Integrability as Duality

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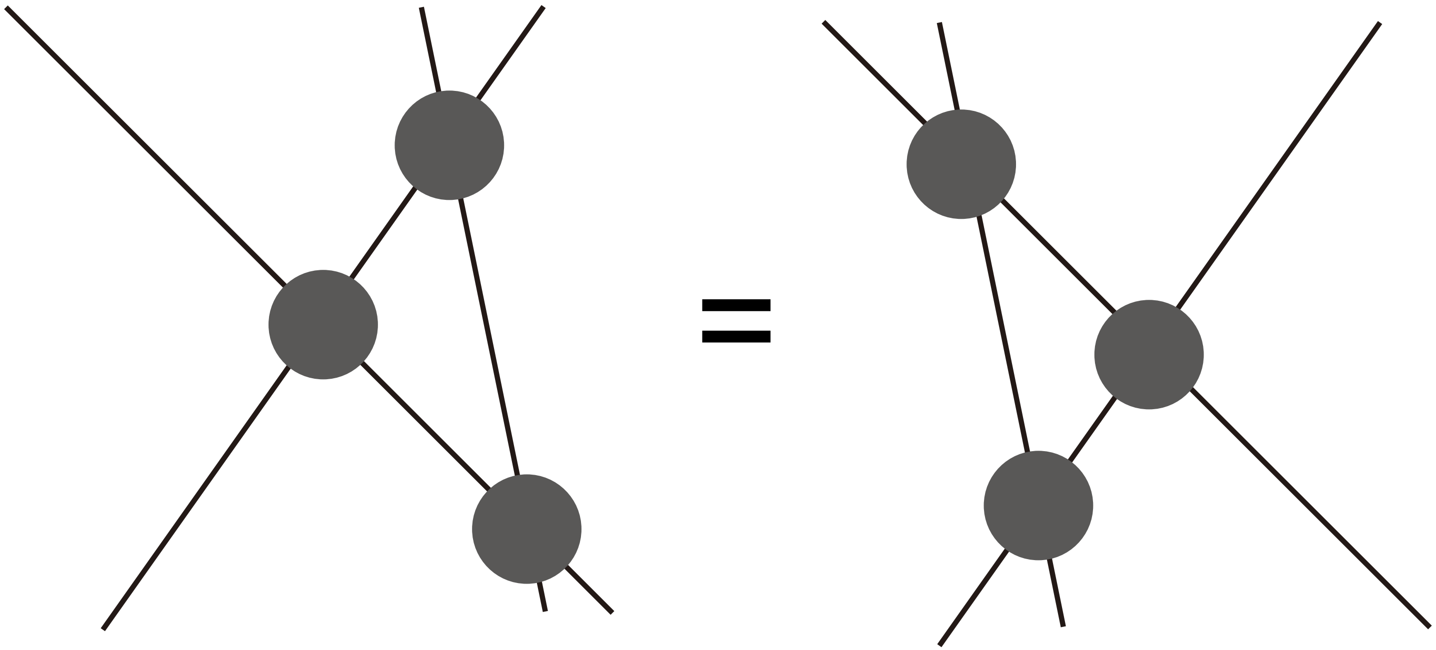
String-Math 2018, Tohoku University

TOHOKU

NEVER GIVE UP

figure from https://storage.googleapis.com/

integrable models is characterized by
Yang-Baxter equation with spectral parameters



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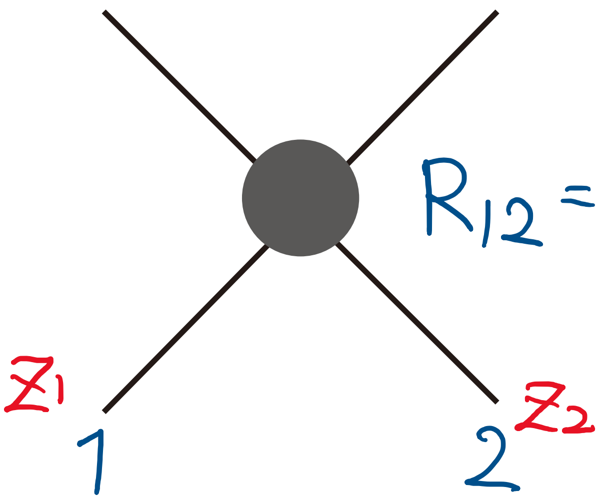
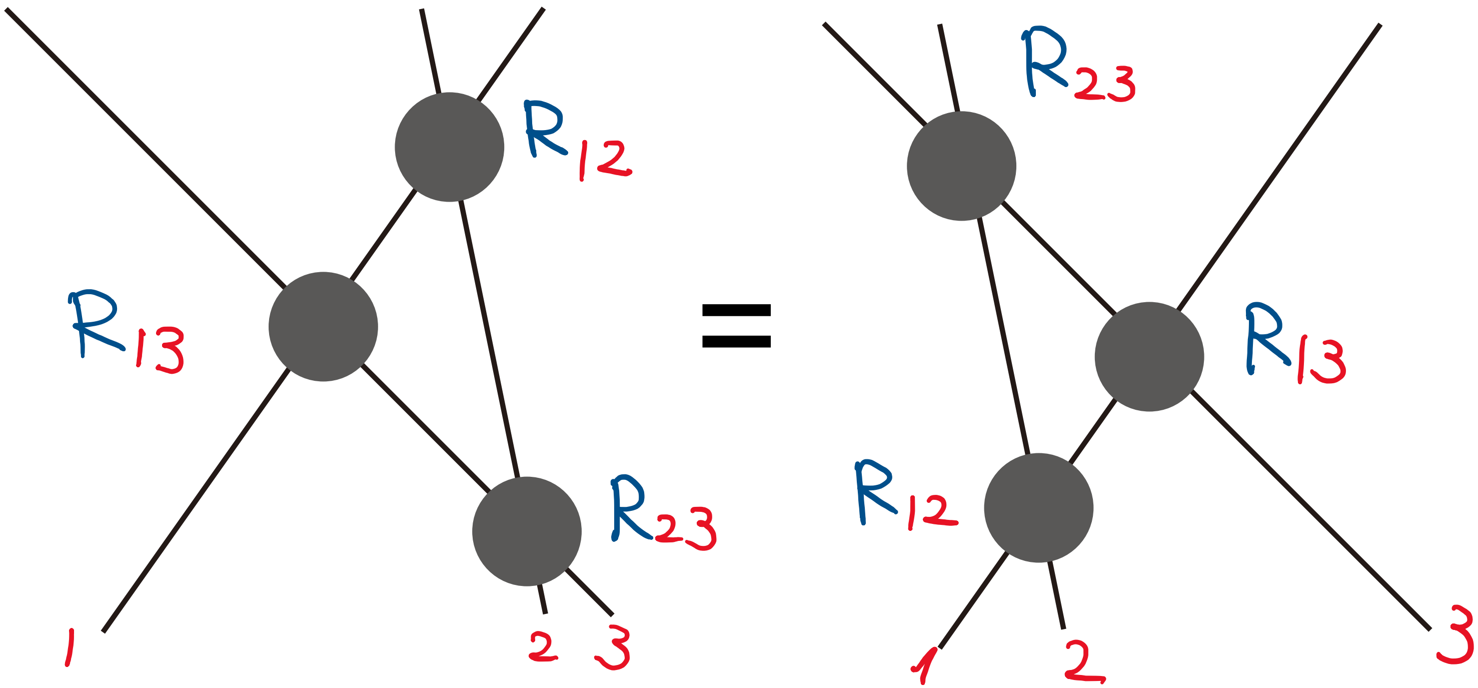


$$R_{12} = R_{12}(\underbrace{z_1 - z_2}_{\text{spectral parameter}}) \in \mathrm{End}(V_1 \otimes V_2)$$

integrable models is characterized by
Yang-Baxter equation with spectral parameters



$$R_{23}(z_2 - z_3)\, R_{13}(z_1 - z_3)\, R_{12}(z_1 - z_2)$$
$$= R_{12}(z_1 - z_2)\, R_{13}(z_1 - z_3)\, R_{23}(z_2 - z_3) \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$$

Why integrable models exist?

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Perspectives from QFT?

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Perspectives from QFT?

Origin of spectral parameter?

New integrable models?

I myself have worked mainly on two approaches:

1. 4d N=1 supersymmetric quiver gauge theories

   (Gauge/YBE correspondence)

   [Y, Terashima-Y] (‘12), [Y] (‘13),….

2. ”4d Chern-Simons”

   [Costello] (‘12),
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   (see also MY’s talk at Strings 2018 next week)

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Integrability

from

4d Quiver Gauge Theories

Initiated in 2012-2013, partly with Yuji Terashima

[Y] [Terashima-Y] ('12) and [Y] ('13)

and inspired in particular by

[Bazhanov-Sergeev] ('10) ('11)

"master solution"

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[Y] [Terashima-Y] (‘12) and [Y] (’13)

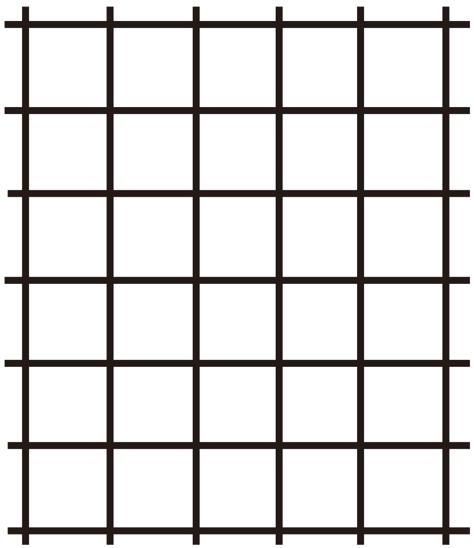
and inspired in particular by
[Bazhanov-Sergeev] (‘10) (’11)

Since then more works in collaboration with
Andrew P. Kels, Wenbin Yan and others

Related works by e.g. Bazhanov, Chicherin, Derkachov, Dolan, Gahramanov, Jafarzade, Mangazeev, Maruyoshi, Nazari, Osborn, Rains, Sergeev, Spiridonov (in particular [Spiridonov] (‘10)), Yagi, Zabrodin,….

