Integrability as Duality

Masahito Yamazaki

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integrable models is characterized by Yang-Baxter equation with spectral parameters
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Yang-Baxter equation with spectral parameters

\[ R_{12} = R_{12}(z_1 - z_2) \in \text{End}(V_1 \otimes V_2) \]
integrable models is characterized by Yang-Baxter equation with spectral parameters $R_{12}(z_1-z_2) R_{13}(z_1-z_3) R_{23}(z_2-z_3) \in \text{End}(V_1 \otimes V_2 \otimes V_3)$.
Why integrable models exist?
Why integrable models exist?

Perspectives from QFT?
Why integrable models exist?

Perspectives from QFT?

Origin of spectral parameter?

New integrable models?
I myself have worked mainly on two approaches:

1. 4d N=1 supersymmetric quiver gauge theories
   (Gauge/YBE correspondence)

2. ”4d Chern-Simons”
   [Costello] (‘12),
   (see also MY’s talk at Strings 2018 next week)
I myself have worked mainly on two approaches:

1. 4d N=1 supersymmetric quiver gauge theories
   (Gauge/YBE correspondence)
   [Y, Terashima-Y] ('12), [Y] ('13),

2. "4d Chern-Simons"
   [Costello] ('12),
   [Costello-Witten-Y] ('17,'18): Part I-IV
   (see also MY’s talk at Strings 2018 next week)
Integrability from 4d Quiver Gauge Theories
Initiated in 2012-2013, partly with Yuji Terashima [Y] [Terashima-Y] (‘12) and [Y] (‘13)

and inspired in particular by [Bazhanov-Sergeev] (‘10) (‘11)

“master solution”
Initiated in 2012-2013, partly with Yuji Terashima [Y] [Terashima-Y] (‘12) and [Y] (‘13)

and inspired in particular by [Bazhanov-Sergeev] (‘10) (‘11)

Since then more works in collaboration with Andrew P. Kels, Wenbin Yan and others

Related works by e.g. Bazhanov, Chicherin, Derkachov, Dolan, Gahramanov, Jafarzade, Mangazeev, Maruyoshi, Nazari, Osborn, Rains, Sergeev, Spiridonov (in particular [Spiridonov] (‘10)), Yagi, Zabrodin,....
Three Basic Ingredients
1. statistical lattice as a quiver diagram

spin $S_{\nu}$ at vertex $\nu$
1. statistical lattice as a quiver diagram

spin \( S_v \) at vertex \( v \)

partition function

\[
Z = \sum_{\{s_v\}} e^{-\mathcal{E}(\{s_v\})}
\]
1. statistical lattice as a quiver diagram

partition function

\[ Z = \sum_{\{s_v\}} e^{-E(\{s_v\})} \]

Boltzmann weight

\[ E(\{s_v\}) = \sum_v E^v(s_v) + \sum_e E^e(\{s_v\}_v \in e) \]
1. statistical lattice as a quiver diagram

gauge field $A_\nu$ for vertex $\nu$

bifundamental matter $\Phi_e$ for each edge $e$
1. statistical lattice as a quiver diagram

- gauge field $A_v$ for vertex $v$
- bifundamental matter $\phi_e$
- partition function

$$Z = \int dA^v d\phi_e e^{-L}$$
1. statistical lattice as a quiver diagram

gauge field $A_\nu$ for vertex $\nu$

bifundamental matter $\Phi_e$

for each edge $e$

partition function

$$Z = \int dA_\nu d\Phi_e e^{-L}$$

Lagrangian

$$L\left(\{A_\mu, \Phi_e\}\right) = \sum_\nu L^\nu\left(A_\nu\right) + \sum_e L^e\left(\Phi_e, \{A_\nu\}_\nu\right)$$
2. supersymmetric localization

Once super-symmetrize the setup, the difference goes away:

\[ Z = \int D A_v \; D \Phi_e \; e^{-L} \]

\[ \text{path-integral} \]
2. supersymmetric localization

Once super-symmetrize the setup, the difference goes away:

\[ Z = \int \mathcal{D} A_v \mathcal{D} \Phi_e \ e^{-L} \]

\[ = \int \mathcal{D} \delta_v e^{-\varepsilon} \]

\(< \text{path-integral} \)

\(< \text{finite-dim integral} \)

