Integrability and Four-Dimensional Chern-Simons Theory

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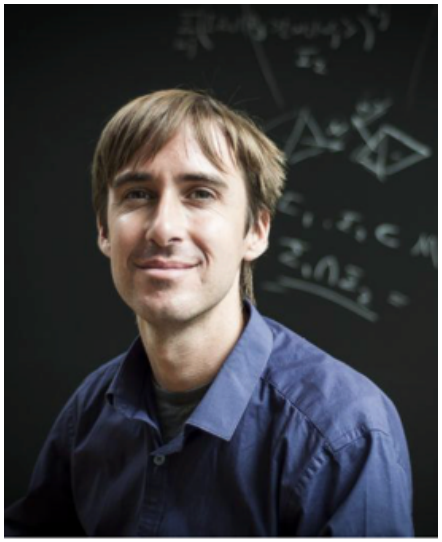
ANZAMP 2020, Tweed Heads

Based on collaboration w/



Kevin Costello
(Perimeter)

Edward Witten
(IAS)

Costello – Witten – Y

Part I 1709

II 1802

Costello – Y

Part III 1908

IV to appear

Y 1904

Costello – Witten – Y

Part I 1709

II 1802

integrable lattice models

Costello – Y

classical

Part III 1908

IV to appear

quantum

integrable field theories

Y 1904

Costello – Witten – Y

$\boxed{\text{Part I} \quad 1709}$

II 1802

integrable lattice models

Costello – Y

$\boxed{\text{Part III} \quad 1908}$ ← classical

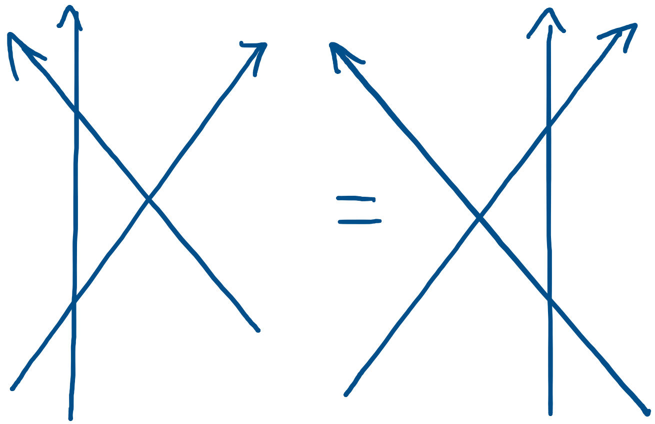
IV to appear ← quantum

integrable field theories

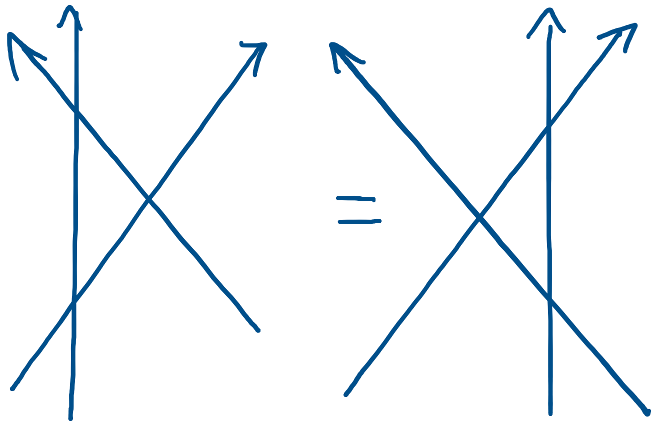
Y $\boxed{1904}$

Motivation

integrable
models

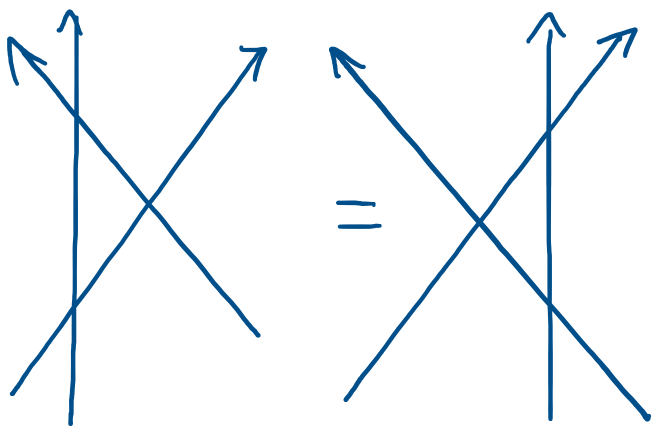


integrable models



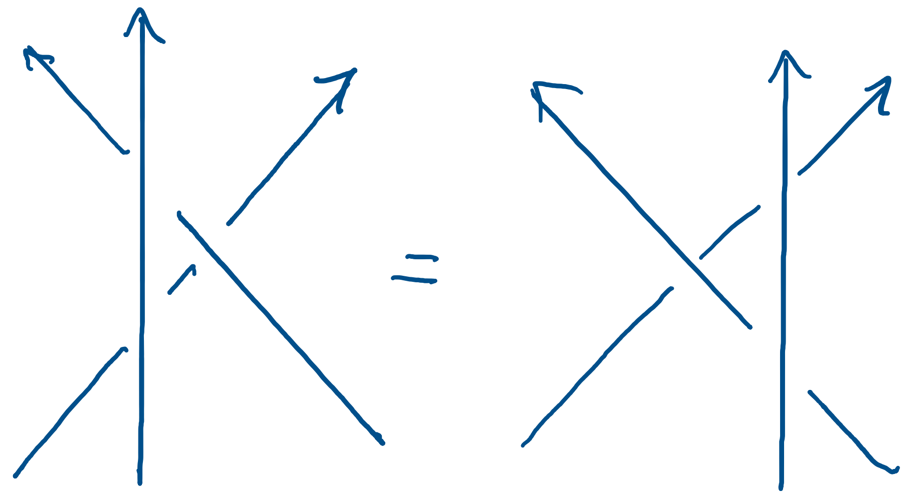
Some QFT?

integrable models

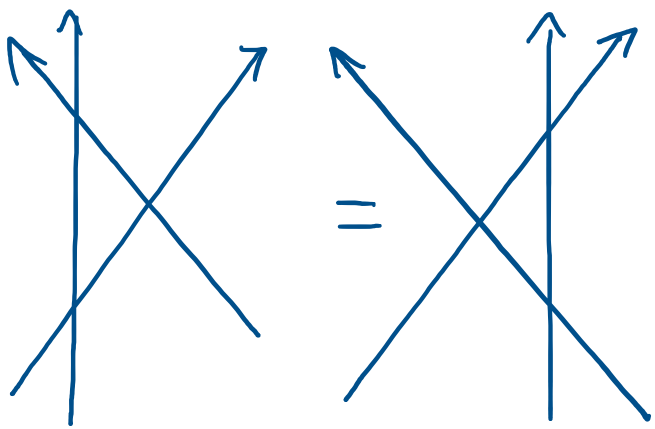


Some QFT?

knots

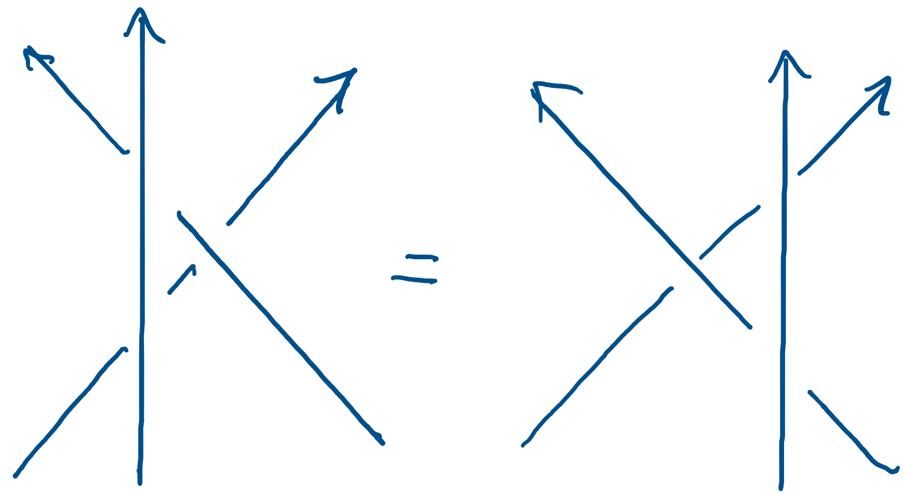


integrable models



Some QFT?

