks for $D > 2$ CFTs

やまざき まさひと (Kavli IPMU)

5·13·31年 1月 2^2·7日 KEK

Based on work in progress with excellent collaborators

J. Penedones 

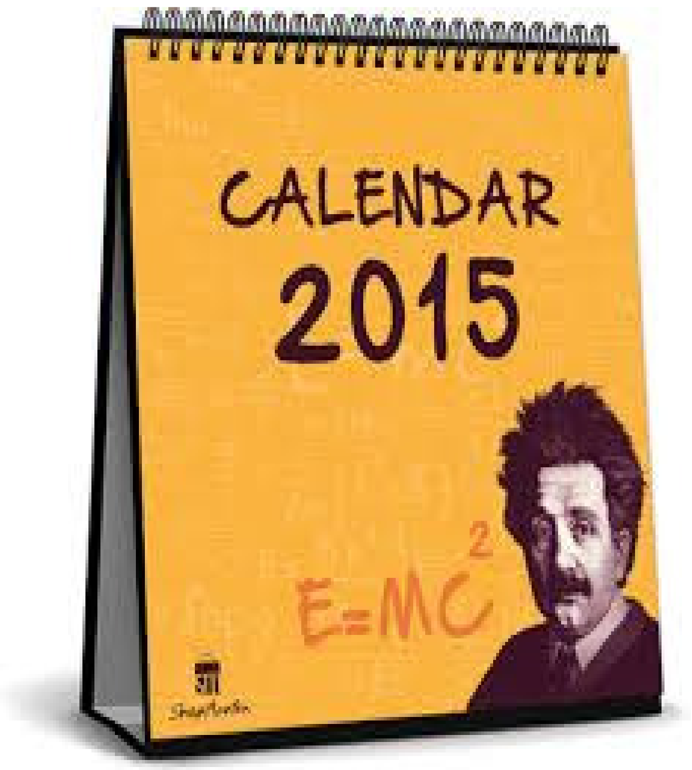
(Univ. Porto)

E. Trevisani 

and also another work by M. Y.

My first time in KEK
was 10 years ago, 2005

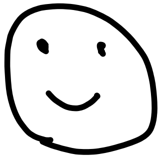
Long live the
KEK-theory workshop!!

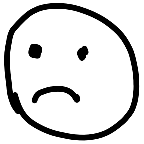


100 th
anniversary

Symmetry $\longrightarrow$ Physics

Symmetry $\longrightarrow$ Physics

 highly non-trivial constraints

 not all theories are not symmetric

Conformal Symmetry $\longrightarrow$ Physics

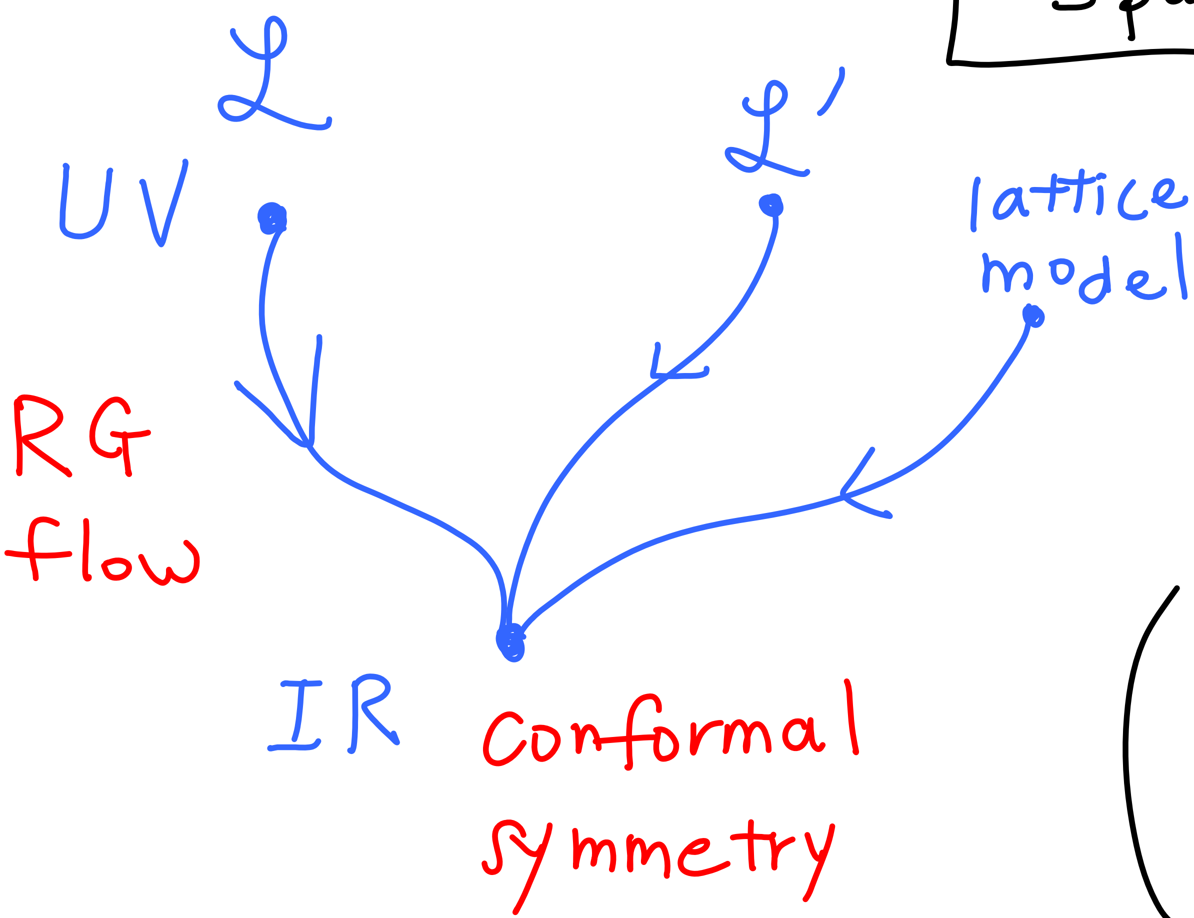
highly non-trivial constraints

AND

rather generic

☺ conformal symmetry is rather ~generic~

parameter space



Caveat

- Scale inv. but non-conformal theory
- limit cycle [Efimov effect]

☺ highly non-trivial constraints?