\(< \text{parametrize saddle point locus} \)
2. supersymmetric localization

Once super-symmetrize the setup, the difference goes away:

\[ Z = \int \mathcal{D} A_v \mathcal{D} \Phi_e \, e^{-L} \]
\[ = \int d\sigma_v \, e^{-\mathcal{E}} \]

\[ \mathcal{E} = \sum_v \mathcal{E}^v(\sigma_v) + \sum_e \mathcal{E}^e(\{\sigma_v\}_v \in e) \]

\( v \): vertex \hspace{1cm} \( e \): edge

One-loop determinant
3. Seiberg(-like) duality
3. Seiberg(-like) duality

$\text{Phys: 4d N}=1$ Seiberg duality [Seiberg] ('94)
(or their cousins in lower dimensions
[Aharony] ('97), [Hori-Tong] ('06), [Gadde-Gukov] ('13))
3. Seiberg(-like) duality

Math: (special example of) quiver mutation [Fomin-Zelevinsky] (‘01)
3. Seiberg(-like) duality

This (in a different disguise) is known as star-star relation in integrable models, originally in the context of tetrahedron equation [Baxter ('86), Bazhanov-Baxter ('92)]

Interestingly, the connection to Seiberg duality (or mutation) was noticed only recently
Since star-star relation implies YBE [Baxter (‘86), Bazhanov-Baxter (‘92)], once we solve SSR we have an integrable model.

We also know that star-star relation is Seiberg duality, so their partition function (in the IR) should coincide.

By combing these two observations we will automatically land on integrable models!
To explain YBE in more detail, it is useful to represent SSR more symmetrically, by adding “half-arrows” [Yan-Y] (‘15)
We can cancel half and full arrows:
The **R-matrix** is identified with a simple quiver ("theory for the R-matrix")
Products of R-matrix is obtained by gluing:

- Phys: gluing three theories by gauging flavor symmetries (as in "class S" theories)
- Math: gluing three quivers by quiver amalgamation
The YBE reads
The YBE reads

\[
\begin{array}{c}
\text{or equivalently}
\end{array}
\]

This is the Yang-Baxter duality \([Y]\) (‘13)
The YBE reads

or e

This is the Yang-Baxter duality
The YBE reads

or equivalently

This is the Yang-Baxter duality

I’m not a duality!
YBE follows from star-star repeated four times

= 

= 

= 

= 

= 

YBE follows from star-star-star repeated four times
Spectral Parameter?
“spectral parameter = R-charge”

The spectral parameter in integrable models matches with the R-charge in quiver gauge theories, found in [Hanany-Vegh] (‘05).

Both are associated with “zig-zag path”, discussed also in mathematical literature [Thurston] (‘04), [Goncharov-Kenyon] (‘11) …. 
spectral parameter: associated with “zig-zag path”
“Z-invariant lattice” [Baxter]
doubled rapidity line
Now go back to Yang-Baxter duality
Now go back to Yang-Baxter duality
“doubled Yang-Baxter equation”
Side remark: zig-zag paths are actually branes!

NS5-D5 brane realizations studied by MY’s master thesis [Y] (’08); T-dual to [Feng-He-Kennaway-Vafa] (’05), see also [Imamura] (’07), [Imamura-Isono-Kimua-Y] (’07)
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Basically, we have $N$ D5 wrapping torus, divided by zig-zag path = NS5-brane

We have an SU(N) quiver node for each region of D5-branes

(We actually need $(N, \pm 1)$-branes, in addition to $(N,0)$-branes; I will suppress this point now)

[Imamura] ('07), [Imamura-Isono-Kimua-Y] ('07), [Y] ('08)
The question I addressed in [Y] (‘08) was to relate the smooth holomorphic curve to the combinatorics of zig-zag path, by certain degeneration process.

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Very similar idea discussed as "Lagrangian skeletons" in [Shende-Treumann-Williams-Zaslow] (‘15)
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The brane realization could also be a starting point for exploring the relation with “4d Chern-Simons” approach to integrable models studied in [Costello] ('12), [Costello-Witten-Y] ('17, '18).

This requires further study, see e.g. [Vafa-Y] (to appear), [Costello-Yagi] (forthcoming); cf. [Ashwinkumar-Tan-Zhao] ('18) ....
Summarizing....
integrable model

YBE
QFT ("categorification")

Yang-Baxter duality

integrable model

YBE
QFT ("categorification")

Yang-Baxter duality

integrable model

partition function = functor

See my Japanese book (‘15)
QFT ("categorification")

Yang-Baxter duality

dim red, mass def.