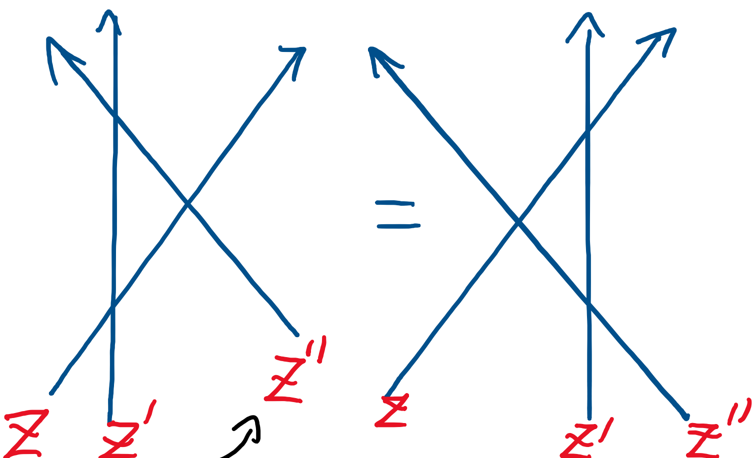
knots



Witten '89

3d Chern-Simons Theory

integrable models

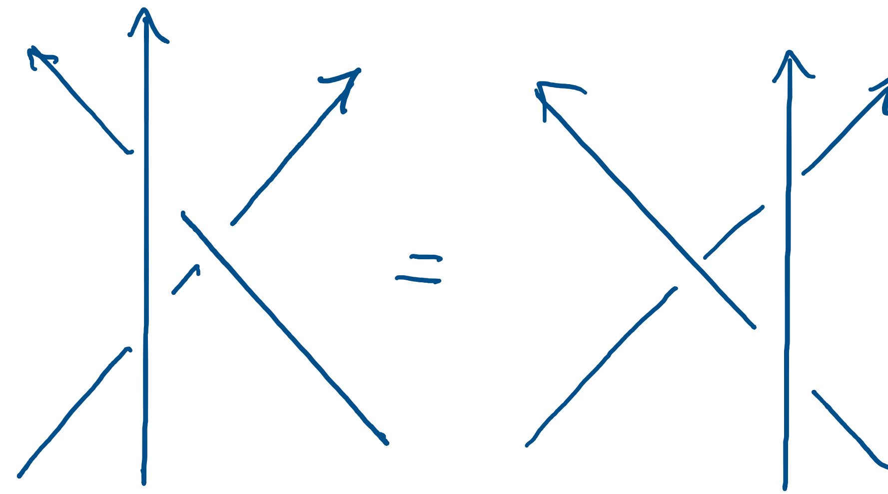


spectral parameters

Costello '13

4d Chern-Simons Theory

knots

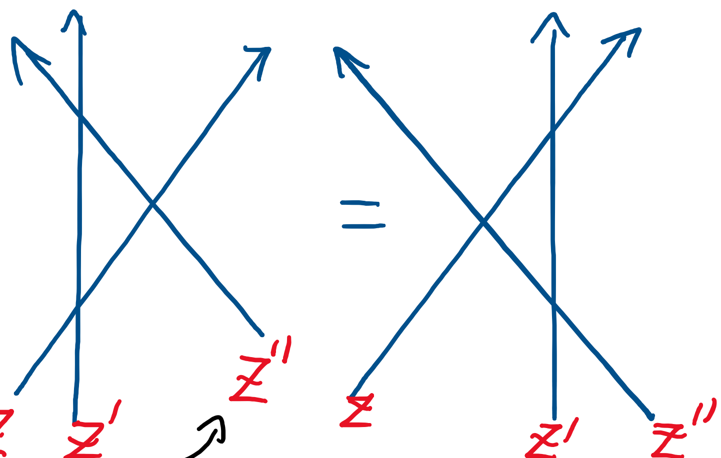


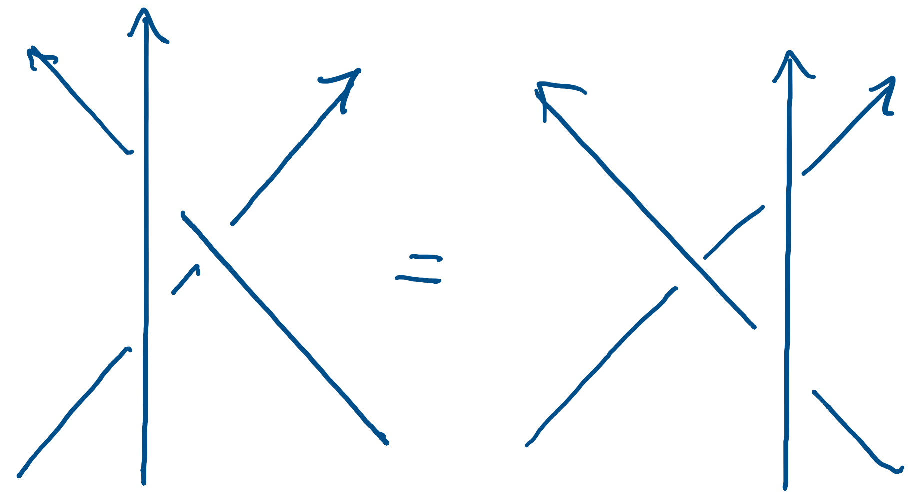
Witten '89

3d Chern-Simons Theory

integrable models

knots





spectral parameters

Costello '13

Witten '89

4d Chern-Simons Theory ⟵ 3d Chern-Simons Theory

Y '19

(novel) T-duality

4D Chern-Simons Theory

4d theory [Costello '13]

$$\mathcal{L} = \frac{1}{2\pi K} \int_{\Sigma \times C} \omega \wedge \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

4d theory [Costello '13]

$$\mathcal{L} = \frac{1}{2\pi K} \int_{\Sigma \times C} \omega \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

* Defined on 4-mfd of the form

$$\underbrace{\Sigma}_{\substack{\text{topological} \\ x,\, y}} \times \underbrace{C}_{\substack{\text{holomorphic} \\ z,\, \bar{z}}}$$

4d theory [Costello '13]

$$\mathcal{L} = \frac{1}{2\pi K} \int_{\Sigma \times C} \overbrace{\omega}^{1\text{-form}} \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

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* ω: hol. 1-form on C

[$\omega = dz$ locally]

4d theory [Costello '13]

$$\mathcal{L} = \frac{1}{2\pi K} \int_{\Sigma \times C} \underbrace{\overbrace{\omega}^{1\text{-form}} \wedge \overbrace{\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)}^{\text{Chern-Simons 3-form}}}_{4\text{-form}}$$

* Defined on 4-mfd of the form

$$\underbrace{\Sigma}_{\substack{\text{topological} \\ x,\, y}} \times \underbrace{C}_{\substack{\text{holomorphic} \\ z,\, \bar{z}}}$$

* ω : hol. 1-form on C

  e.g. $\omega = dz$ for $C = \mathbb{C}$

4d theory [Costello '13]

$$\mathcal{L} = \frac{1}{2\pi K} \int_{\Sigma \times C} \underbrace{\omega \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)}_{\text{complex Lagrangian}}$$