For $D=2$ Virasoro symmetry : ∞ - dim.

huge literature

For $D>2$ $SO(D,2)$: finite - dim.

nevertheless highly constraining

<u>next</u>

$\mathrm{CFT}_{D\,(=3)}$

$SO(3,2)$

P_μ, $J_{\mu\nu}$, D, K_μ (Cartan J_3, D)

$\mathcal{O}$: operator with spin ℓ

dimension Δ.

Given $\mathcal{O}$ $\rightsquigarrow$ descendant $P_{\mu_1} P_{\mu_2} \cdots\cdots \mathcal{O}$

$$\langle P_\mu \mathcal{O}(x) \cdots \rangle = -i\partial_\mu \langle \mathcal{O}(x) \cdots\cdots \rangle$$

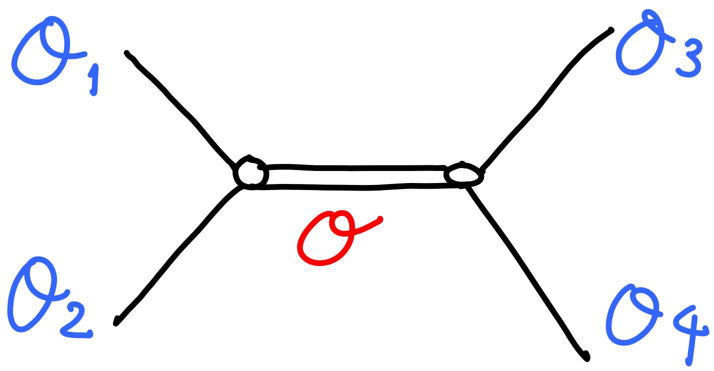
only primary is sufficient $[P, K] \sim J + D$

$K_\mu \mathcal{O} = 0$

Consider the 4-pt function
for scalar primaries $\mathcal{O}_1, \cdots, \mathcal{O}_4$

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3)\, \mathcal{O}_4(x_4) \rangle$$

$$= \sum_{\mathcal{O}}$$



Sum over conformal primaries

$$= \sum_{\mathcal{O}} C_{12\mathcal{O}}\, C_{34\mathcal{O}}\, G_{\Delta,\ell}(x_1, x_2, x_3, x_4)$$

structure constants determining 3-pts

$$\langle \mathcal{O}_1 \mathcal{O}_2\, \mathcal{O} \rangle \langle \mathcal{O}\, \mathcal{O}_3 \mathcal{O}_4 \rangle$$

$\mathcal{O}$ has dimension Δ
spin(s) ℓ

For example, consider the 4-pt function for scalar primaries $\mathcal{O}_1, \cdots, \mathcal{O}_4$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$$
$$= \sum_{\mathcal{O}} C_{12\mathcal{O}}\, C_{34\mathcal{O}}\, G_{\Delta,\ell}(x_1, x_2, x_3, x_4)$$

$$G_{\Delta,\ell}(x_1, x_2, x_3, x_4) = \frac{\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\, g_{\Delta,\ell}(r, \eta)$$

Conformal block

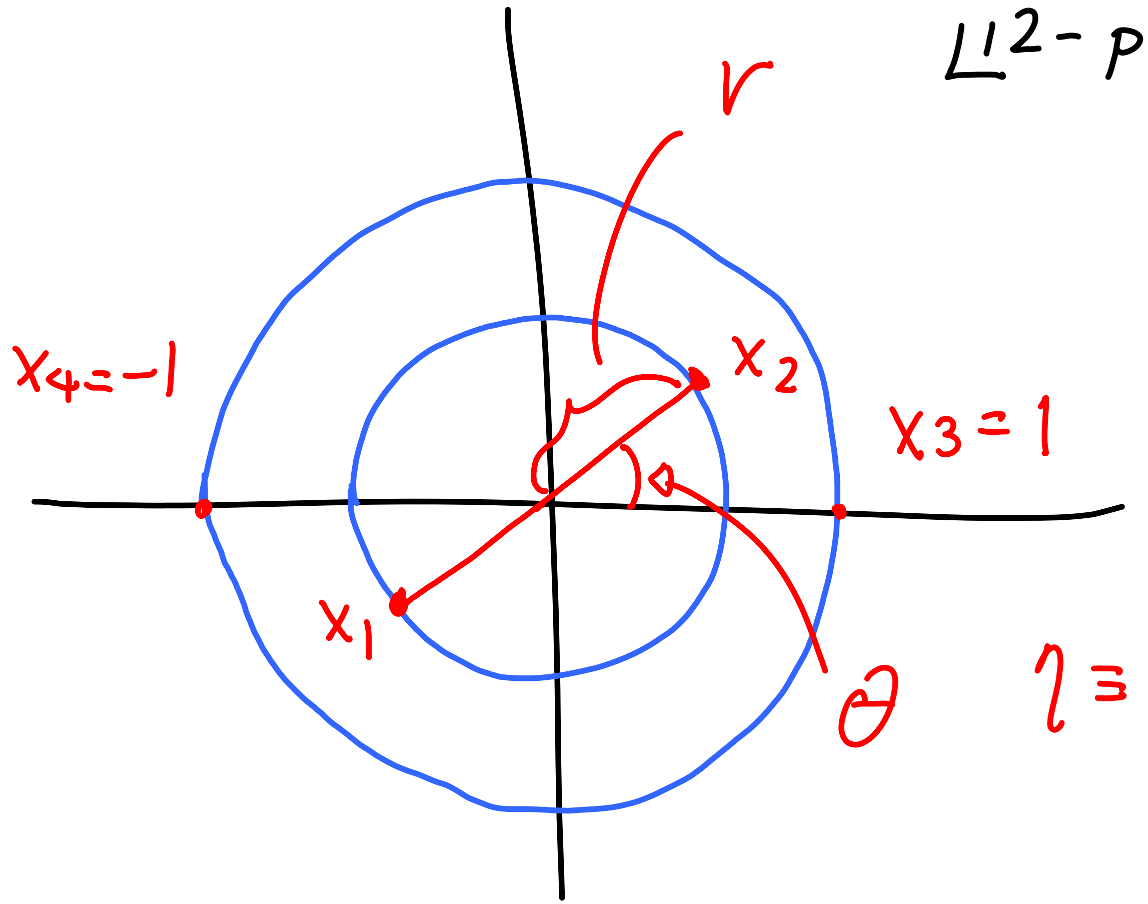
$\left[\begin{array}{l} \Delta_i : \text{dimension of } \mathcal{O}_i \\ \Delta_{ij} \equiv \Delta_i - \Delta_j \end{array}\right]$

conformally invariant cross ratios

r, η

radial coordinate [Hogervorst-Rychkov '13]