YBE

integrable model

YBE2

$\mathbb{Z}$

$\mathbb{Z}_2$
QFT ("categorification")

Yang-Baxter duality

integrable model

YBE

YBE2

YBE3

YBE4

Yang-Baxter duality
QFT (“categorification”) \[\sim\] Yang-Baxter duality\[\sim\] integrable model \[\sim\] YBE \[\sim\] YBE2 \[\sim\] YBE3 \[\sim\] YBE4
New Integrable Models
We obtain many new solutions of the R-matrix from partition functions

\[ 4d \ N = 1 \quad S' \times S^3/\mathbb{Z}_r, \ S \times T^2, \ldots \]

\[ 3d \ N = 2 \quad S' \times S^2, \ S^3/\mathbb{Z}_r, \ldots \]

\[ 2d \ N = (2,2) \quad S^3, \quad T^2, \ldots \]

We obtain many new solutions of the $R$-matrix from partition functions.

$4d \, N=1$

(lens) elliptic

$S' \times S^3/\mathbb{Z}_r, \ S_x \mathbb{T}^2, \ ...$

(lens) elliptic gamma: $\Gamma'(x;i,p,\ell)$

$3d \, N=2$

trigonometric

$S' \times S^2, \ S^3/\mathbb{Z}_r, \ ...$

$\theta$-dilog: $(x; q)_\infty, \ S_b(x)$

$2d \, N=(2,2)$

rational

$S^3, \ \ \ \ \ \ \ \mathbb{T}^2, \ ...$

$\Gamma(x)$

$\Theta(x; 12)$

They do NOT fit into [Belavin-Drinfeld] (‘82) classification.
We obtain many new solutions of the R-matrix from partition functions

4d $\mathcal{N}=1$

3d $\mathcal{N}=2$

2d $\mathcal{N}=(2,2)$

dimensional reduction

[Gadde-Yan] [Dolan-Spiridonov-Vartanov] [Imamura] ('11) [Y] [Benini-Cremonesi] ('13),
We obtain many new solutions of the R-matrix from partition functions

\[ \begin{align*}
4d \ N = 1 & \quad S' \times S^3/\mathbb{Z}_r, \ S^2 \times T^2, \ \ldots \ldots \\
3d \ N = 2 & \quad S' \times S^2, \ S^3/\mathbb{Z}_r, \ \ldots \ldots \\
2d \ N = (2,2) & \quad S^3, \quad T^2, \ \ldots \ldots
\end{align*} \]

mixture of cluster algebra & YBE
"cluster-enriched YBE"

[Y] ('16), based on [Benini-Park-Zhao] ('14)
We obtain many new solutions of the R-matrix from partition functions

4d $\mathcal{N} = 1$

$S' \times S^3/\mathbb{Z}_r, S^2 \times T^2, \ldots$

3d $\mathcal{N} = 2$

$S' \times S^2, S^3/\mathbb{Z}_r, \ldots$

2d $\mathcal{N} = (2,2)$

$S^3, T^2, \ldots$

Jeffrey-Kirwan residue

“super-master solution” [Y] (‘13) (for $r=1$ “master solution” of [Bazhanov-Sergeev] (‘10), and contains all known solutions of STR with positive Boltzmann weight)

\[
\begin{align*}
4d \quad N=1 & \quad S^1 \times S^3/\mathbb{Z}_r, \quad S^2 \times T^2, \ldots \\
3d \quad N=2 & \quad S^1 \times S^2, \quad S^3/\mathbb{Z}_r, \ldots \quad \ldots \\
2d \quad N=(2,2) & \quad S^3, \ldots \quad T^2, \ldots
\end{align*}
\]
“super-master solution” from 4d N=1 on $S^1 \times S^3/\mathbb{Z}_r$ with gauge group $SU(N)$

$\text{[Y]}$ (‘13), based on $\text{[Benini-Nishioka-Y]}$ (‘11)

Spins take values in discrete/continuous variables

$\left( \mathbb{R} \times \mathbb{Z}_r \right)^{N-1}$

Elliptic parameter $p, q$ arises from complex structure of $S^1 \times S^3/\mathbb{Z}_r$