* the action is Complex

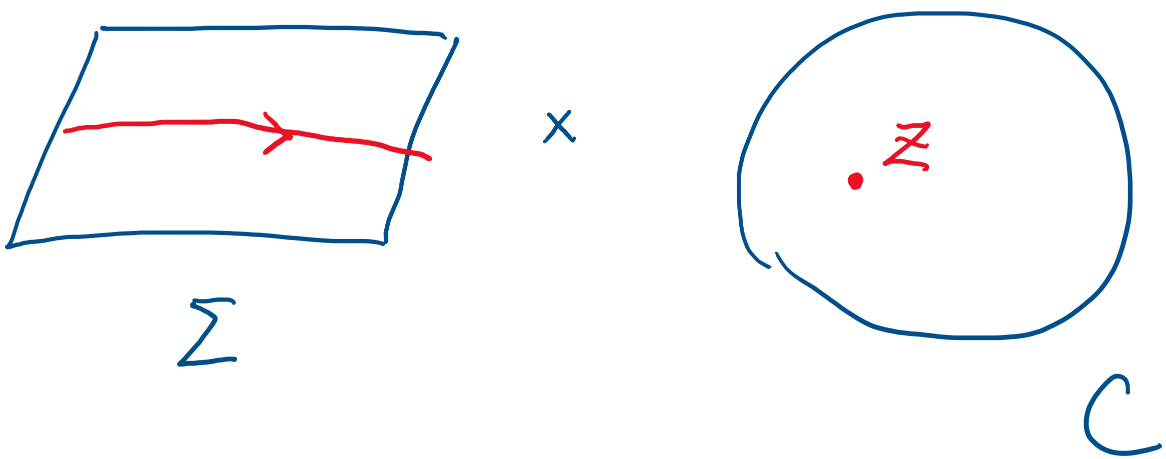
* here we do perturbation in $\hbar$
  around isolated classical contribution

integrability

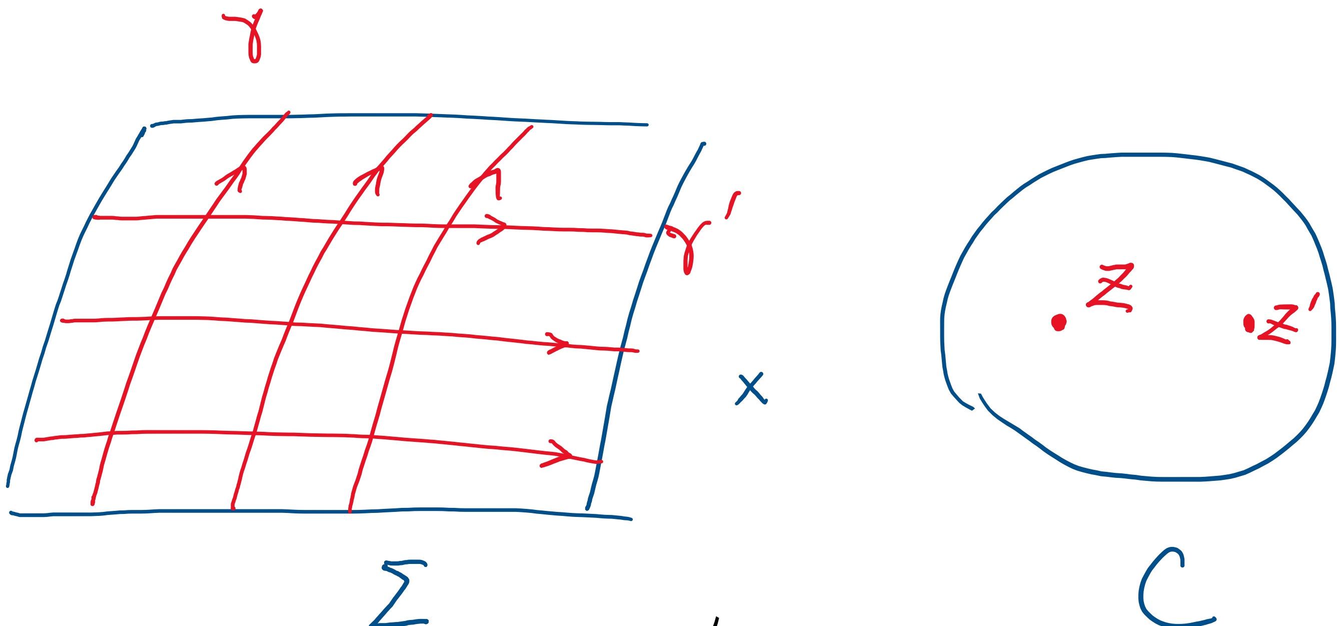
Wilson line

$$\underset{\sim}{W_\gamma} = \left\langle \mathrm{Tr}\, P \exp \int_\gamma A \right\rangle$$