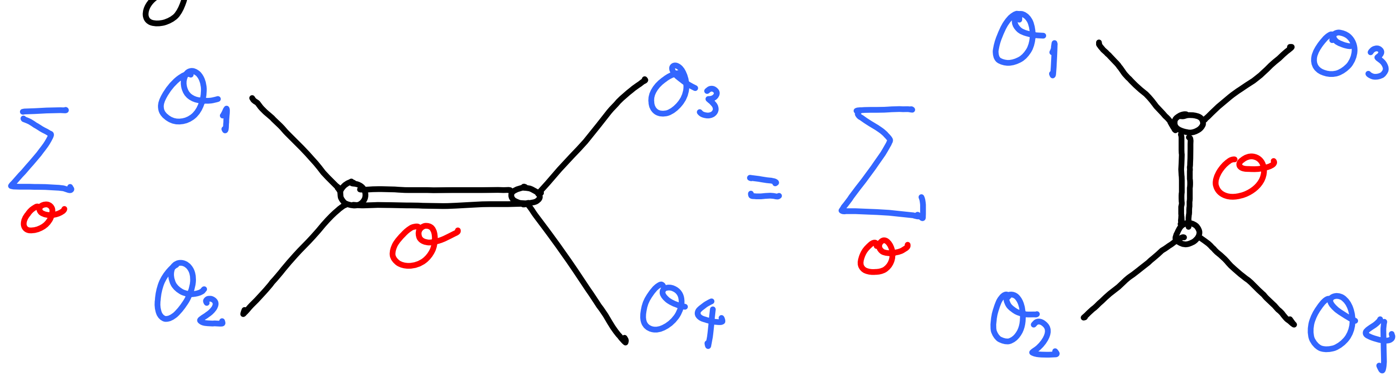
$\perp$ 12-plane



$\eta \equiv \cos\theta$

conformal block $g_{\Delta,\ell}(r,\eta)$

crossing relation



leads to constraints on CFTs

(conformal bootstrap) [Ferrara-Gatto-Grillo '73, Polyakov '74]

Recent revival (partly numerical)

[Rattazi-Tonni-Rychkov-Vichi '08, ⋯
$\mathcal{O}(100)$ papers, contributions by many authors]

We will not discuss these exciting developments today.

Instead, the question here is

how to compute the conformal block $g_{\Delta,\ell}(u,v)$,

which is the input for the numerical bootstrap

There are of course partial results in the literature;

the prototypical case is 4D CFT,

the 4-pt for equal scalars $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle$

[Dolan-Osborn, ...]

The natural next step is to study conformal blocks

- in general spacetime dimension $D > 2$
- in general # of SUSY $\mathcal{N}$
- for (unequal) operators of general spin

  e.g current J^{μ}

  stress-energy tensor $T^{\mu\nu}$

Here we propose a systematic approach, and work out the details for the 4-pt of (conserved) currents $\langle J^{\mu_1} J^{\mu_2} J^{\mu_3} J^{\mu_4} \rangle$ for $D=3$ $(\mathcal{N}=0)$ CFTs

Basic Idea: Recursion

Result

(for scalar block for simplicity; our main focus is really for vector blocks)

for scalars [Kos + Simmons-Duffin + Poland ('13)]

derivation & generalization [Penedones-Trevisani-Y]

$$g_{\Delta,\ell}(r,\eta) = (4r)^{\Delta} h_{\Delta,\ell}(r,\eta)$$

$$h_{\Delta,\ell}(r,\eta) = h_{\Delta=\infty,\ell}(r,\eta) + \sum_A \frac{R_A}{\Delta-\Delta_A^*}(4r)^{n_A} h_{\Delta_A,\ell_A}(r,\eta)$$

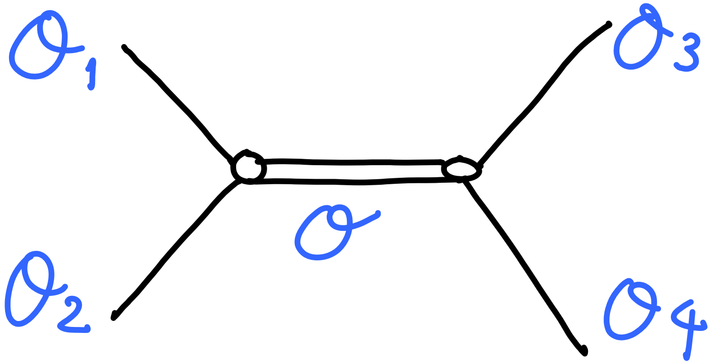
$n_A > 0$

similar result for D=2 [Zamolodchikov]

Scala

$$\langle O_1(x_1)\, O_2(x_2)\, O_3(x_3)\, O_4(x_4)\rangle$$

$= \sum_{O}$



$= \sum_{O} C_{12O}\, C_{34O}\, G_{\Delta,\ell}$

$$= \sum_{O} \left[\sum_{\alpha \in \mathcal{H}_O} \frac{\langle 0| O_1 O_2 |\alpha\rangle \langle \alpha | O_3 O_4 | 0\rangle}{\langle \alpha | \alpha \rangle} \right]$$