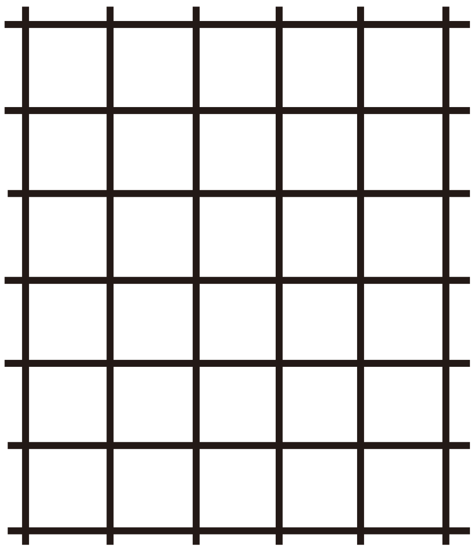
Three Basic Ingredients

1. statistical lattice as a quiver diagram



spin S_v at vertex v

1. statistical lattice as a quiver diagram



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partition function

$$Z = \sum_{\{s_v\}} e^{-\mathcal{E}(\{s_v\})}$$

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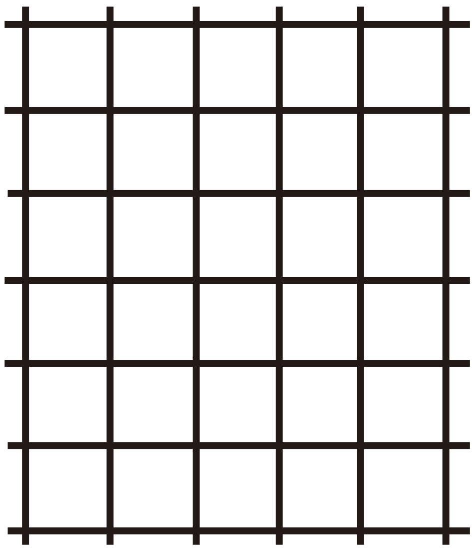

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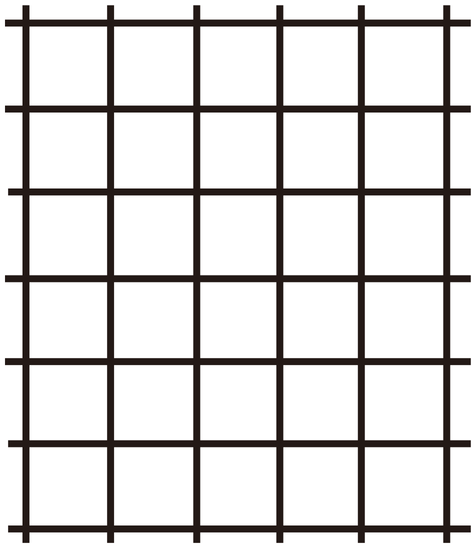
partition function

$$Z = \sum_{\{S_v\}} e^{-\mathcal{E}(\{S_v\})}$$

Boltzmann weight

$$\mathcal{E}(\{S_v\}) = \sum_{v:\ \text{vertex}} \mathcal{E}^{v}(S_v) + \sum_{e:\ \text{edge}} \mathcal{E}^{e}(\{S_v\}_{v\in e})$$

1. statistical lattice as a quiver diagram



gauge field A_v for vertex v

bifundamental matter Φ_e

for each edge e

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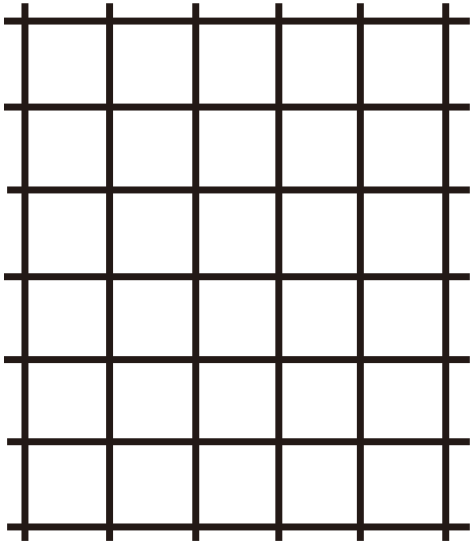

gauge field A_v for vertex v

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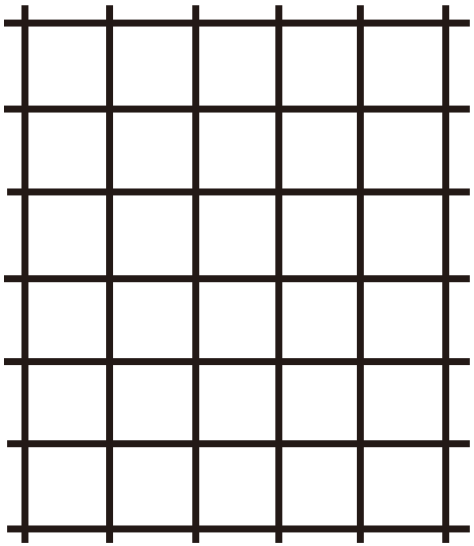
partition function

$$Z = \int \mathcal{D}A^v \mathcal{D}\Phi_e \, e^{-\mathcal{L}}$$

1. statistical lattice as a quiver diagram



gauge field A_v for vertex v

bifundamental matter Φ_e

for each edge e

partition function

$$Z = \int \mathcal{D}A^v \mathcal{D}\Phi_e \, e^{-\mathcal{L}}$$

Lagrangian

$$\mathcal{L}(\{A_\mu^v, \Phi_e\}) = \sum_{v\,:\,\text{vertex}} \mathcal{L}^v(A_v) + \sum_{e\,:\,\text{edge}} \mathcal{L}^e(\Phi_e, \{A_v\}_{v \in e})$$

2. supersymmetric localization

Once super-symmetrize the setup, the difference goes away:

$$Z = \int \mathcal{D}A_\nu \, \mathcal{D}\Phi_e \, e^{-\mathcal{L}}$$

← path-integral

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$$Z = \int \mathcal{D}A_V \, \mathcal{D}\Phi_e \, e^{-\mathcal{L}}$$ ← path-integral

$$= \int d\underbrace{\sigma_V}_{\text{parametrize saddle point locus}} \, e^{-\mathcal{E}}$$ ← finite-dim integral

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$$Z = \int \mathcal{D}A_v \, \mathcal{D}\Phi_e \, e^{-\mathcal{L}}$$

path-integral

$$= \int d\underbrace{\sigma_v}_{} \, e^{-\mathcal{E}}$$

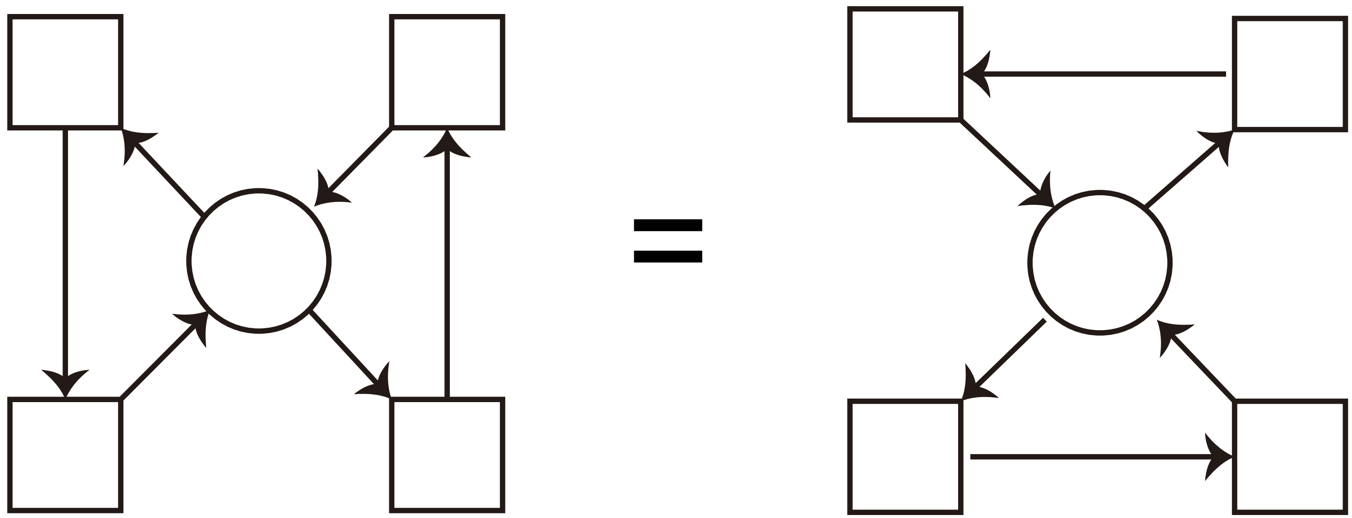
finite-dim integral

parametrize saddle point locus

$$\mathcal{E} = \sum_{v\text{: vertex}} \mathcal{E}^v(\sigma_v) + \sum_{e\text{: edge}} \mathcal{E}^e(\{\sigma_v\}_{v\in e})$$

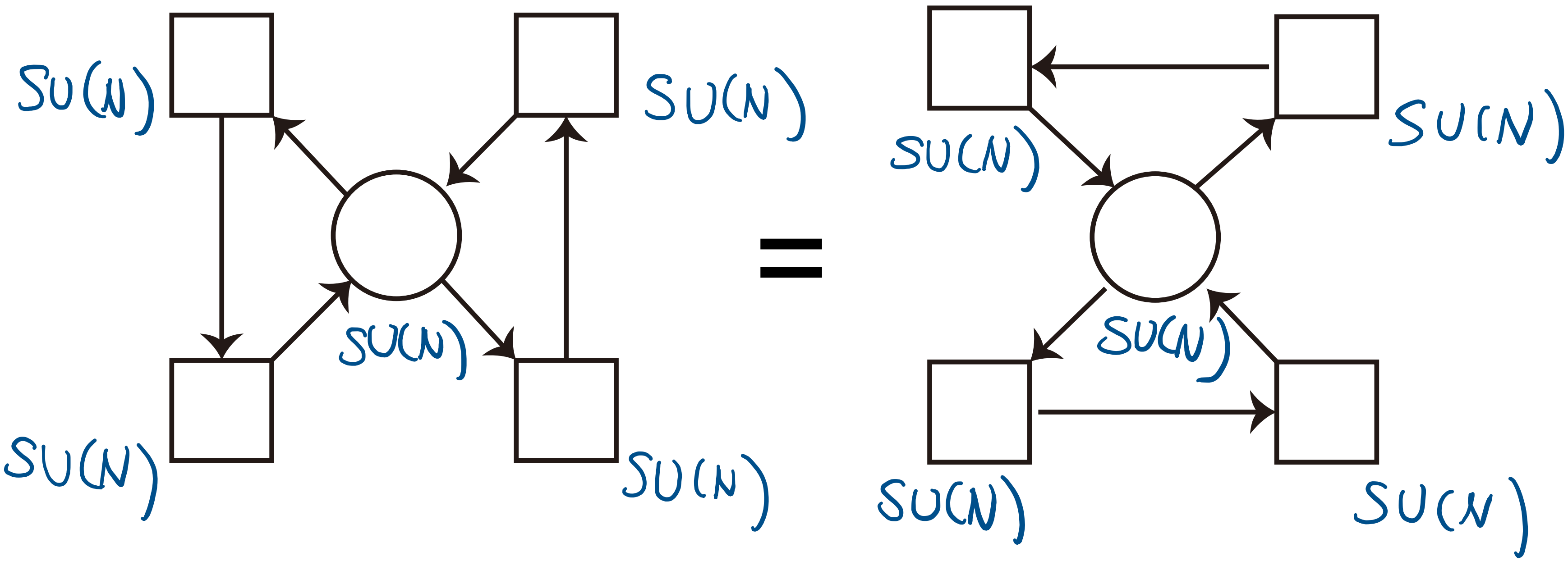
One-loop determinant

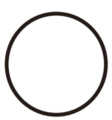
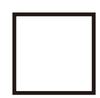
3. Seiberg(-like) duality



3. Seiberg(-like) duality

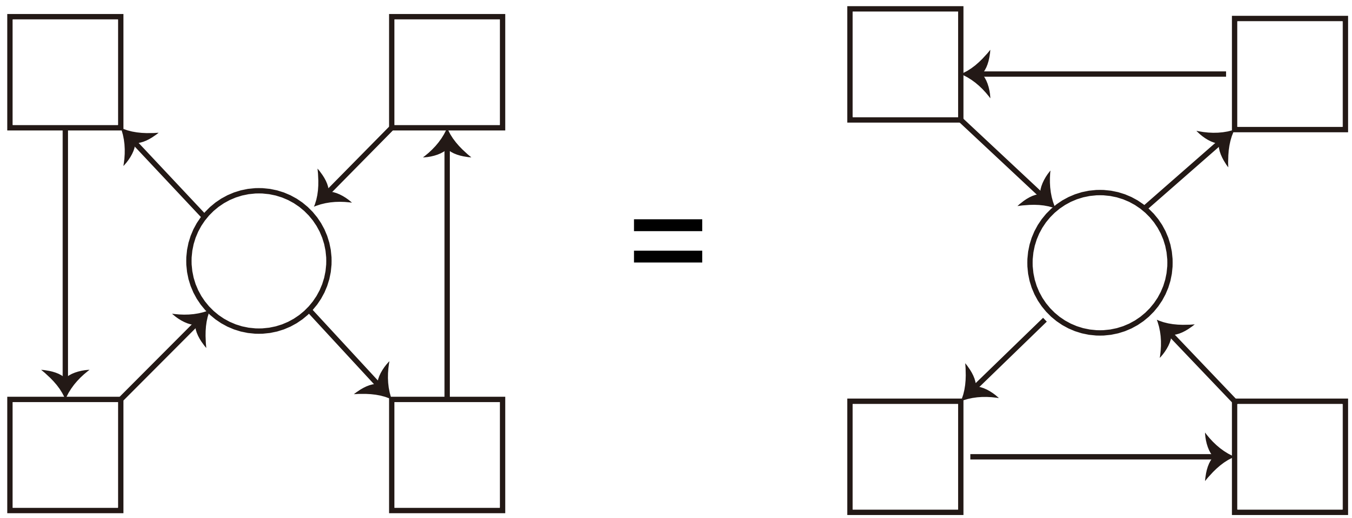
$(SU(N),\ N_f = 2N)$

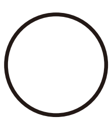
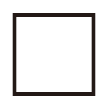


 gauge sym.    flavor sym

Phys: 4d N=1 Seiberg duality [Seiberg] ('94)
(or their cousins in lower dimensions
[Aharony] ('97), [Hori-Tong] ('06), [Gadde-Gukov] ('13))