path along Σ, located at point z



generate the statistical lattice from Wilson lines

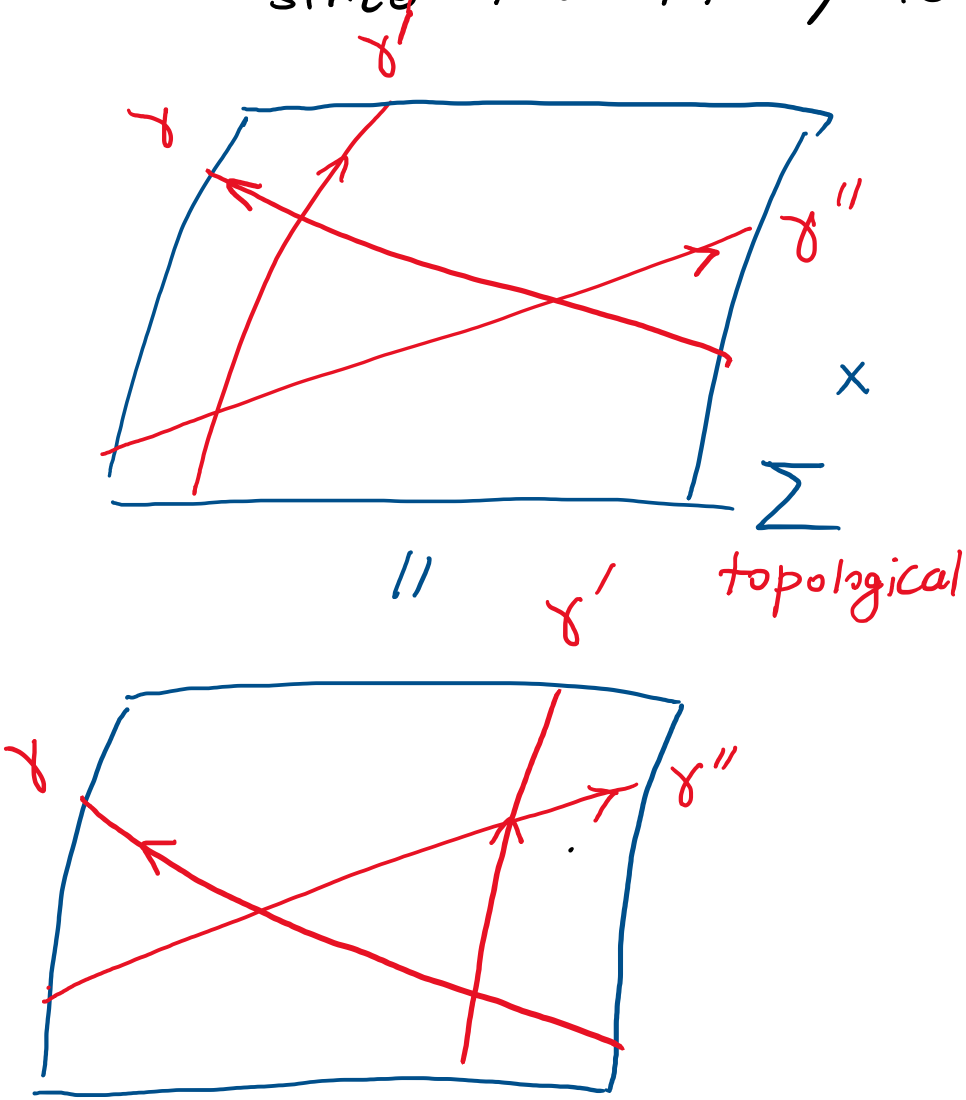


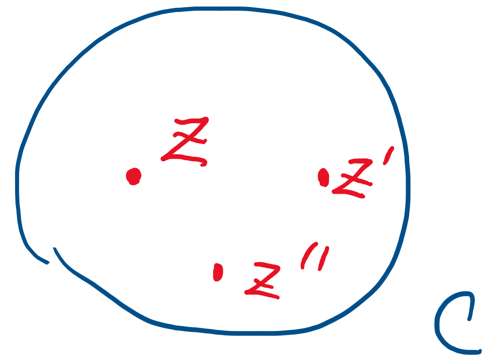
integrable lattice model

"extra dimension" for spectral parameter

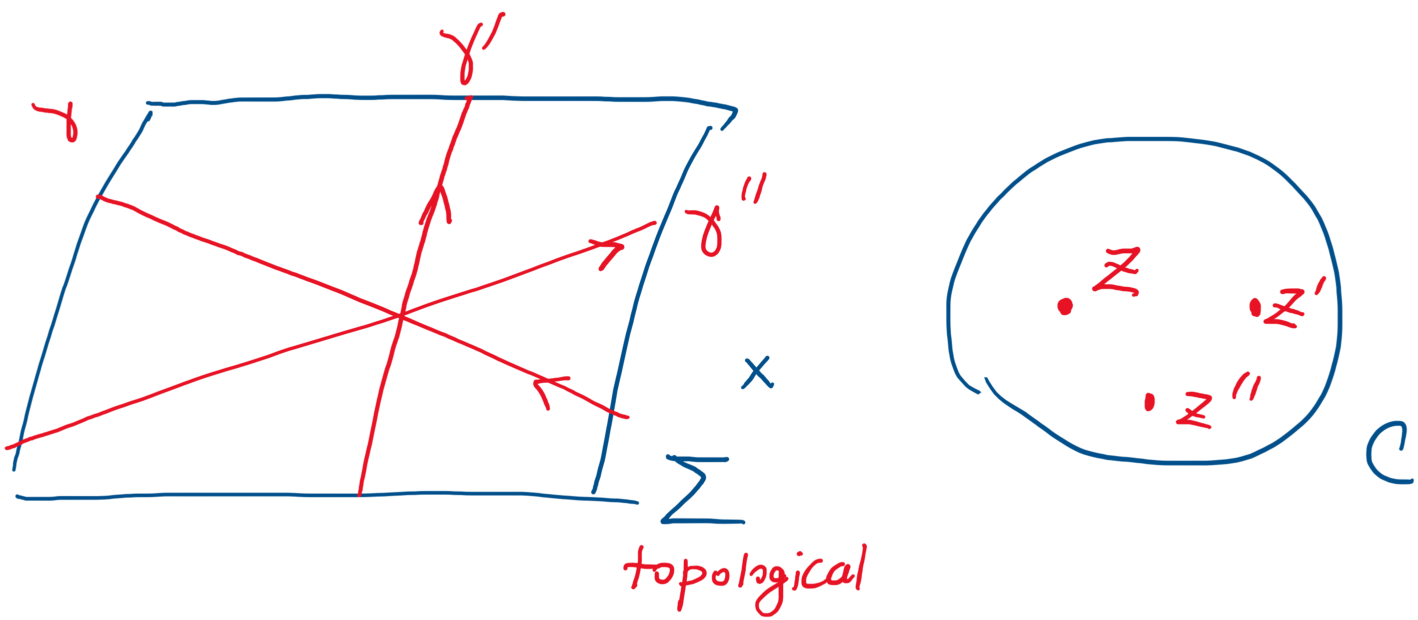
Yang-Baxter equation follows

since the theory is topological along Σ

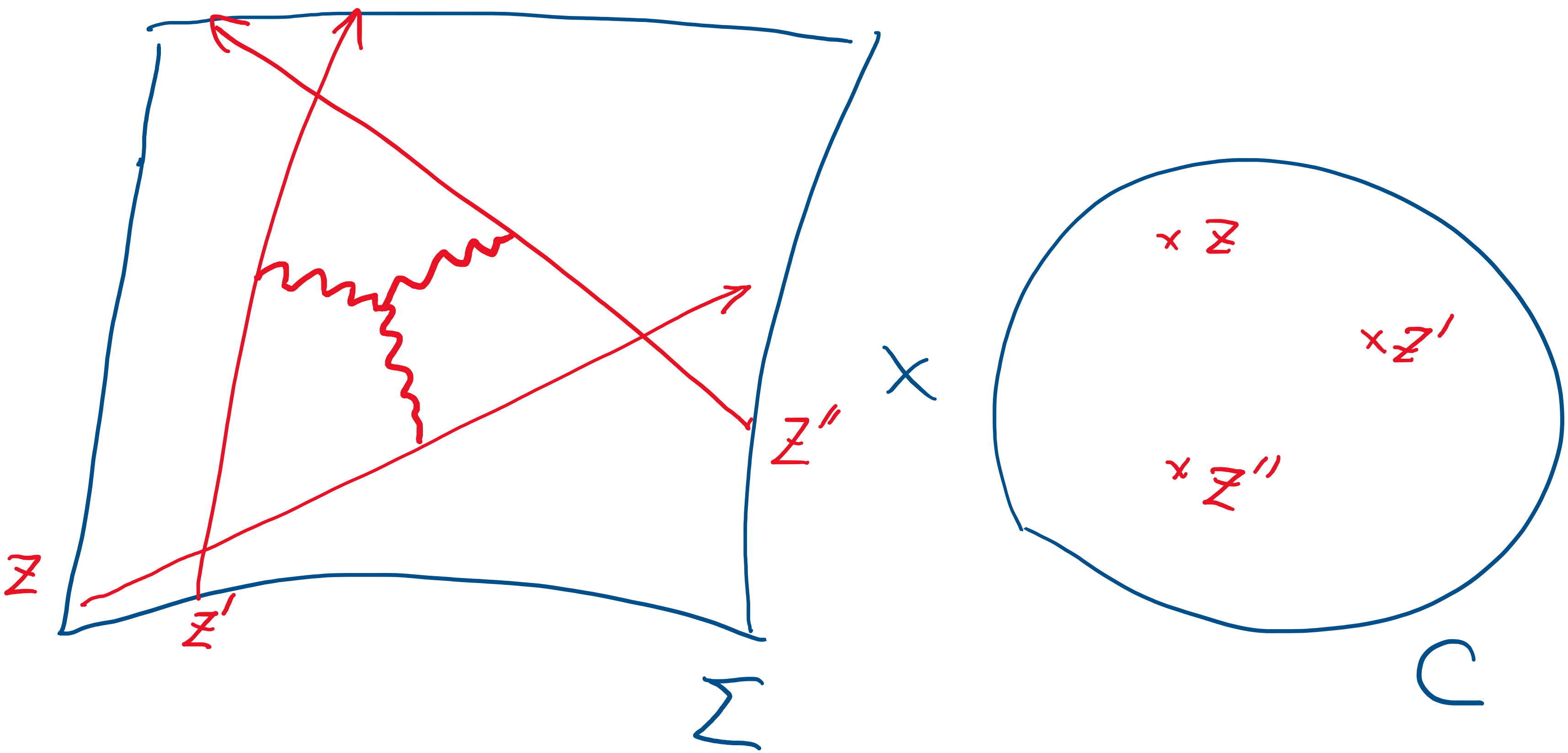




※ no singularity when 3 line cross,
since the Wilson lines are separate
along C and never touch



... But what about non-factorizable contribution?