primary

descendant of O

$K_\mu O = 0$

$\alpha = P_{\mu_1} \cdots P_{\mu_n} O$

lowers Δ

raises Δ

This diverges when $|\alpha\rangle$ is a null state

i.e. $\langle \alpha | \alpha \rangle = 0$

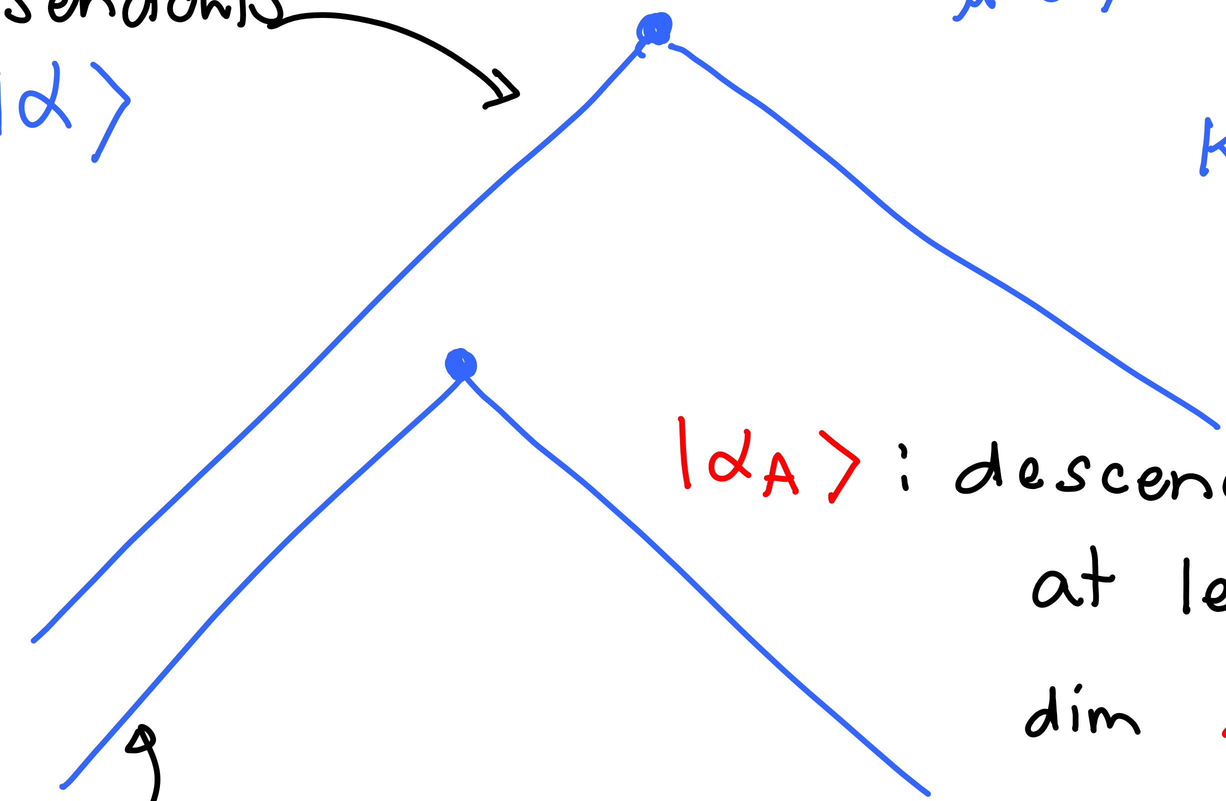
Schematically (more refined version later)

$|0\rangle$: primary dim $\Delta = \Delta_A^*$

$K_\mu |0\rangle = 0$

$K_\mu \uparrow \downarrow P_\mu$

tower of desendants $|\alpha\rangle$



$|\alpha_A\rangle$: descendant <u>and</u> primary at level n_A, i.e, dim $\Delta_A = \Delta_A^* + n_A$

$\rightsquigarrow \alpha_A$: null

$\left[\because \langle \alpha_A | \alpha_A \rangle = (\langle \, | P_\mu)^\dagger | \alpha_A \rangle = \langle \, | K_\mu | \alpha_A \rangle = 0 \right]$

tower of descendant of $|\alpha_A\rangle$

$$G_{\Delta,\ell} \xrightarrow{\Delta \to \Delta_A} \frac{R_A}{\Delta - \Delta_A^*} G_{\Delta_A = \Delta_A^* + n_A,\, \ell_A}$$

where $R_A = M_A^{(L)} Q_A M_A^{(L)}$

$$\frac{\langle 0| \mathcal{O}_1 \mathcal{O}_2 |\alpha\rangle \langle \alpha| \mathcal{O}_3 \mathcal{O}_4 |0\rangle}{\langle \alpha | \alpha \rangle}$$

(explicit form of null state $\rightsquigarrow$ $M_A^{(L)}$ $M_A^{(R)}$, Q_A $\rightsquigarrow$ R_A)

By collecting these,

$$G_{\Delta,\ell} \sim \sum_A \frac{R_A}{\Delta - \Delta_A^*} G_{\Delta_A = \Delta_A^* + n_A,\, \ell_A}$$

At $\Delta \to \infty$, we have

$$g_{\Delta,\ell}(r,\eta) \sim (4r)^{\Delta}$$

This gives

radial coordinate

sum over null states

$$g_{\Delta,\ell}(r,\eta) =: (4r)^{\Delta}\, h_{\Delta\ell}(r,\eta)$$

conformal block

$$\boxed{h_{\Delta,\ell}(r,\eta) = h_{\Delta\to\infty,\eta}(r,\eta) + \sum_{A} \frac{R_A}{\Delta-\Delta_A^*} (4r)^{n_A \nearrow 0}\, h_{\Delta_A \ell_A}(r,\eta)}$$

determined by Conformal Casimir eq.