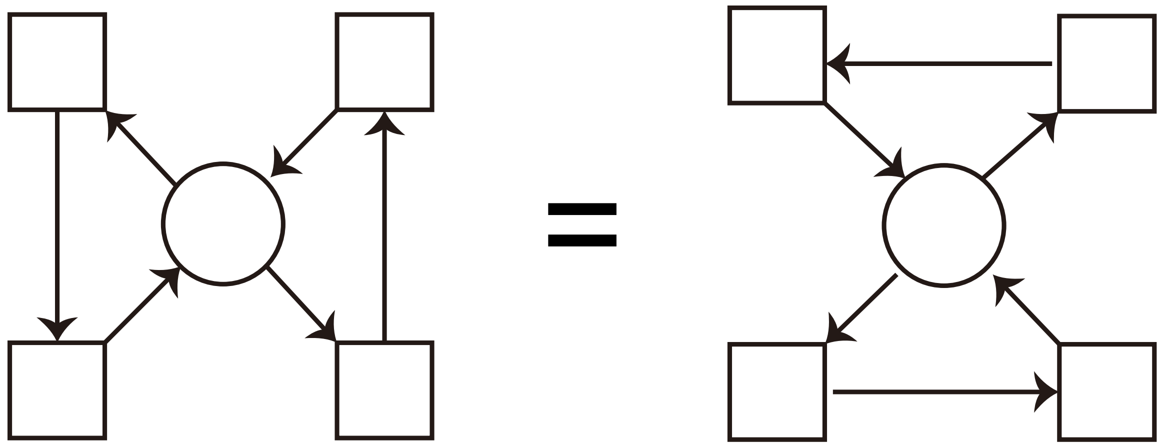
3. Seiberg(-like) duality



 unfrozen node  frozen node

Math: (special example of) quiver mutation [Fomin-Zelevinsky] ('01)

3. Seiberg(-like) duality



This (in a different disguise) is known as star-star relation in integrable models, originally in the context of tetrahedron equation [Baxter ('86), Bazhanov-Baxter ('92)]

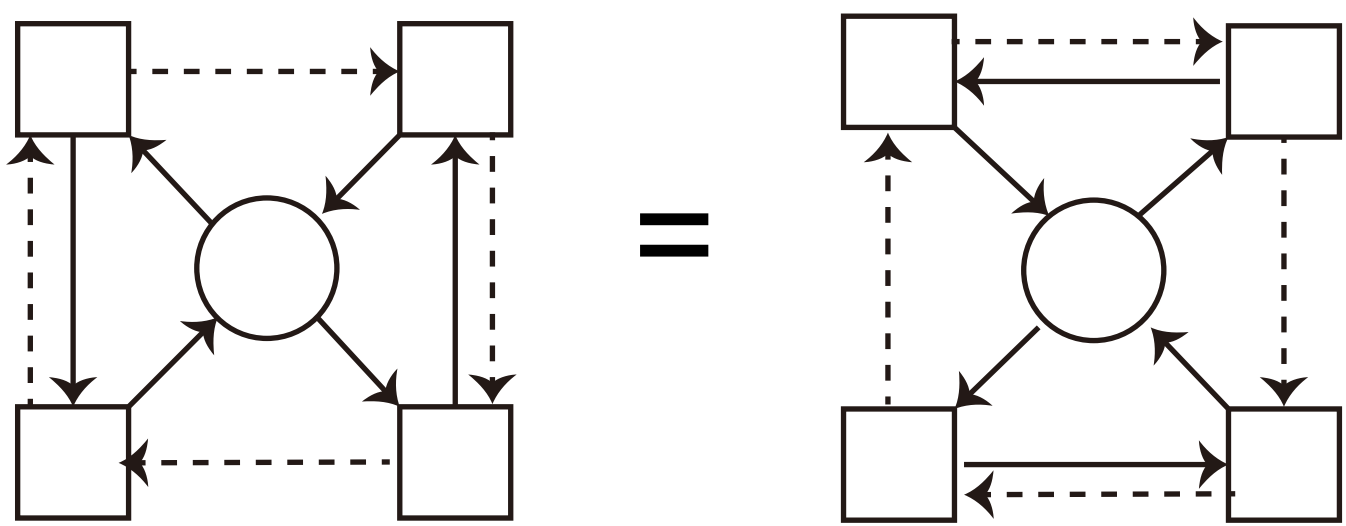
Interestingly, the connection to Seiberg duality (or mutation) was noticed only recently

Since star-star relation implies YBE [Baxter ('86), Bazhanov-Baxter ('92)], once we solve SSR we have an integrable model

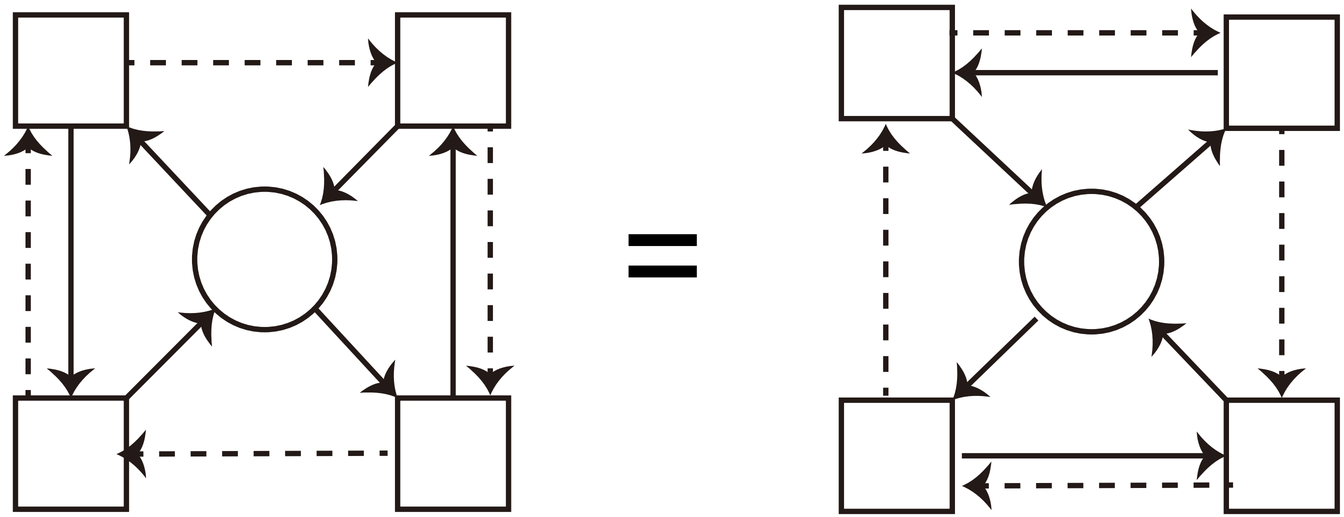
We also know that star-star relation is Seiberg duality, so their partition function (in the IR) should coincide.

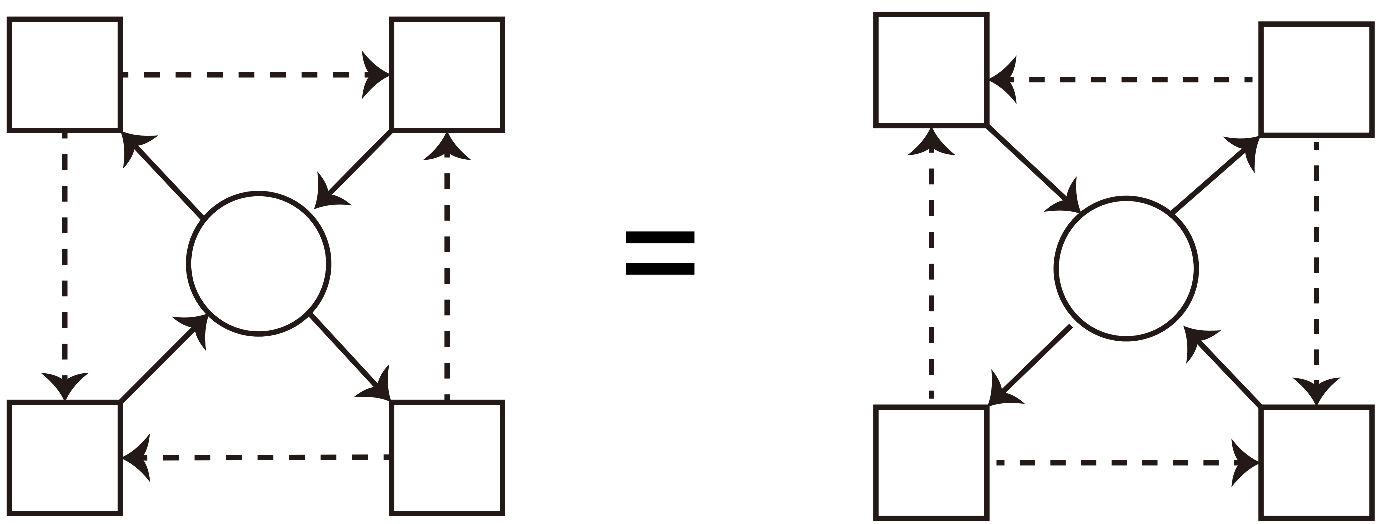
By combing these two observations we will automatically land on integrable models!