Recall

$$\mathcal{L} = \frac{1}{2\pi\hbar} \underbrace{\int_{\Sigma \times C}}_{4\,d\text{ integral}} \omega \wedge \overbrace{\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)}^{[\text{mass}]^3}$$

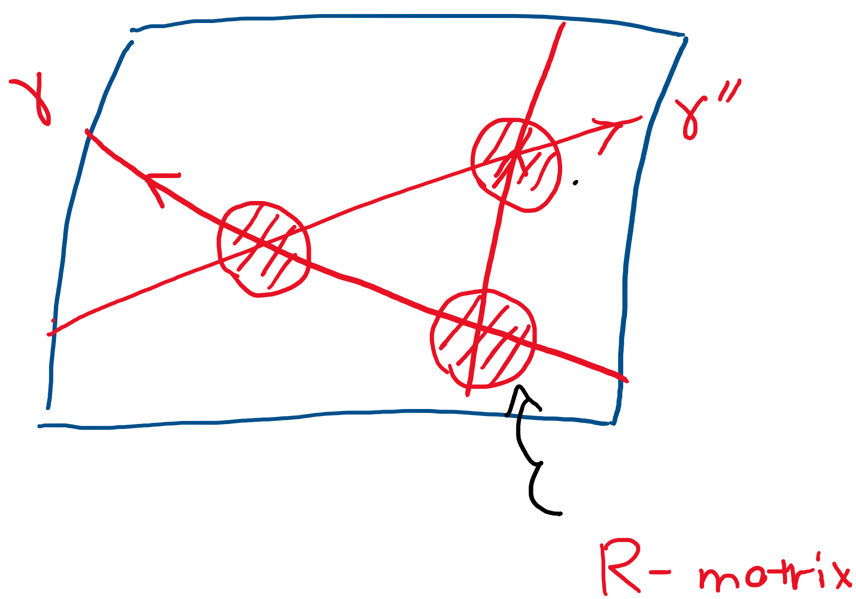
* expansion parameter $\hbar$ is dimensionfull

$[\hbar] = [\text{mass}]^{-1}$

$\rightsquigarrow$ theory is IR free

Our theory : IR free & topological along Σ,

factorized contribution from each crossing

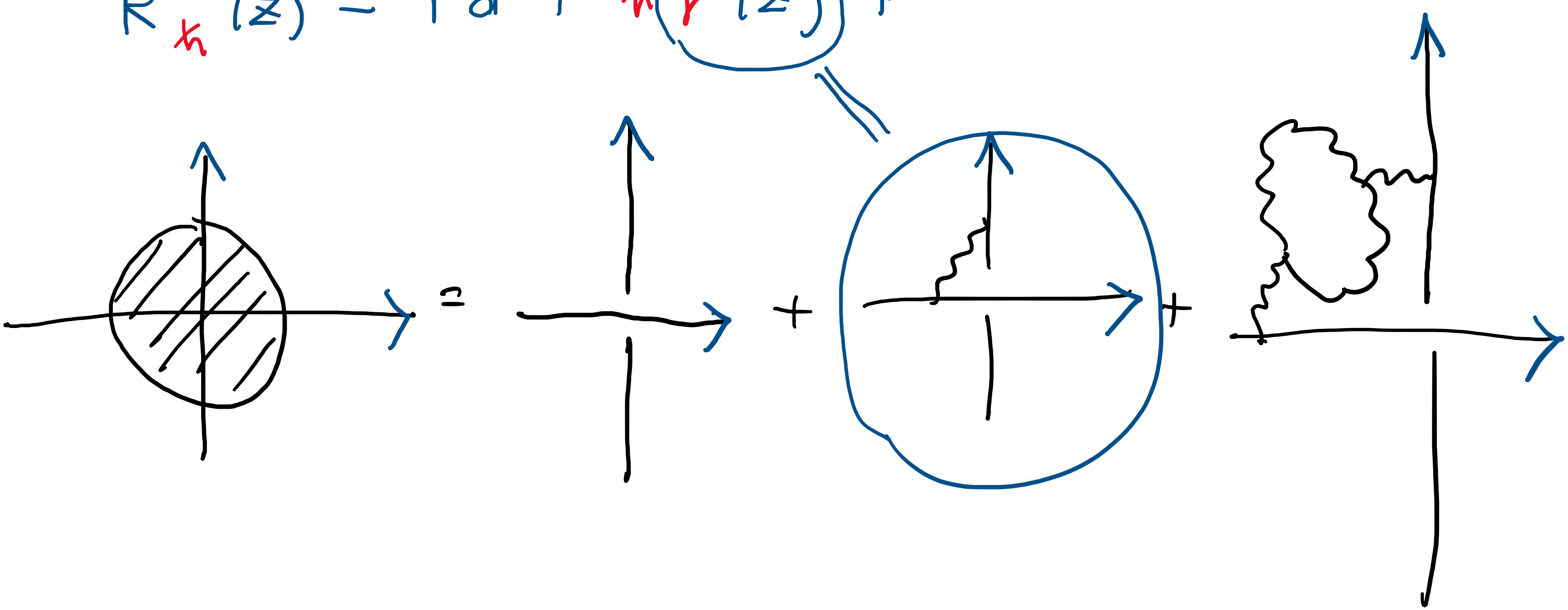


R-matrix

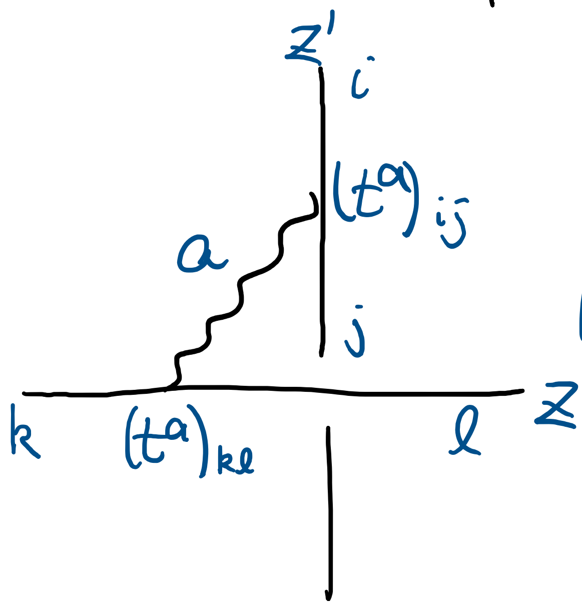
perturbative Feynman diagram computation

reproduce perturbative R-matrix

$$R_{\hbar}(z) = \mathrm{id} + \hbar\, r(z) + \cdots$$



lowest order computation gives classical R-matrix $r(z)$



$$\boxed{r(z) = \frac{(t^a)_{ij}(t^a)_{kl}}{z - z'}}$$

reproduces the known answer

this is enough for reproducing the full R-matrix, all order in $\hbar$

together w/ YBE

[Drinfeld, also CWY II]

perturbative

classification

$$\mathcal{L} = \frac{1}{2\pi\hbar}\int_{\Sigma\times C} \omega \wedge \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A^3\right)$$

ω and $\hbar$ appear in combination

$\hbar \to \infty$ around zero of ω: $\omega \to 0$

let's therefore impose ω has no zero,

for perturbation in $\hbar$

C in general has boundary points

$$C = \underbrace{\bar{C}}_{\text{compactification}} \setminus \{\text{points}\}$$