$$\left(R_A = M_A^{(L)} Q_A M_A^{(R)} \right)$$

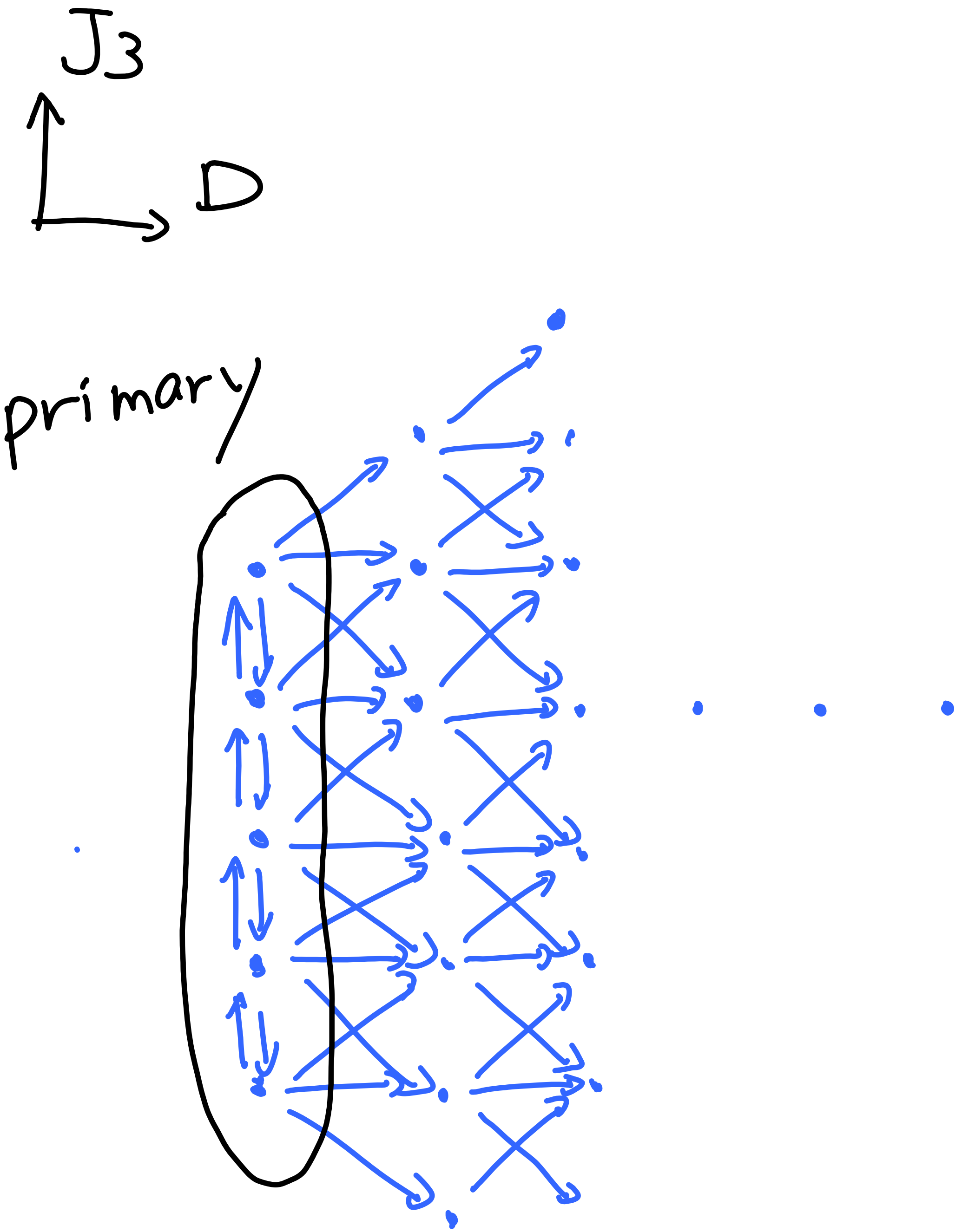
norm

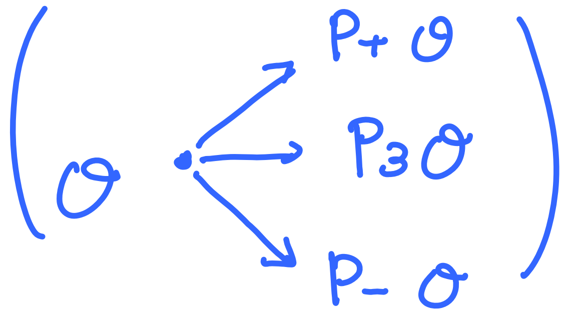
3-pt normalization

$h_{\Delta,\ell}(r,\eta)$ can be computed efficiently order by order in r

Null States

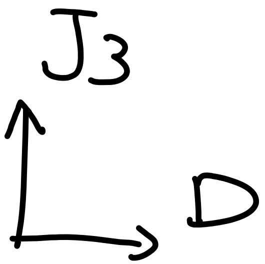
← analog of Kac determinant formula for parabolic Verma module