To explain YBE in more detail, it is useful to represent SSR more symmetrically, by adding “half-arrows” [Yan-Y] (‘15)

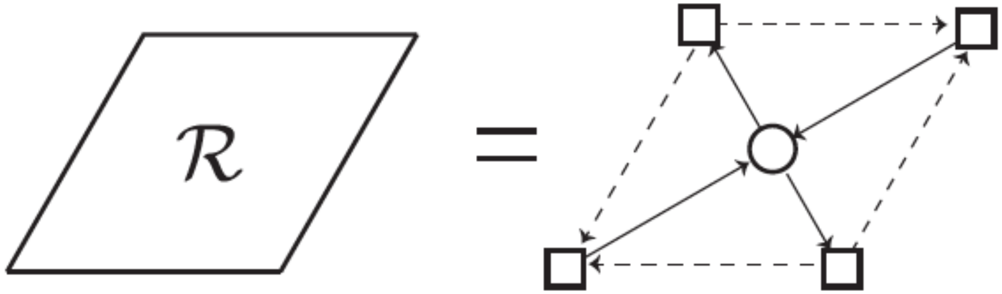


We can cancel half and full arrows:

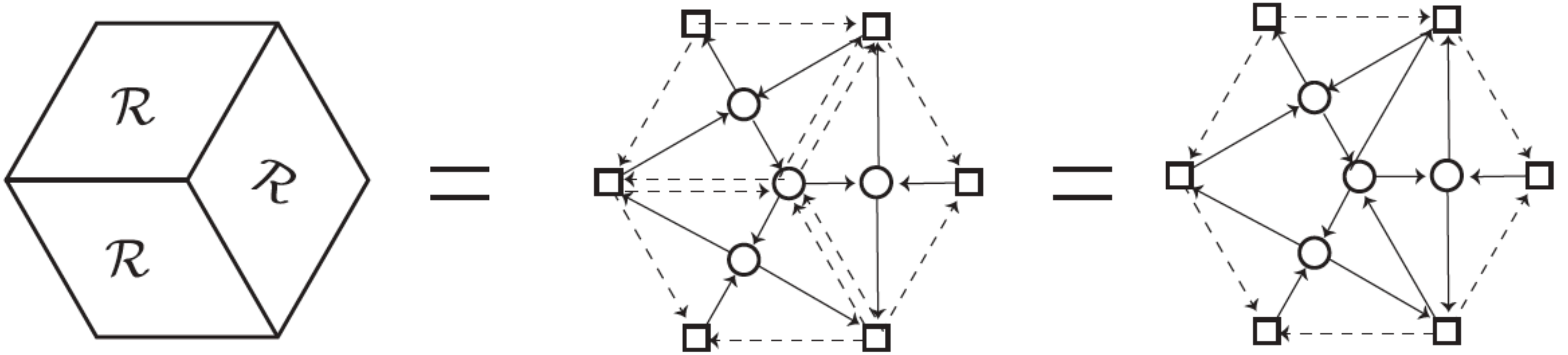




The R-matrix is identified with a simple quiver ("theory for the R-matrix")



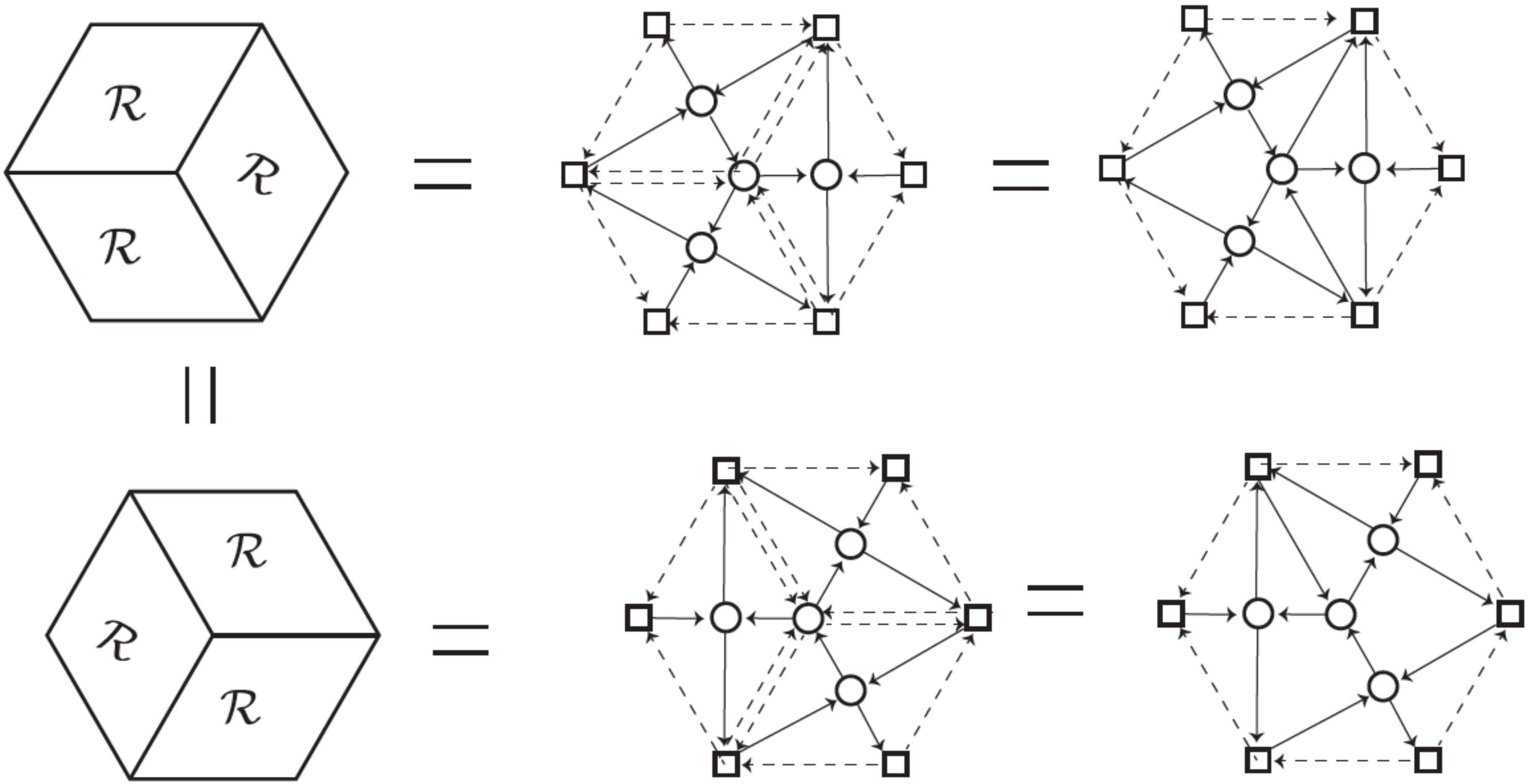
Products of R-matrix is obtained by gluing:



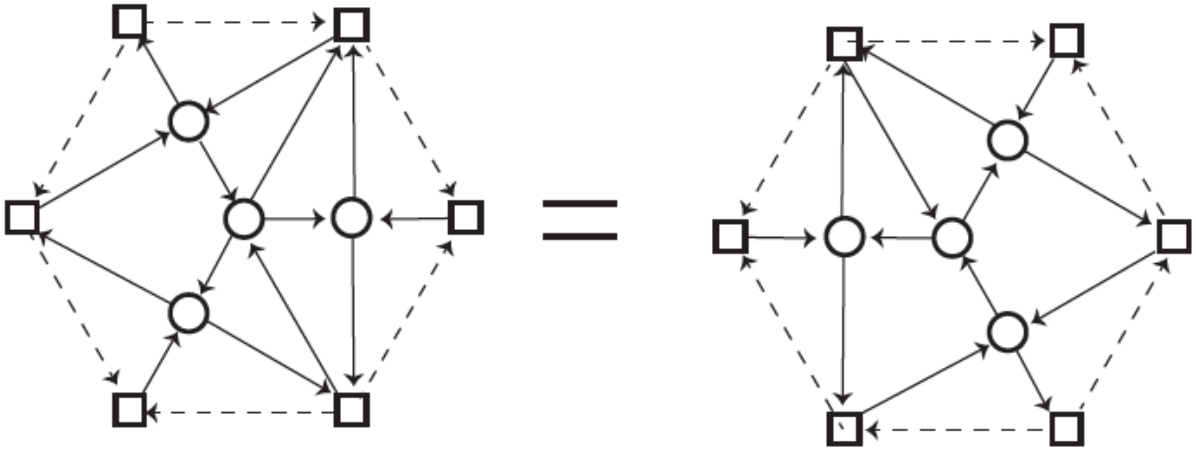
- Phys: gluing three theories by gauging flavor symmetries (as in “class S” theories)

- Math: gluing three quivers by quiver amalgamation

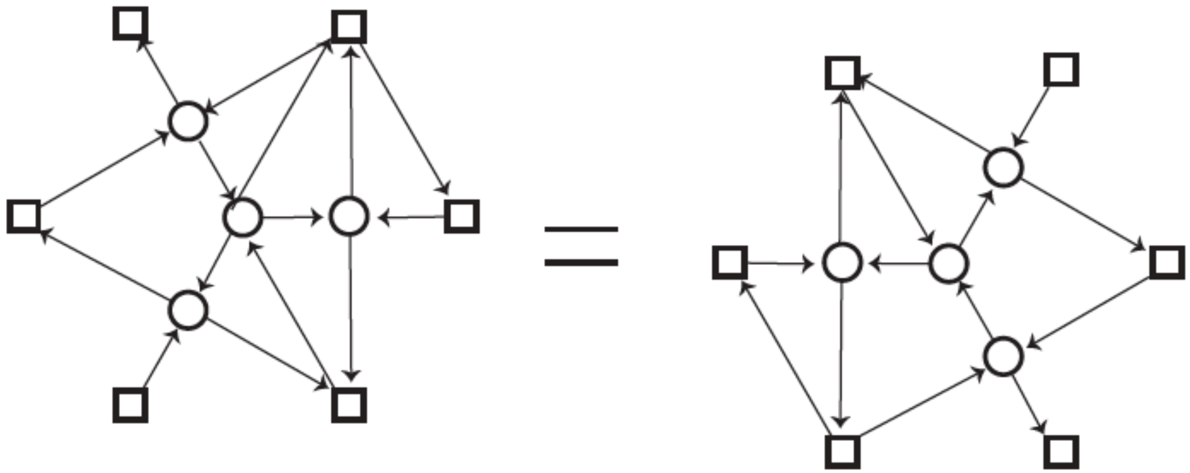
The YBE reads



The YBE reads



or equivalently



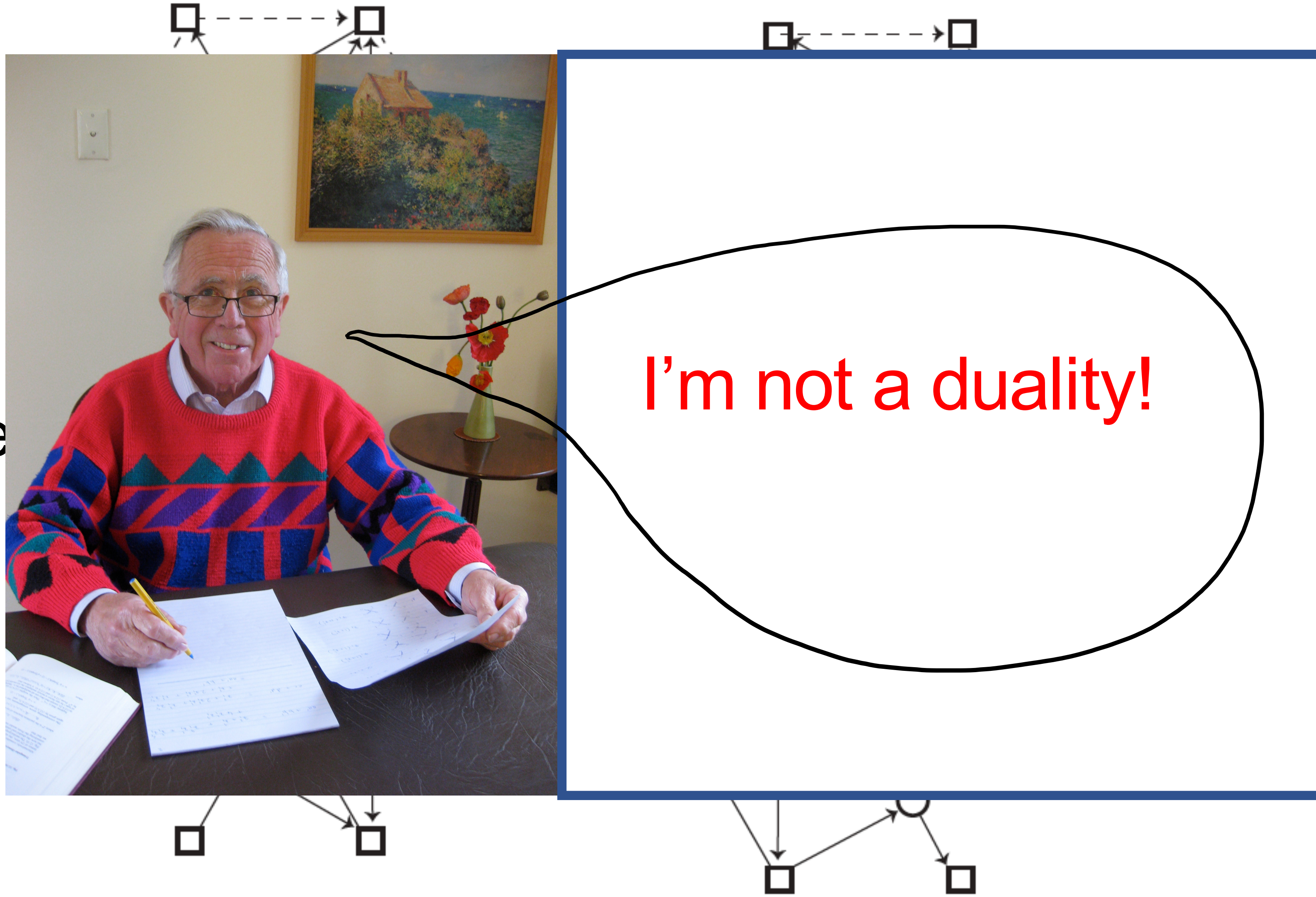
This is the Yang-Baxter duality [Y] ('13)