ω : hol. 1-form on C, no zero

$\downarrow$ Riemann-Roch theorem

3 possibilities

$\omega = dz$	$C = \mathbb{C}$	rational
$\omega = \frac{dz}{z}$	$C = \mathbb{C}^\times$	trigonometric
$\omega = dz$	$C = E$	elliptic

matches with classification of Belavin & Drinfeld

identify w/ $\hbar$ in $\mathcal{L}$

$$\underbrace{R_{\hbar}(z)}_{\text{quasi-classical R-matrix}} = \mathrm{Id} + \hbar \underbrace{r(z)}_{\text{classical R-matrix}} + \hbar^2 r'(z) + \cdots$$

elliptic solution known only for

$G = PGL_N$ [Belavin, Belavin–Drinfeld]

(rigid G-bundle ;
isolated critical pt of 4d CS)

elliptic solution known only for

$G = PGL_N$ [Belavin / Belavin–Drinfeld]

$\left(\begin{array}{l}\text{rigid } G\text{-bundle};\\ \text{isolated critical pt of 4d CS}\end{array}\right)$

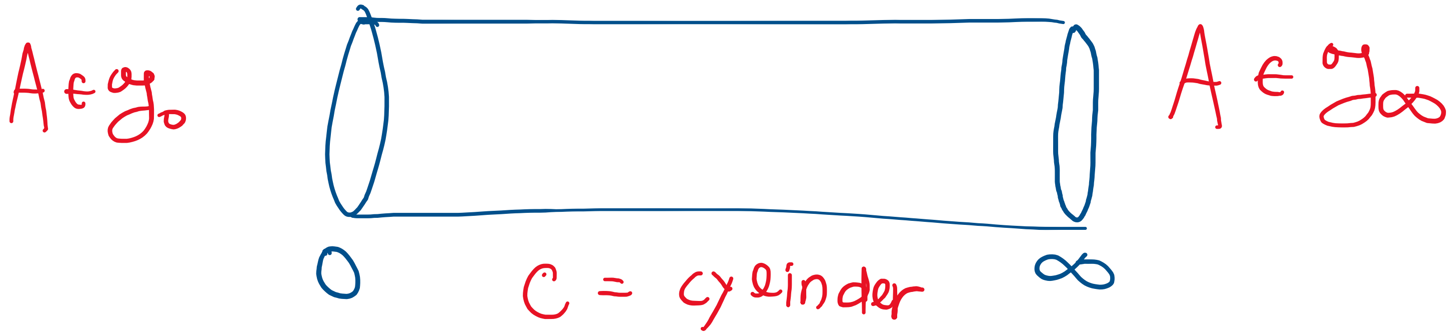
for general G

we have dynamical YBE

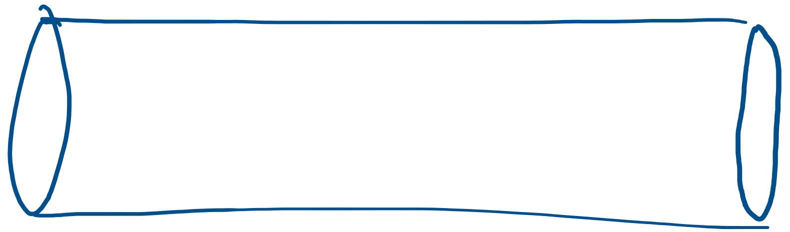
$R(z, \eta)$

"dynamical parameter"

trigonometric case:



trigonometric case:

$A \in \mathfrak{g}_0$  $A \in \mathfrak{g}_\infty$

0 ∞

C = cylinder

non-dynamical YBE (isolated solution)

iff
$$\mathfrak{g}_0 \oplus \mathfrak{g}_\infty = \hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{t}$$
$$\mathfrak{g}_0 \cap \mathfrak{g}_\infty = \phi$$

(Manin triple)

$\mathfrak{t}$: Cartan
(e.g. 6v electric deformation)

Yangian

Consider $C = \mathbb{C}$, $\omega = dz$ (rational case)

"level n" generator

the symmetry algebra here is

Yangion $Y_\hbar(\mathfrak{g}) \xrightarrow[\hbar \to 0]{} \mathcal{U}(\mathfrak{g}[[z]])$

$\{ t_{a,n} = t_a z^n \}$

$\hbar$ deformation

$$\left(\begin{array}{l} [t_{a,m}, t_{b,n}] \\ \quad = i f_{abc}\, t_{c,m+n} \end{array} \right)$$

The Wilson line allows for derivative couplings

$$\langle W_\gamma \rangle = \left\langle \mathrm{Tr}_{\hat{\rho}} \exp \int_{\gamma \times \{z_0\}} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} \frac{\partial^n A}{\partial z^n}(z_0) \right\rangle$$

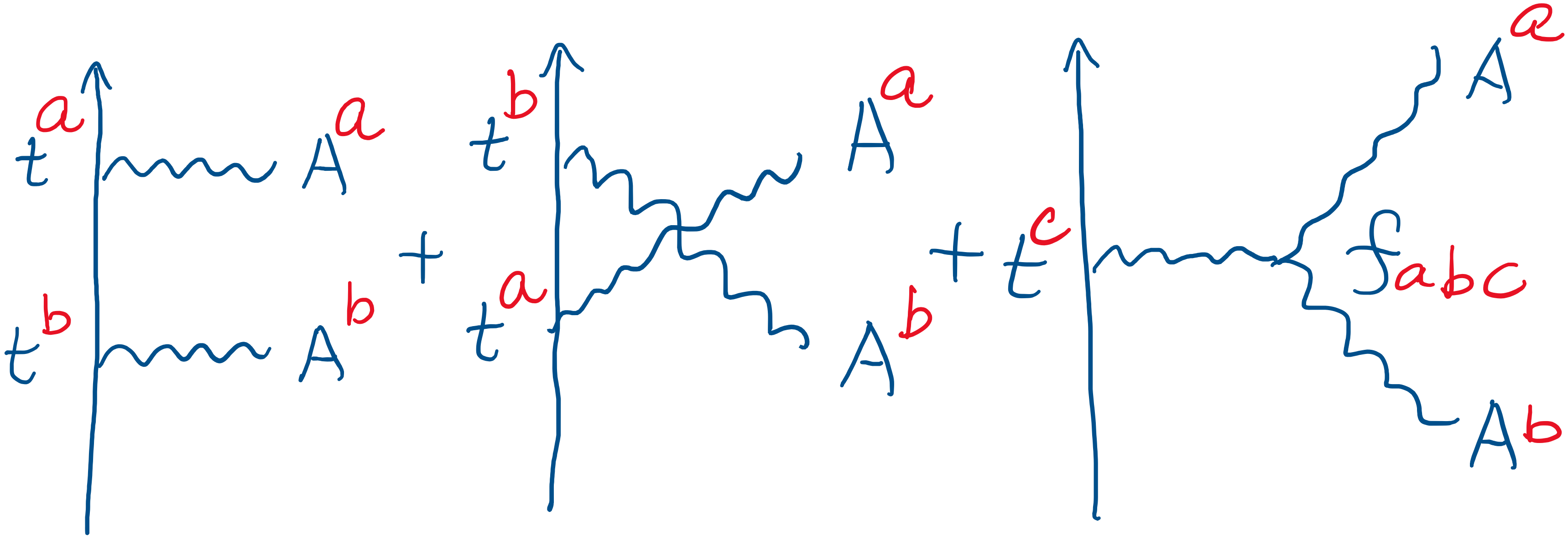
(along Σ
not along C)