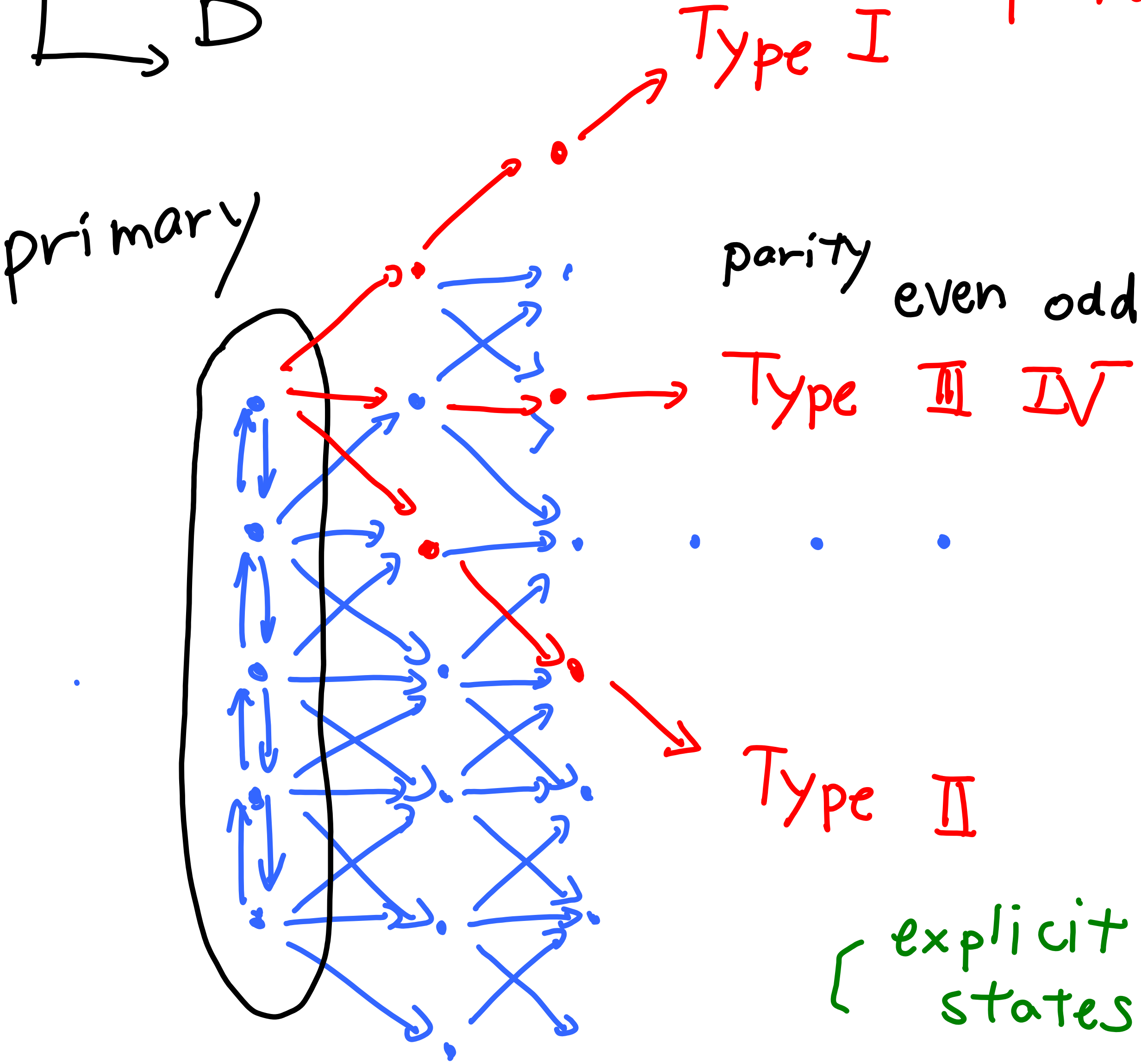


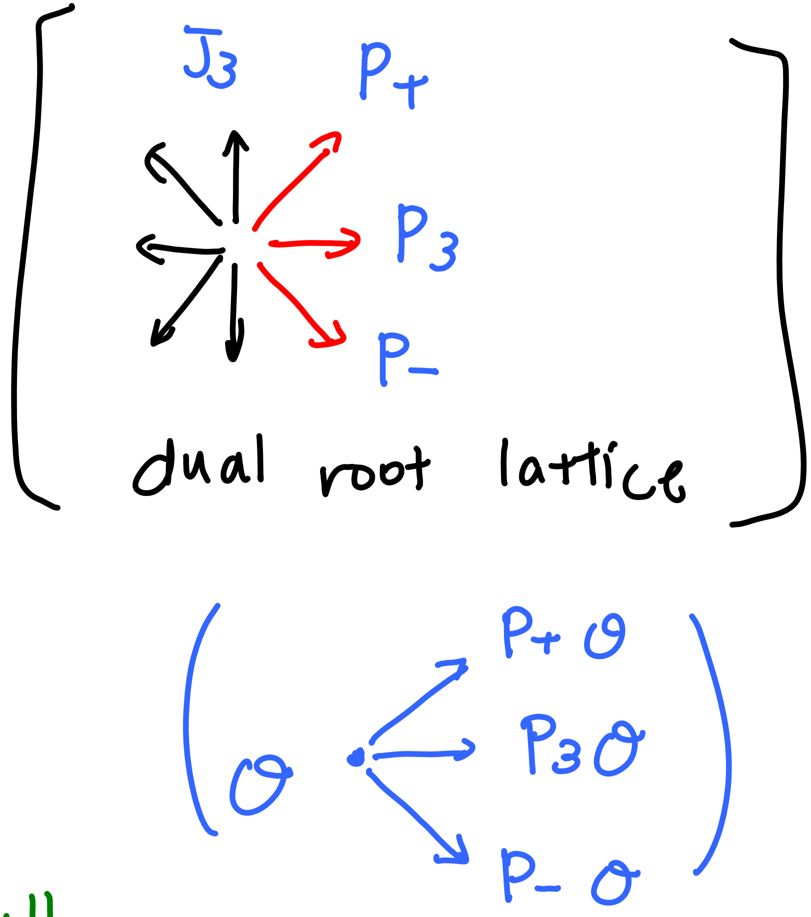


Null States

⟵ analog of Kac determinant formula for parabolic Verma module







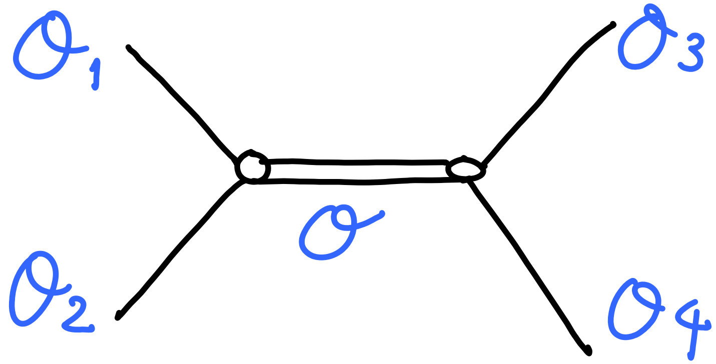
[explicit null states in our work]

Our main focus was $\langle J^\mu J^\nu J^\rho J^\sigma \rangle$

We then have many "conformal blocks"

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3)\, \mathcal{O}_4(x_4) \rangle \qquad \mathcal{O}_i : \text{spin } 1$$

$$= \sum_{\mathcal{O}}$$



Sum over conformal primaries

$$= \sum_{p,q}^{5} \sum_{\mathcal{O}} C^{(p)}_{12\mathcal{O}}\, C^{(q)}_{34\mathcal{O}}\, G^{(p,q)}_{\Delta,\ell}(x_1, x_2, x_3, x_4)$$

can be expanded in 43 different structures

$\langle J_{\mu_1} J_{\mu_2} J_{\mu_3} J_{\mu_4} \rangle$ 43

$\langle T_{\mu_1\nu_1} T_{\mu_2\nu_2} T_{\mu_3\nu_3} T_{\mu_4\nu_4} \rangle$ 633

works for $D>3$

looks messy, but doable ✓

- encoding sym. traceless tensors by polynomials

$f_{\mu_1 \cdots \mu_n} z^{\mu_1} \cdots z^{\mu_n}$

$(z^\mu z_\mu = 0)$

- embedding space formalism

[many papers from 70's

Costa + Penedones + Poland + Rychkov '11]

recursion relation fast for numerics

$\rightsquigarrow$ application ?