$\text{[Kodaira]}$ (‘66)
$\text{[Closset-Dumitrescu-Festuccia-Komargodski]}$ (‘13)
While integrability is a consequence of gauge theory duality, integrability was recently directly proven mathematically by [Kels-Yamazaki] ('17), by generalizing earlier works [Spiridonov] ('01) (r=1, N=2), [Rains] ('03) (r=1, N>2), [Kels] ('15) (r>1, N=2),...
While integrability is a consequence of gauge theory duality, integrability was recently directly proven mathematically by [Kels-Yamazaki] (’17), by generalizing earlier works [Spiridonov] (’01) (r=1, N=2), [Rains] (’03) (r=1, N>2), [Kels] (’15) (r>1, N=2),…

Intertwiner of (very likely new) quantum-group type structure?

\[ U_{p,q}^{r \ (sl_N)} \] ???

Hint: for \( r=1, N=2 \) this comes from Sklyyanin algebra [Sklyyanin] (’83) [Cherednik] (’85) \[ U_{p,q}^{(sl_N)} \]
Particularly interesting limit: root of unity limit

(Lens) elliptic gamma function diverges:

\[ \Phi(z; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2i\pi j} p^j q^k}{1 - e^{-2i\pi j} p^{j+1} q^{2k+1}} \]

\[ \varepsilon \to 0 \quad (p = e^{i\pi \varepsilon}, \quad q = e^{-\frac{\varepsilon}{2N^2}} \varepsilon, \quad \varepsilon^{2N} = 1) \]

\[ \Phi(z; p, q) = \exp \left( \frac{i}{\varepsilon} 2N \int_0^2 du \ln \overline{\Theta_3} \left( Nu/Nz \right) \right) \times \text{(subleading finite piece)} \]

[Bazhanov-Sergeev] (‘10) [Kels-Y] (‘17)
Particularly interesting limit: root of unity limit

This requires saddle point analysis

\[
\text{[Bazhanov-Sergeev]} \ (\text{'10}) \ [\text{Kels-Y}] \ (\text{'17})
\]

Schematically,

\[
z \rightarrow \sum \int \text{d} \sigma \ e^{\frac{1}{\epsilon} W^{(\sigma)} + W^{(\epsilon)} + O(\epsilon)}
\]

saddle point in $\epsilon \rightarrow 0$

\[
\frac{\partial W^{(\sigma)}}{\partial \sigma} = 0
\]
\[
\frac{\partial W^{(\sigma)}}{\partial \sigma} = 0
\]

Saddle point equation (for N=2): \textit{discrete classical integrable equation (Q4)} of [Adler-Bobenko-Suris] (‘02)

In their classification, (Q4) is the most general, and everything else is its degeneration
Saddle point equation (for N=2): discrete classical integrable equation (Q4) of \([\text{Adler-Bobenko-Suris}]\) (‘02)

In their classification, (Q4) is the most general, and everything else is its degeneration

The saddle point equation is also the Bethe Ansatz equation for the dimensionally-reduced theory, in Gauge/Bethe correspondence \([\text{Nekrasov-Shatashvili}]\) (‘02), see also \([\text{Kels-Y}]\) (‘17) for related comments
We obtain YBE by evaluating the subleading corrections at a (leading) saddle point.
We obtain YBE by evaluating the subleading corrections at a (leading) saddle point.

Several discrete integrable models reproduced, including **chiral Potts model**
[Bazhanov-Sergeev] ('10) [Kels-Y] ('17)

has higher-genus spectral curve, and do not have "rapidity-difference property"

\[ R(z_1, z_2) \neq R(z_1 - z_2) \]
GAUGE THEORIES, VERTEX MODELS, AND QUANTUM GROUPS

Edward WITTEN*

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

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There are several obvious areas for further investigation. In terms of statistical mechanics, one compelling question is to understand the origin of the spectral parameter (and the elliptic modulus) in IRF and vertex models; this is essential for explaining the origin of integrability. Another question, which may or may not be related, is to understand the spin models formulated only rather recently [24] in which the spectral parameter is not an abelian variable (as in previous construc-

chiral Potts
Summary
Why integrable models exist?
Perspective from QFT?
Why integrable models exist? Perspective from QFT?

Because of

Gauge Theory Duality
Why integrable models exist? Perspective from QFT?

Because of Gauge Theory Duality

Because of Locality, Unitarity,...
Origin of **spectral parameter:**
R-charge
Origin of spectral parameter: R-charge

New integrable models, and new mathematics and physics