repr. of $\mathcal{U}(\mathfrak{g}[[z]])$

$\rightsquigarrow$ We have repr. of $\mathcal{U}(\mathfrak{g}[[z]])$ classically

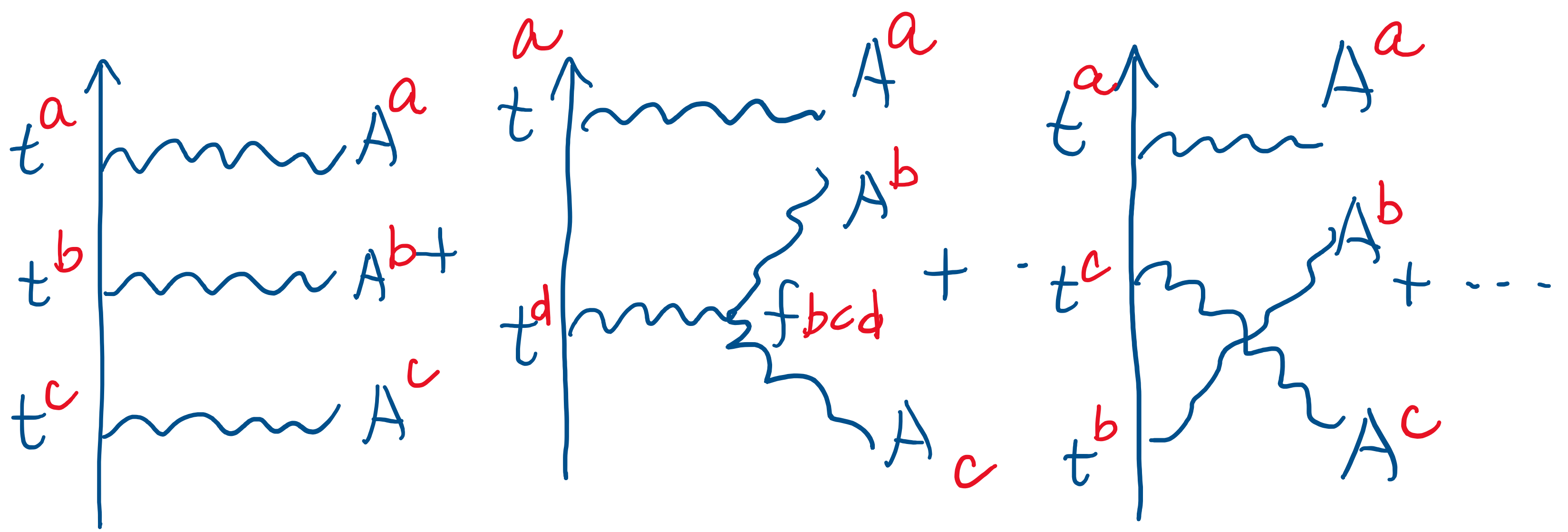
.... however anomalies in quantization

classically



gauge-invariant only if

$$[t^a, t^b] = i f^{abc} t_c$$

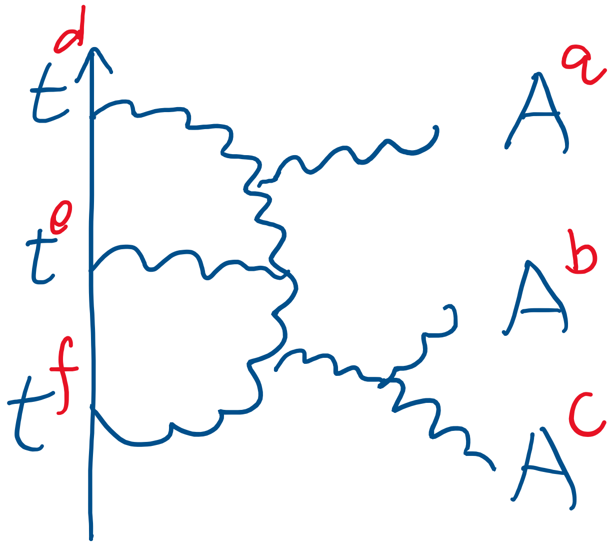


Jacobi identity for $\mathfrak{g}$

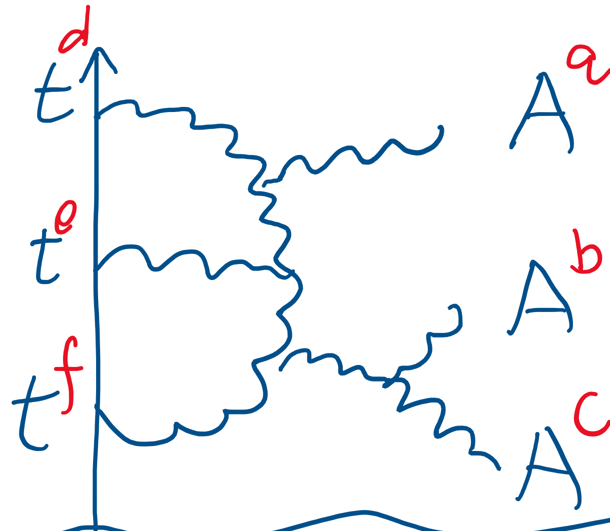
$$[t^a, [t^b, t^c]] + (\text{cyclic}) = 0$$

anomaly quantum mechonically

ⓐ 2-loop



anomaly quantum mechonically
@ 2-loop



Yongian relation!

$$\left[J(t_a) = J(t_{a,0}) = t_{a,1} = t_a z \right]$$

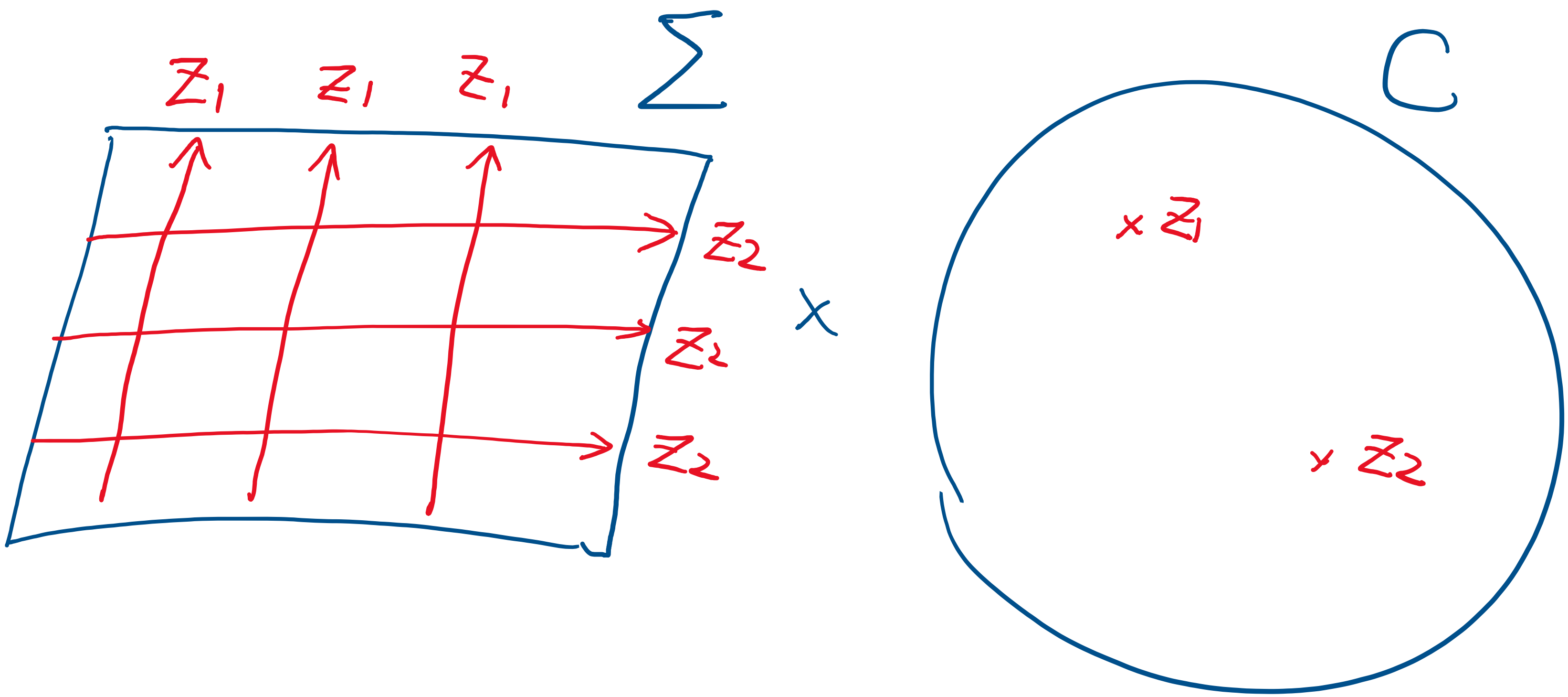
$$[J(t_a), J[t_b, t_c]] + (\text{cyclic})$$