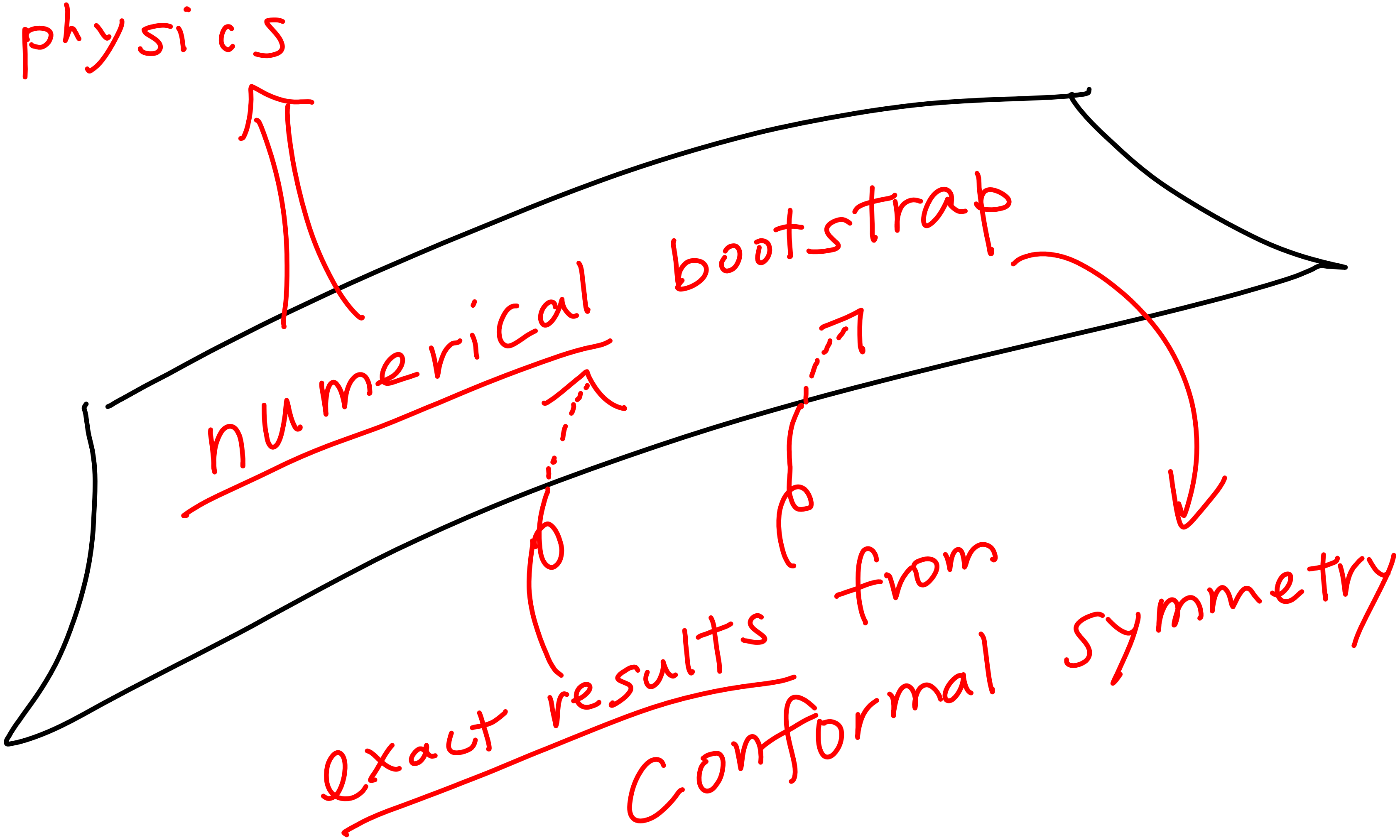
Summary

- recursion relation for conformal blocks for exchange of sym. traceless tensor in $D>3$ dim $\langle J^\mu J^\nu J^\rho J^\sigma \rangle$

$$g_{\Delta,\ell}(r,\eta) =: (4r)^{\Delta} h_{\Delta\ell}(r,\eta)$$

$$h_{\Delta,\ell}(r,\eta) = h_{\Delta\to\infty,\eta}(r,\eta) + \sum_A \frac{R_A}{\Delta-\Delta_{A*}} (4r)^{n_A} h_{\Delta_A \ell_A}(r,\eta)$$

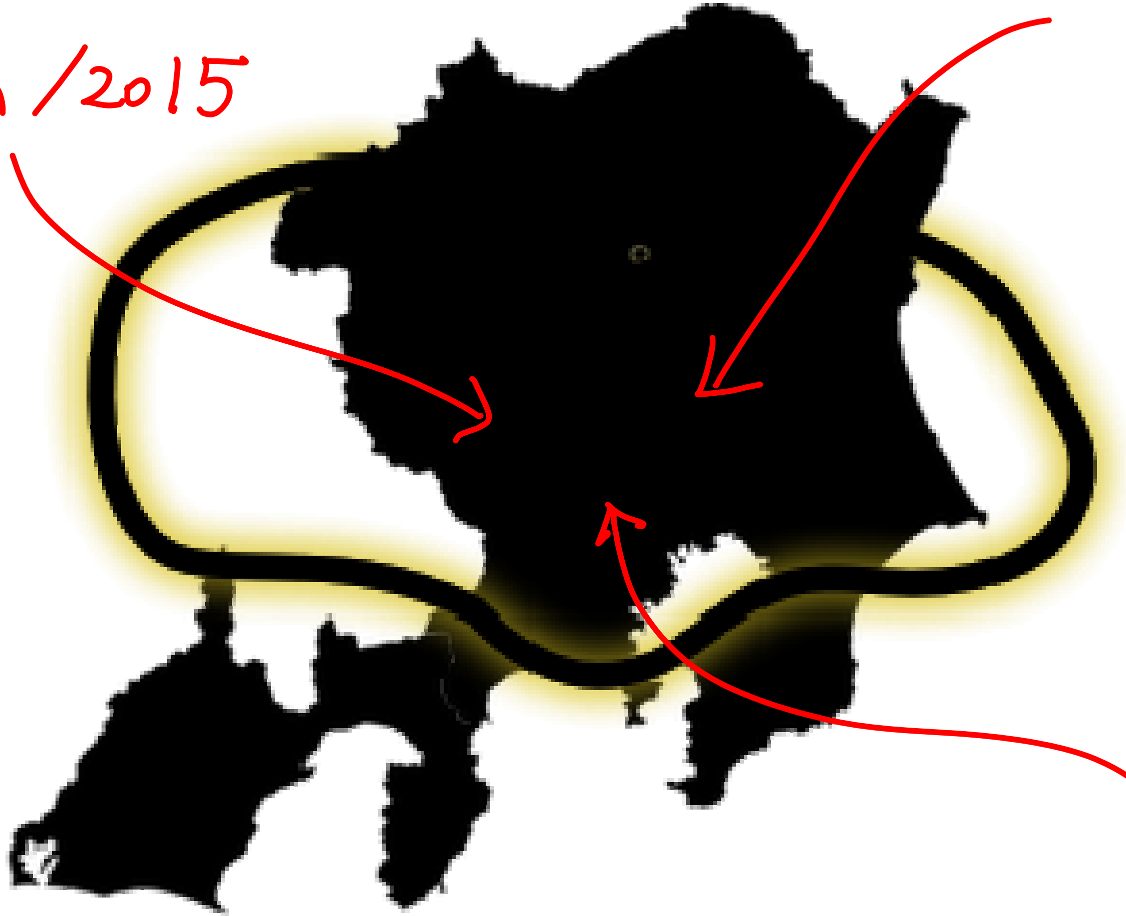
- Systematic & analytic approach to numerical bootstrap


physics
numerical bootstrap
exact results from
Conformal symmetry

"String Theory in Greater Tokyo"