The YBE reads

or e

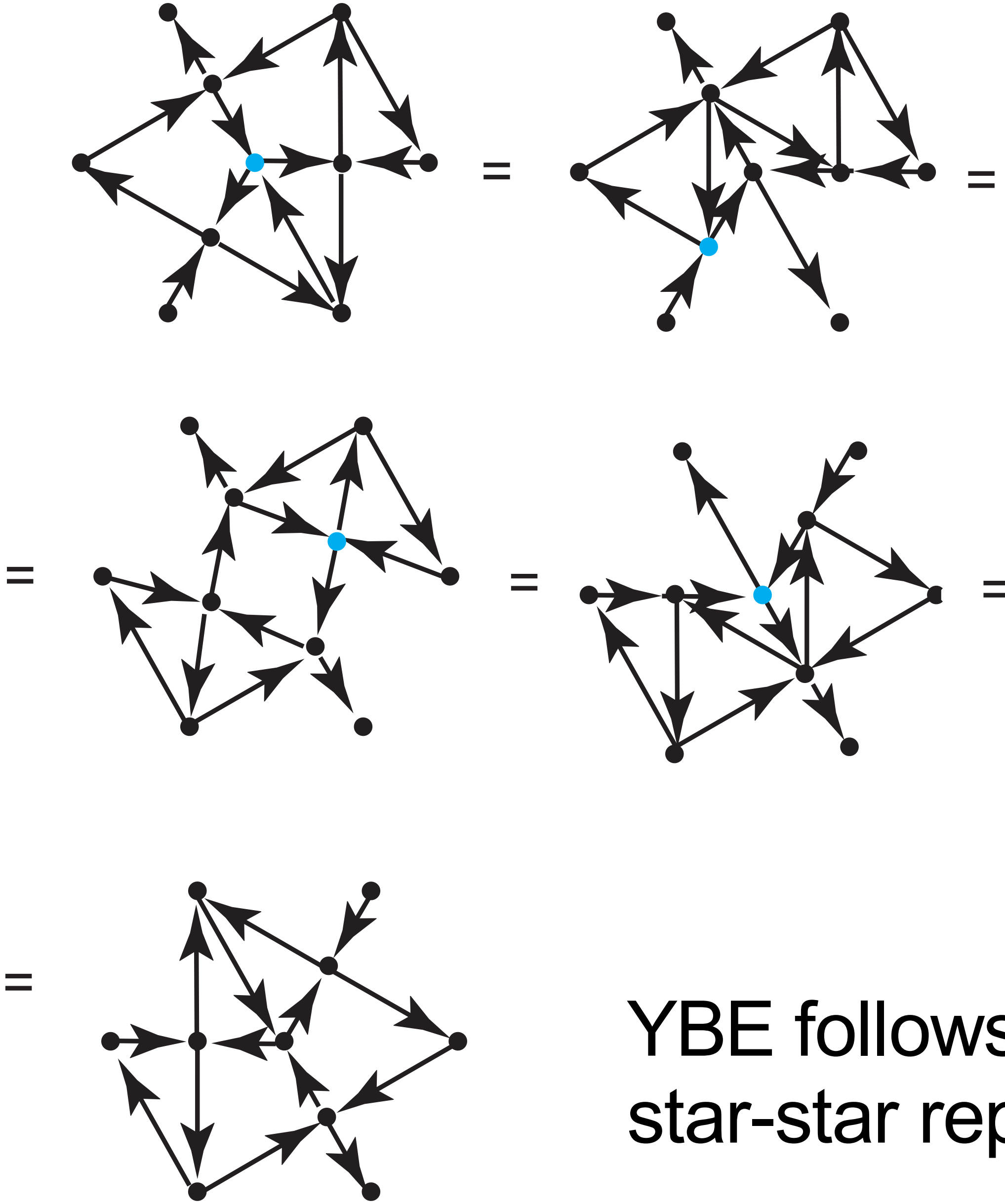
This is the Yang-Baxter duality

The YBE reads



or e

This is the Yang-Baxter duality



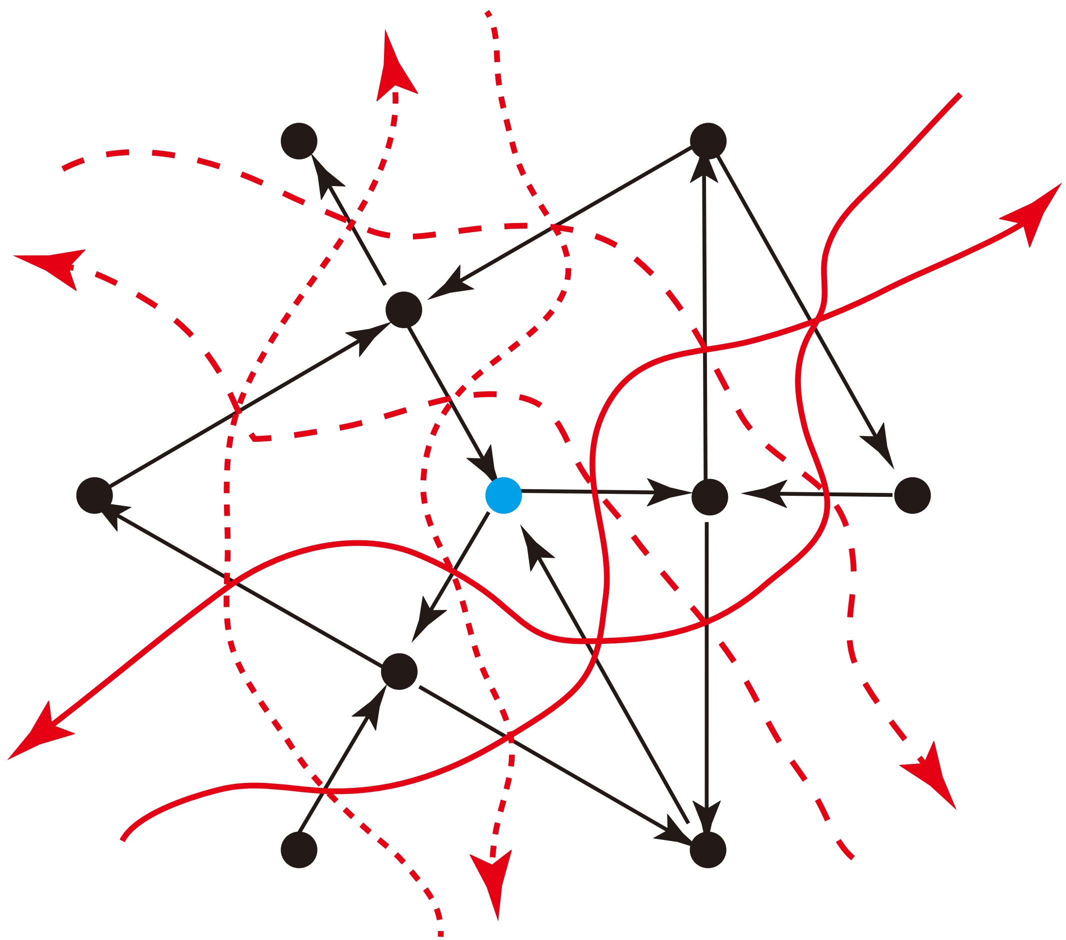
YBE follows from
star-star repeated four times

Spectral Parameter?

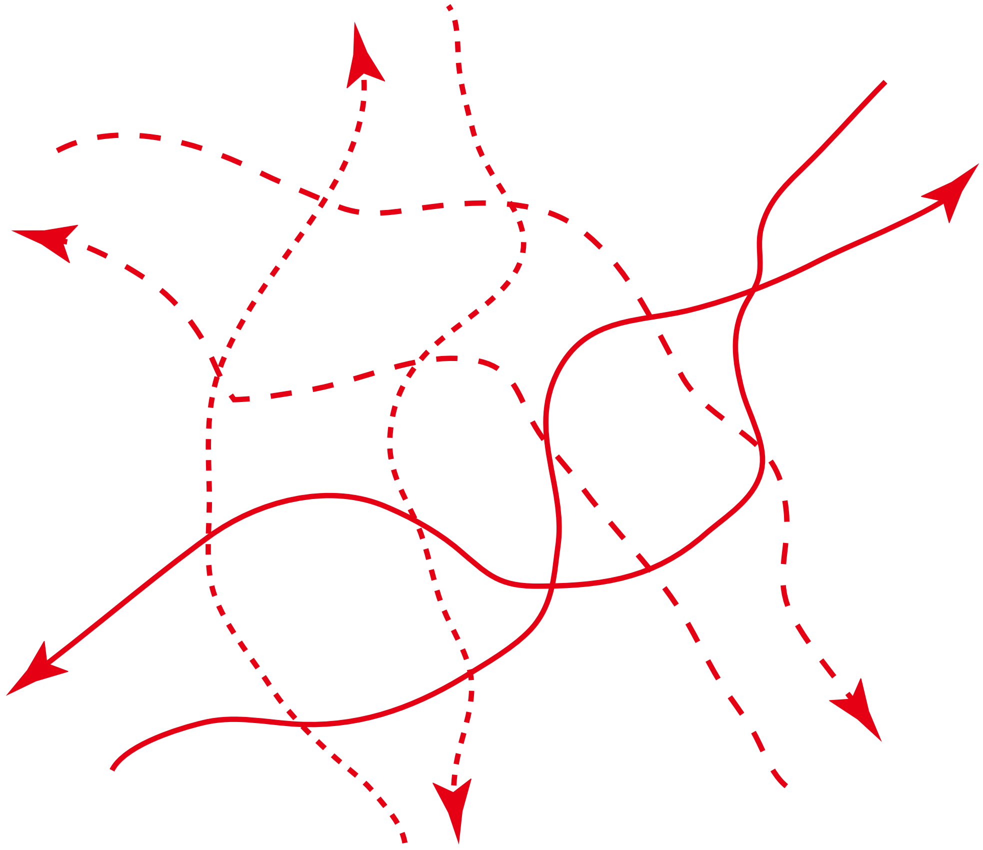
“spectral parameter = R-charge”

The spectral parameter in integrable models matches with the R-charge in quiver gauge theories, found in [Hanany-Vegh] (‘05)

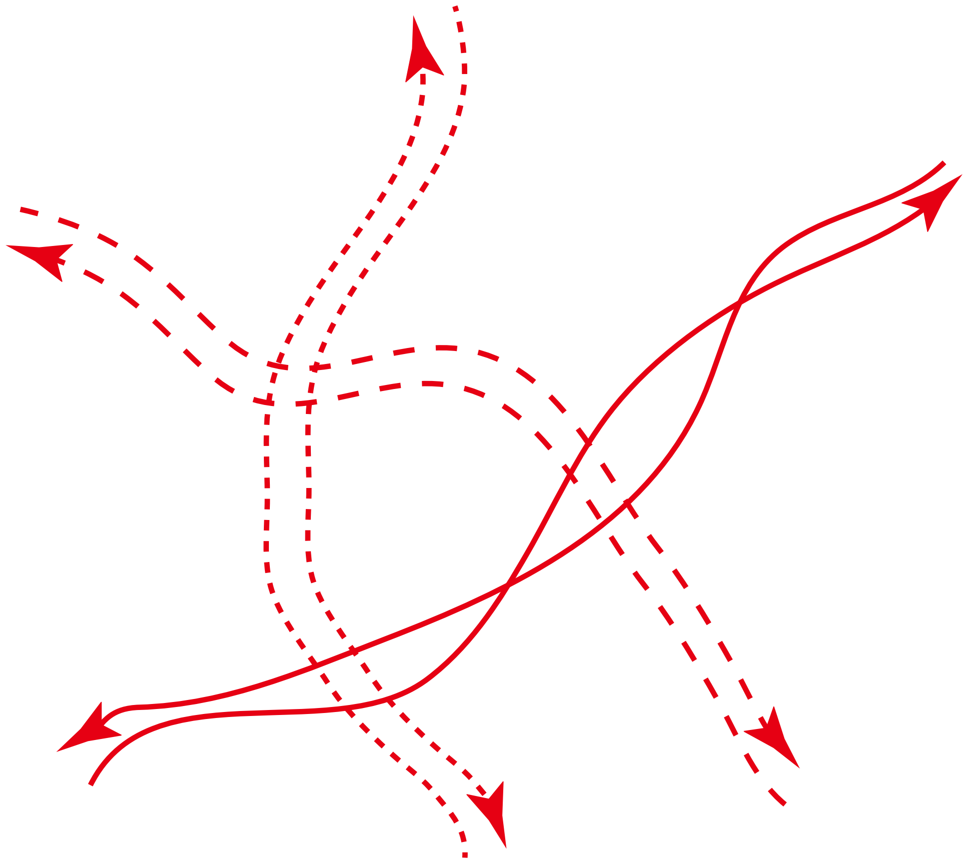
Both are associated with “zig-zag path”, discussed also in mathematical literature [Thurston] (‘04), [Goncharov-Kenyon] (‘11) ….



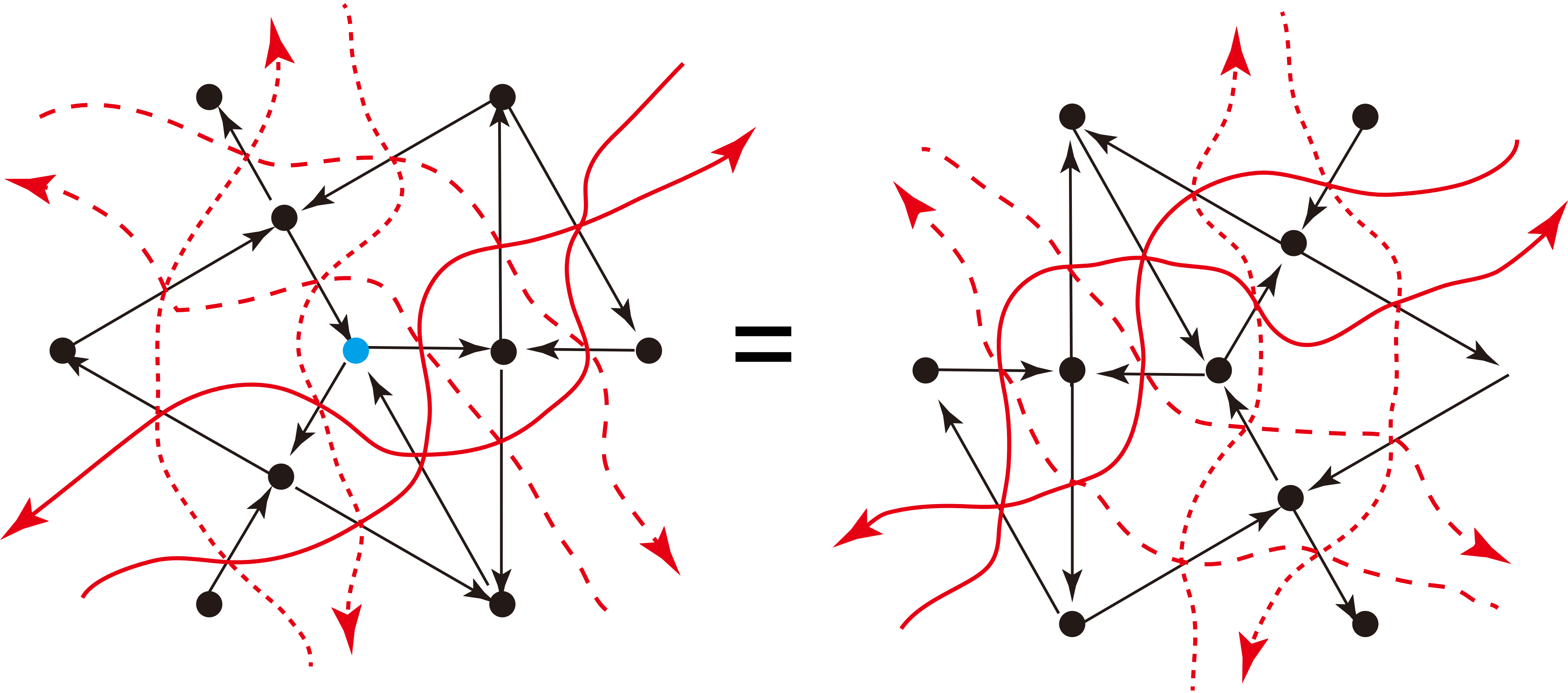
spectral parameter:
associated with “zig-zag path”



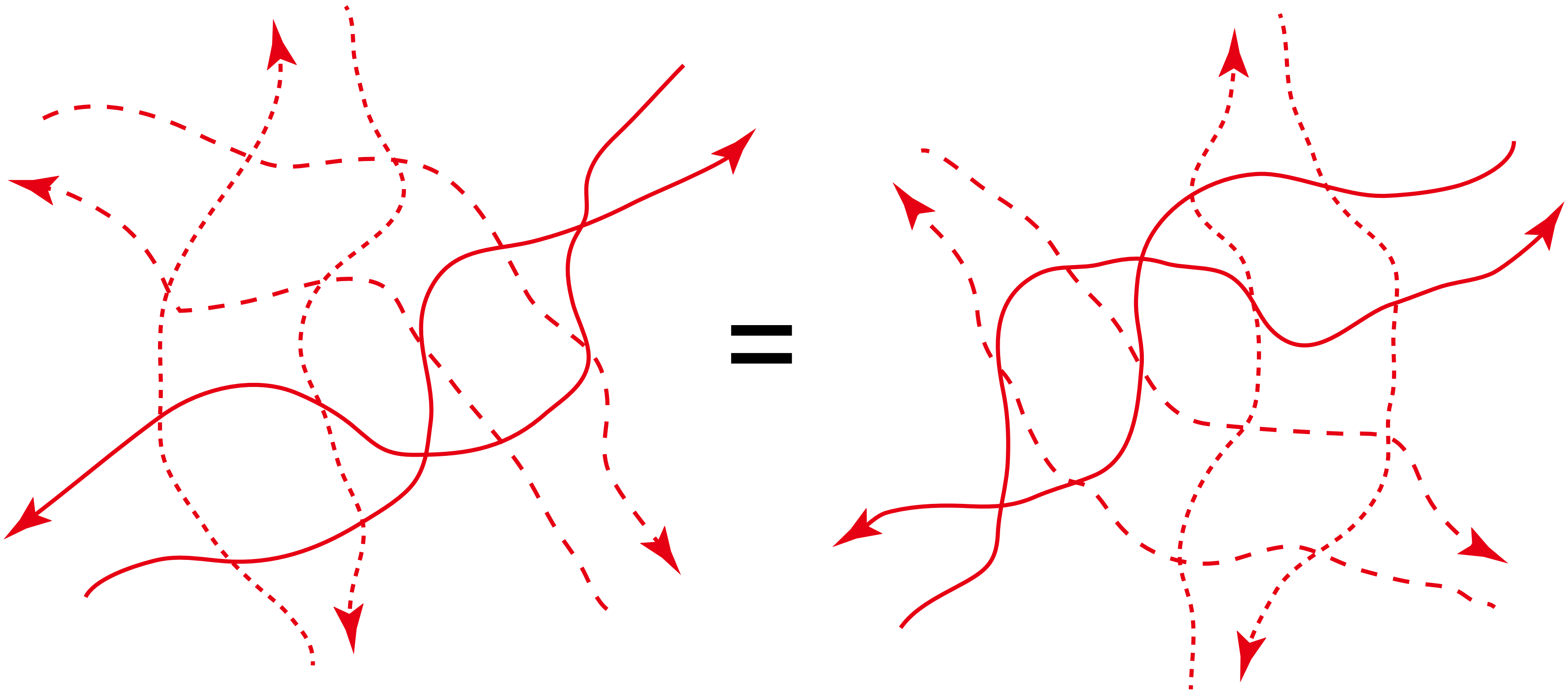
“Z-invariant lattice” [Baxter]



doubled rapidity line



Now go back to Yang-Baxter duality



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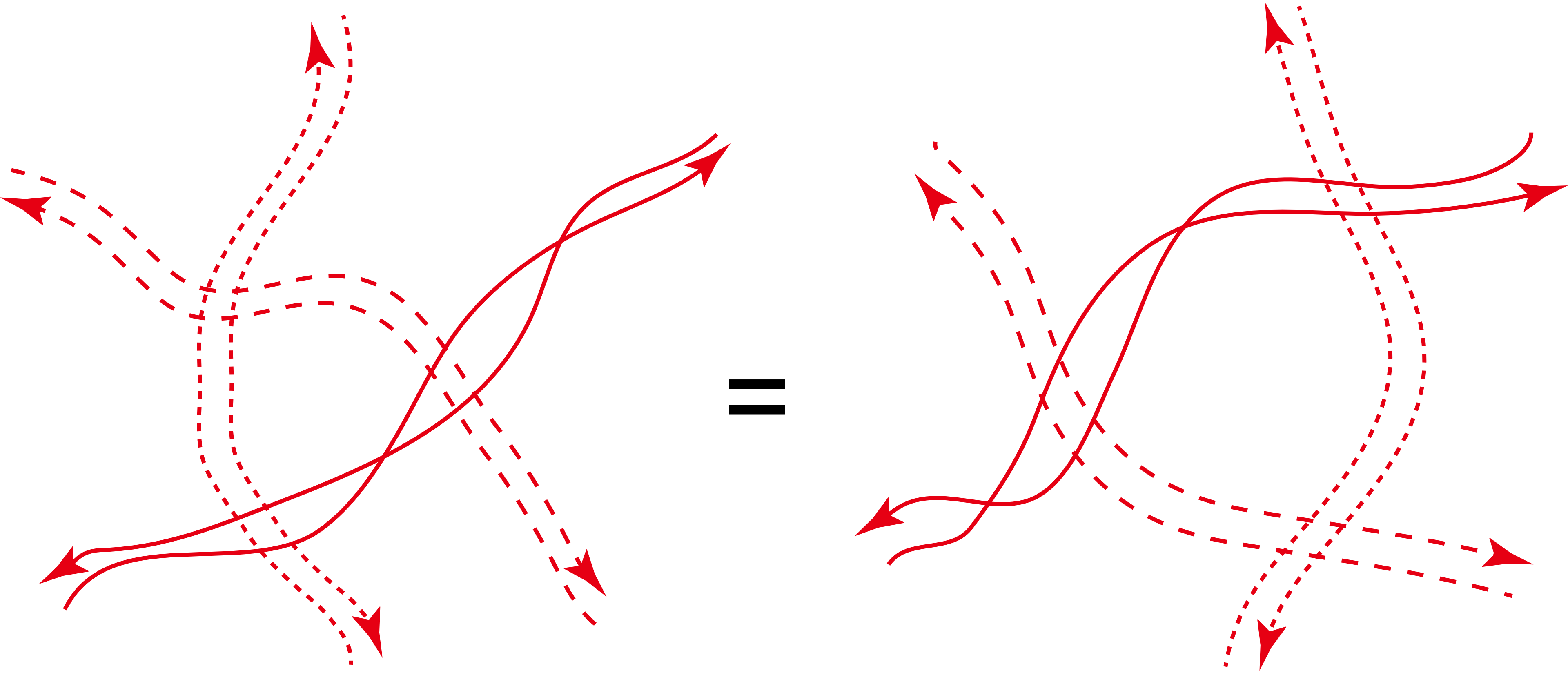


“doubled Yang-Baxter equation”

Side remark: zig-zag paths are actually branes!