2. RIKEN
May-Jun/2015

1. IPMU
Jan/2015



3. Chuo ?
Fall/2015

4, 5, ··· volunteers welcome!


東京大学
THE UNIVERSITY OF TOKYO
International Workshop on
Condensed Matter Physics and AdS/CFT
May 25 - 29, 2015
Kavli IPMU, Kashiwa campus, University of Tokyo
Poster Session & Gong Show!
Register until Feb. 28, 2015!
Invited Speakers include
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Zhong Fang
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Shin-ichi Sasa
Suchitra Sebastian
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Cenke Xu
Matt Visser
Ben Craps
Sumit Das
Johanna Erdmenger
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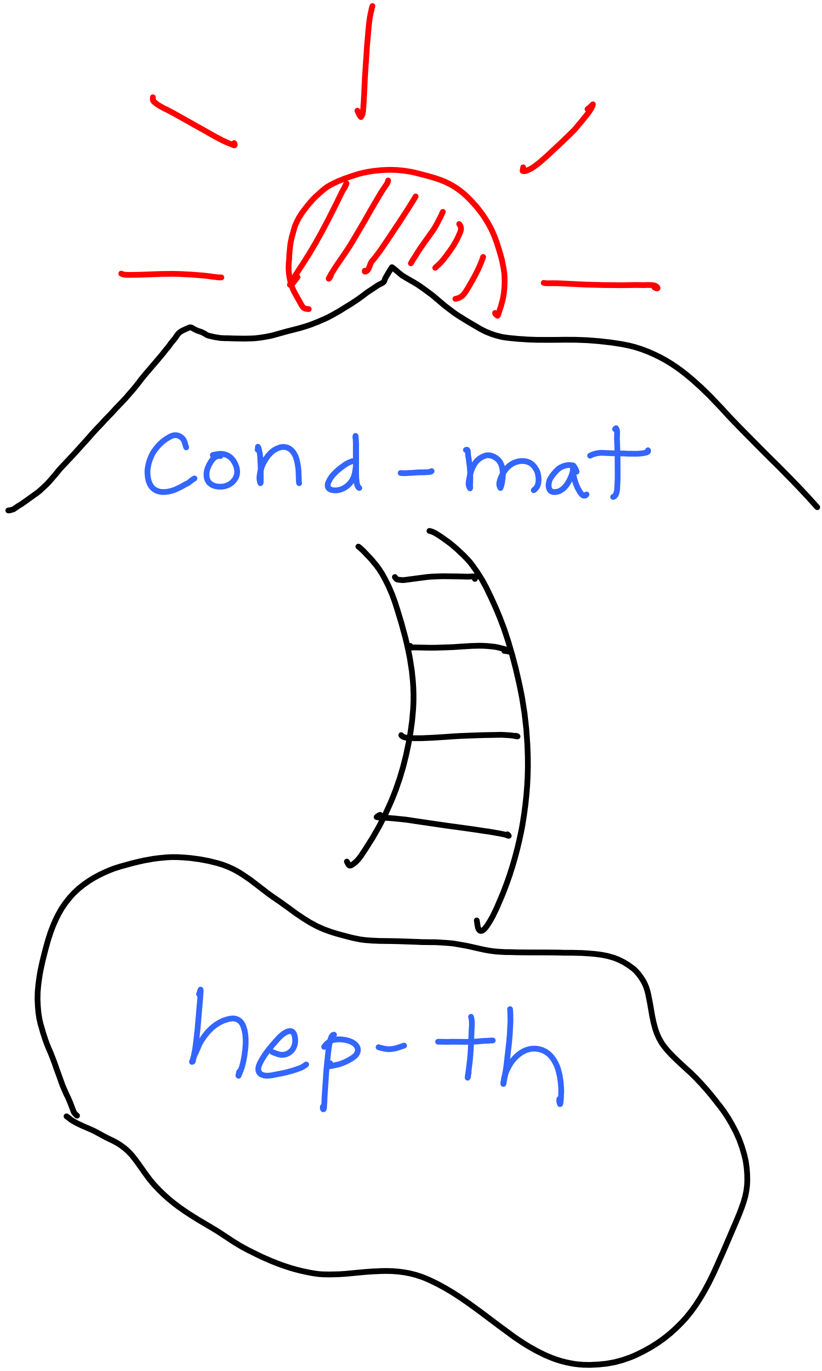
cond-mat
hep-th

$$\boxed{h_{\Delta,\ell}(r,\eta) = h_{\Delta=\infty,\ell}(r,\eta) + \sum_A \frac{R_A}{\Delta-\Delta_A}(4r)^{n_A}\, h_{\Delta_A \ell_A}(r,\eta)}$$

We need

α. null states A ← below

β. Residue R_A

γ. large Δ behavior $h_{\Delta=\infty,\ell}(r,\eta)$

conformal Casimir eq.

$$\boxed{h_{\Delta,\ell}(r,\eta) = h_{\Delta\to\infty,\eta}(r,\eta) + \sum_A \frac{R_A}{\Delta-\Delta_A}(4r)^{n_A} h_{\Delta_A \ell_A}(r,\eta)}$$

$$\left(R_A = M_A^{(L)} Q_A M_A^{(R)} \right)$$

- Similar recursion for 2D CFT, albeit wrt c, not Δ

  [Zamolodchikov]

- For $D=3$ $\quad \langle \phi\phi\phi\phi \rangle$ [Kos Poland Simmons-Duffin] ('13 '14)

  $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$