NS5-D5 brane realizations studied by MY's master thesis [Y] ('08); T-dual to [Feng-He-Kennaway-Vafa] ('05), see also [Imamura] ('07), [Imamura-Isono-Kimua-Y] ('07)

$(\mathbb{C}^\times)^2$

$\mathbb{R}^{3,1}$ T^2 $\mathbb{R}^2$ $U(1)_R$

	x_0	x_1	x_2	x_3	θ_x	θ_y	x	y	x_8	x_9
D5	—	—	—	—	—	—				
NS5	—	—	—	—	Σ					

hol. curve in $(\mathbb{C}^\times)^2$

	x_0	x_1	x_2	x_3	θ_x	θ_y	x	y	x_8	x_9
D5	—	—	—	—	—	—				
NS5	—	—	—	—		Σ				

Basically, we have N D5 wrapping torus, divided by zig-zag path = NS5-brane

We have an SU(N) quiver node for each region of D5-branes

(We actually need (N, ±1)-branes, in addition to (N,0)-branes; I will suppress this point now)
[Imamura] ('07), [Imamura-Isono-Kimua-Y] ('07), [Y] ('08)

	x_0	x_1	x_2	x_3	θ_x	θ_y	x	y	x_8	x_9
D5	$-$	$-$	$-$	$-$	$-$	$-$				
NS5	$-$	$-$	$-$	$-$		Σ				

The question I addressed in [Y] ('08)
was to relate the smooth holomorphic curve
to the combinatorics of zig-zag path, by certain
degeneration process

Very similar idea discussed as
"Lagrangian skeletons"
in [Shende-Treumann-Williams-Zaslow] ('15)

	x_0	x_1	x_2	x_3	θ_x	θ_y	x	y	x_8	x_9
D5	$-$	$-$	$-$	$-$	$-$	$-$				
NS5	$-$	$-$	$-$	$-$		Σ				

The brane realization could also be a starting point for exploring the relation with "4d Chern-Simons" approach to integrable models studied in [Costello] ('12), [Costello-Witten-Y] ('17, '18).

This requires further study, see e.g. [Vafa-Y] (to appear), [Costello-Yagi] (forthcoming); cf. [Ashwinkumar-Tan-Zhao] ('18) ….

Summarizing….

integrable model

YBE

QFT (“categorification”)

integrable model

YBE

Yang-Baxter duality

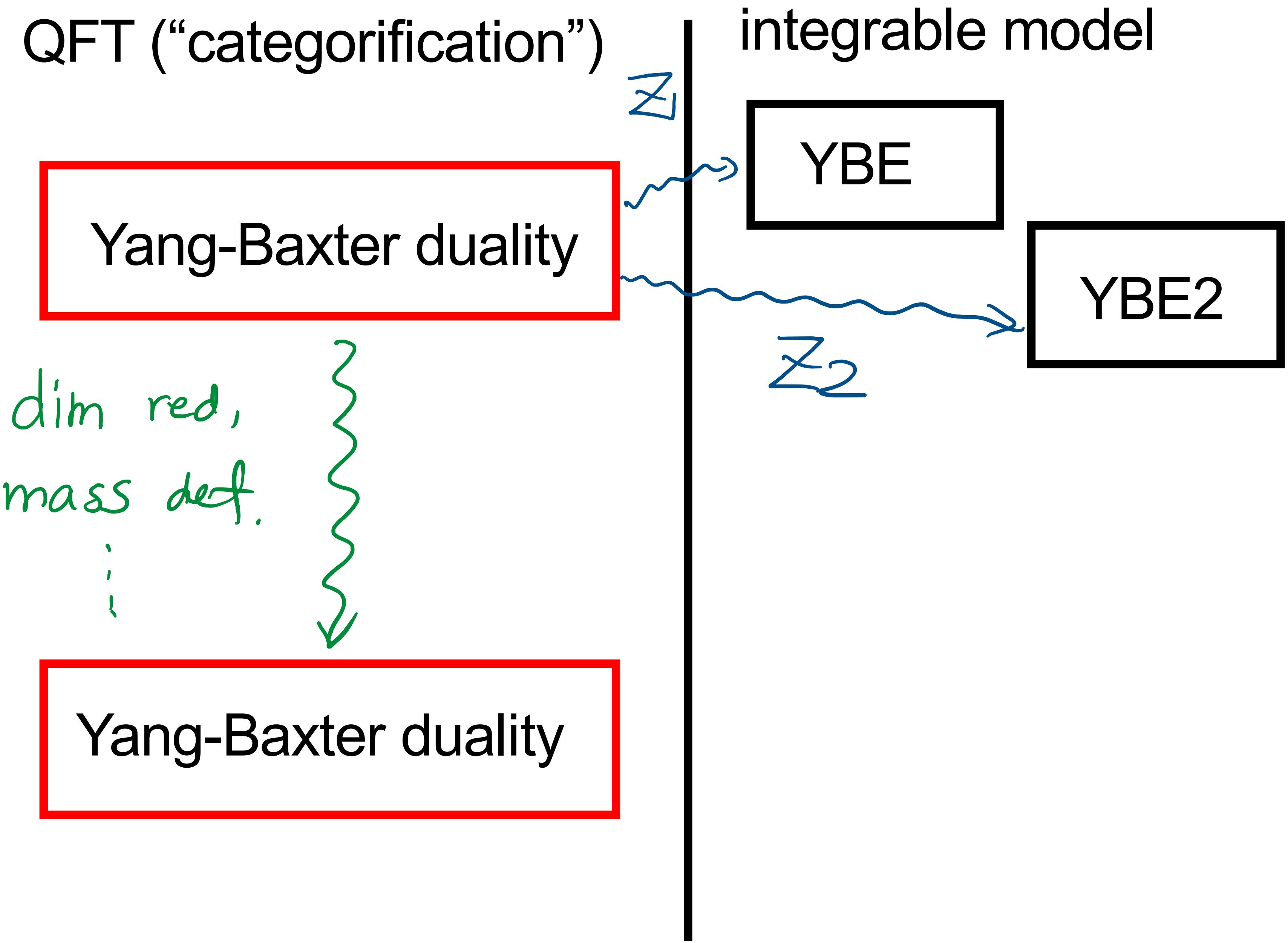
QFT ("categorification") | integrable model

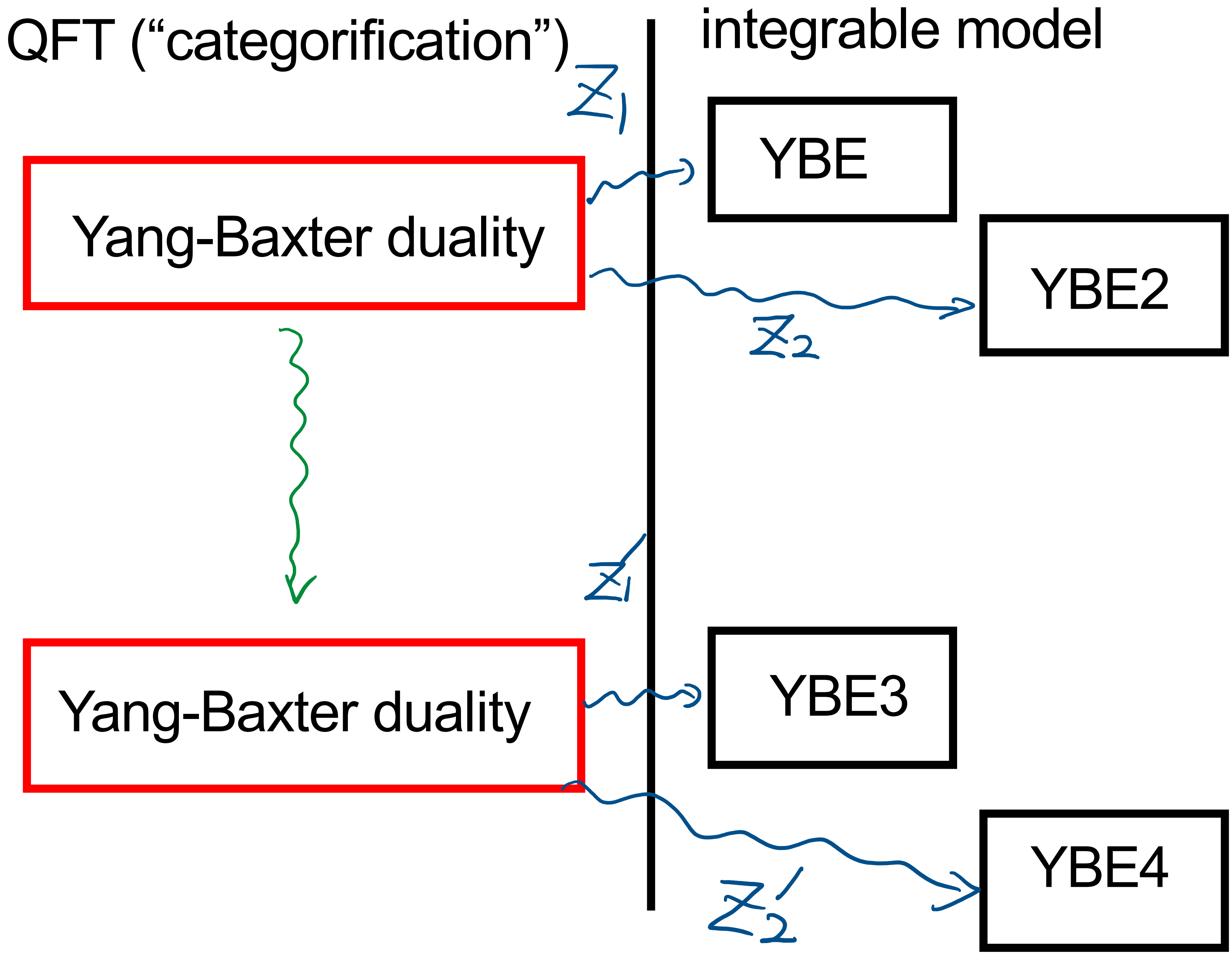
$$\text{Yang-Baxter duality} \xrightarrow{Z_1} \text{YBE}$$

$$\text{Yang-Baxter duality} \xrightarrow{Z_2} \text{YBE2}$$

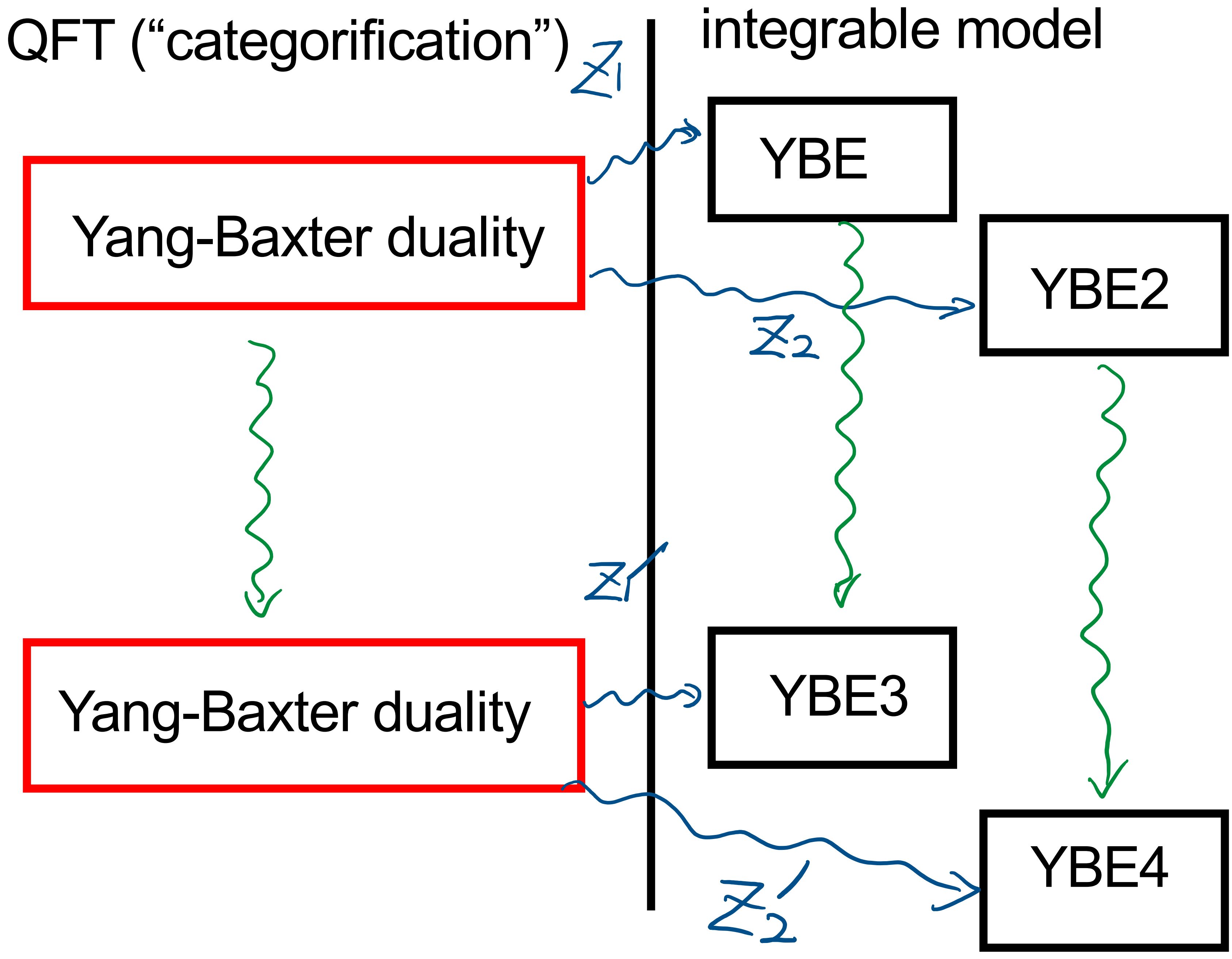
partition function = functor

See my Japanese book ('15)


QFT ("categorification")
integrable model
z_1
YBE
Yang-Baxter duality
YBE2
z_2
dim red,
mass def.
⋮
Yang-Baxter duality



QFT ("categorification")
integrable model
Z_1
YBE
Yang-Baxter duality
YBE2
Z_2
Z_1
YBE3
Yang-Baxter duality
YBE4
Z_2'



QFT (“categorification”)
integrable model
Z_1
YBE
Yang-Baxter duality
YBE2
Z_2
Z_1'
YBE3
Yang-Baxter duality
YBE4
Z_2'

New Integrable Models

We obtain many new solutions of the R-matrix from partition functions

4d $\mathcal{N}=1$ $S^1 \times S^3/\mathbb{Z}_r$, $S^2 \times T^2$, ...

3d $\mathcal{N}=2$ $S^1 \times S^2$, $S^3/\mathbb{Z}_r$, ...

2d $\mathcal{N}=(2,2)$ S^2, T^2, ...

[Y] [Terashima-Y] ('12), [Y] ('13),
[Yagi] ('15), [Yan-Y] ('15), [Y] ('16),…

We obtain many new solutions of the R-matrix from partition functions

4d $\mathcal{N}=1$ $S^1 \times S^3/\mathbb{Z}_r$, $S^2 \times T^2$, ...

(lens) elliptic (lens) elliptic gamma: $\Gamma(x;p,q)$

3d $\mathcal{N}=2$ $S^1 \times S^2$, $S^3/\mathbb{Z}_r$, ...

trigonometric q-dilog: $(x;q)_\infty$, $S_b(x)$

2d $\mathcal{N}=(2,2)$ S^2, T^2, ...

rational $\Gamma(x)$ $\theta(x|\tau)$

They do NOT fit into [Belavin-Drinfeld] ('82) classification

We obtain many new solutions of the R-matrix from partition functions

$$\begin{tikzcd}
4d\ \mathcal{N}=1 & & S^1\times S^3/\mathbb{Z}_r, \arrow[dl] \arrow[dr] & & S^2\times T^2, \arrow[ddlll] \arrow[dd] & \cdots \\
3d\ \mathcal{N}=2 & S^1\times S^2, \arrow[d] & & S^3/\mathbb{Z}_r, \arrow[dll] & \cdots & \\
2d\ \mathcal{N}=(2,2) & S^2, & & & T^2, & \cdots
\end{tikzcd}$$

dimensional reduction

[Gadde-Yan] [Dolan-Spiridonov-Vartanov] [Imamura] ('11) [Y] [Benini-Cremonesi] ('13),

We obtain many new solutions of the R-matrix from partition functions

4d $\mathcal{N}=1$ $\quad S^1 \times S^3/\mathbb{Z}_r,\ S^2 \times T^2,\ \cdots$

3d $\mathcal{N}=2$ $\quad S^1 \times S^2,\ S^3/\mathbb{Z}_r,\ \cdots$

2d $\mathcal{N}=(2,2)$ $\quad S^2,\ T^2,\ \cdots$

mixture of cluster algebra & YBE

"cluster-enriched YBE"

[Y] ('16), based on [Benini-Park-Zhao] ('14)

We obtain many new solutions of the R-matrix from partition functions

4d $\mathcal{N}=1$ $S^1 \times S^3/\mathbb{Z}_r$, $S^2 \times T^2$, ...

3d $\mathcal{N}=2$ $S^1 \times S^2$, $S^3/\mathbb{Z}_r$, ...

2d $\mathcal{N}=(2,2)$ S^2, T^2, ...

Jeffrey-Kirwan residue

[Yan-Y] ('15), cf.
[Benini-Eager-Hori-Tachikawa] ('13)

"super-master solution" [Y] ('13)

(for r=1 "master solution" of [Bazhanov-Sergeev] ('10)),

and contains all known solutions of STR with positive Boltzmann weight)

4d $\mathcal{N}=1$ $\quad S^1 \times S^3/\mathbb{Z}_r$, $S^2 \times T^2$, ...

3d $\mathcal{N}=2$ $\quad S^1 \times S^2$, $S^3/\mathbb{Z}_r$, ... --

2d $\mathcal{N}=(2,2)$ $\quad S^2$, $\quad T^2$, ..

"super-master solution" from 4d N=1 on $S^1 \times S^3/\mathbb{Z}_r$ with gauge group $SU(N)$

[Y] ('13), based on [Benini-Nishioka-Y] ('11)

Spins take values in discrete/continuous variables $(\mathbb{R} \times \mathbb{Z}_r)^{N-1}$

Elliptic parameter p, q arises from complex structure of $S^1 \times S^3/\mathbb{Z}_r$

[Kodaira] ('66)
[Closset-Dumitrescu-Festuccia-Komargodski] ('13)

While integrability is a consequence of gauge theory duality, integrability was recently directly proven mathematically by [Kels-Yamazaki] ('17), by generalizing earlier works [Spiridonov] ('01) ($r=1$, $N=2$), [Rains] ('03) ($r=1$, $N>2$), [Kels] ('15) ($r>1$, $N=2$),…

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Intertwiner of (very likely new) quantum-group type structure?

$\mathcal{U}_{p,q;r}(sl_N)$???

Hint: for r=1, N=2 this comes from Sklyanin algebra [Sklyanin] ('83)

[Cherednik] ('85)

$\mathcal{U}_{p,q}(sl_N)$

Particularly interesting limit: root of unity limit

(Lens) elliptic gamma function diverges:

[Bazhanov-Sergeev] ('10) [Kels-Y] ('17)

$$\Phi(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2iz} p^{2j+1} q^{2k+1}}{1 - e^{-2iz} p^{2j+1} q^{2k+1}}$$

$$\epsilon \to 0 \downarrow \left(p = e^{i\pi\tau},\ q = e^{-\frac{\epsilon}{2N^2}} \zeta,\ \zeta^{2N} = 1 \right)$$

$$\Phi(z;p,q) = \exp\left(\frac{i}{\epsilon}\, 2N \int_0^z du\, \ln \overline{\vartheta_3}(Nu/N\tau)\right)$$

$$\times\ (\text{subleading finite piece})$$

Particularly interesting limit: root of unity limit

This requires saddle point analysis

[Bazhanov-Sergeev] ('10) [Kels-Y] ('17)

Schematically,

$$Z \longrightarrow \sum \int d\sigma \, e^{\frac{1}{\epsilon} W^{(0)} + W^{(1)} + \mathcal{O}(\epsilon)}$$

saddle point in $\epsilon \to 0$

$$\frac{\partial W^{(0)}}{\partial \sigma} = 0$$

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Saddle point equation (for N=2): discrete classical integrable equation (Q4) of [Adler-Bobenko-Suris] ('02)

In their classification, (Q4) is the most general, and everything else is its degeneration

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Saddle point equation (for N=2): discrete classical integrable equation (Q4) of [Adler-Bobenko-Suris] ('02)

In their classification, (Q4) is the most general, and everything else is its degeneration

The saddle point equation is also the Bethe Ansatz equation for the dimensionally-reduced theory, in Gauge/Bethe correspondence [Nekrasov-Shatashvili] ('02), see also [Kels-Y] ('17) for related comments

We obtain YBE by evaluating the subleading corrections at a (leading) saddle point

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Several discrete integrable models reproduced, including chiral Potts model [Bazhanov-Sergeev] ('10) [Kels-Y] ('17)

Chiral Potts models [von Gehlen-Rittenberg] ('85) [Au-Yang-McCoy-Perk-Tan-Yang] ('87) [Baxter-Perk-Au-Yang] ('88) has higher-genus spectral curve, and do not have "rapidity-difference property"

$$R(z_1, z_2) \neq R(z_1 - z_2)$$

GAUGE THEORIES, VERTEX MODELS, AND QUANTUM GROUPS

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There are several obvious areas for further investigation. In terms of statistical mechanics, one compelling question is to understand the origin of the spectral parameter (and the elliptic modulus) in IRF and vertex models; this is essential for explaining the origin of integrability. Another question, which may or may not be related, is to understand the spin models formulated only rather recently [24] in which the spectral parameter is not an abelian variable (as in previous construc-

chiral Potts

Summary

Why integrable models exist?
Perspective from QFT?

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Perspective from QFT?

Because of
Gauge Theory Duality

Why integrable models exist?
Perspective from QFT?

Because of
Gauge Theory Duality

Because of
Locality, Unitarity,...

Origin of spectral parameter: R-charge

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New integrable models, and new mathematics and physics