$$= \frac{\hbar^2}{12}\left([t_a, t_d], [[t_b, t_e], [t_c, t_f]]\right)$$

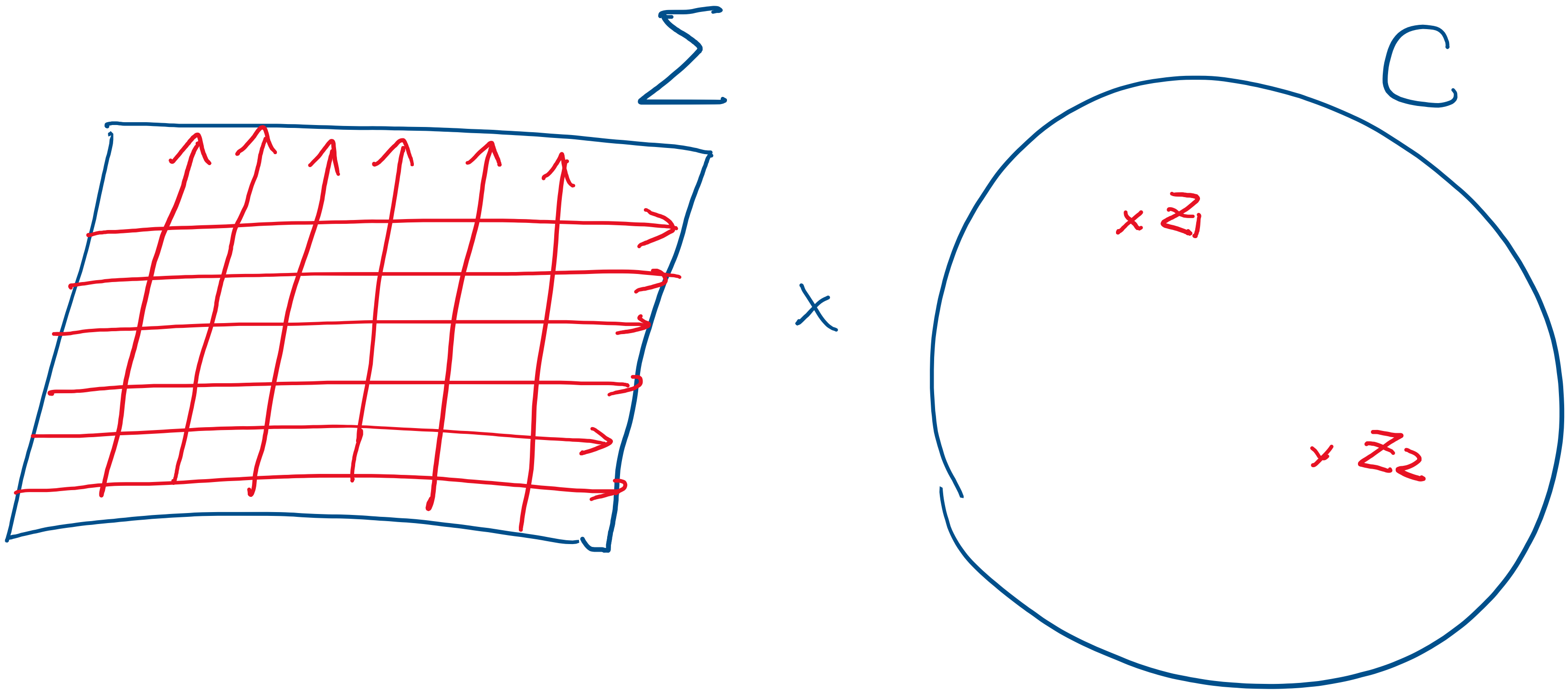
$$\times \left(t_d t_e t_f + (\text{perm})\right)/3!$$

Integrable Field Theories

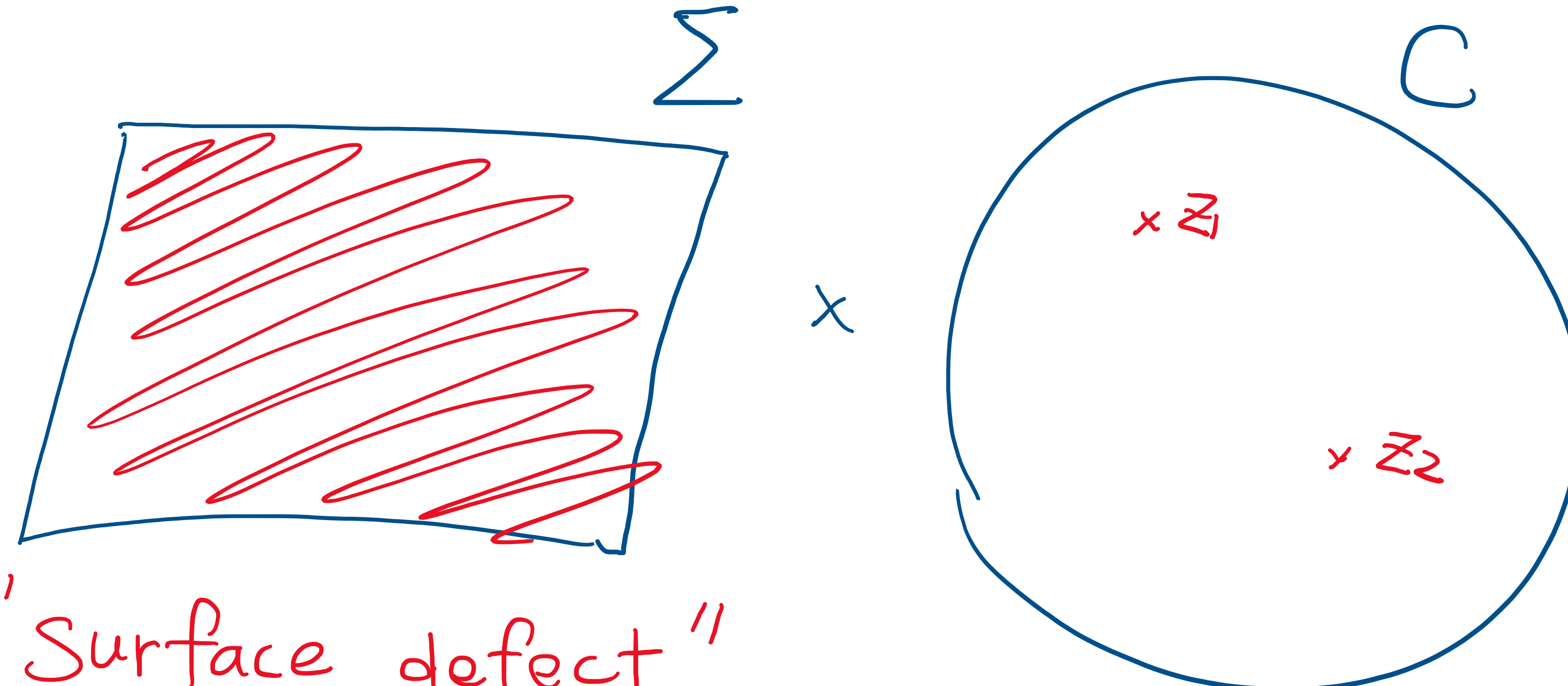
[Costello - Y 19]



z_1 z_1 z_1
Σ
z_2
z_2
z_2
$\times$
C
z_1
z_2

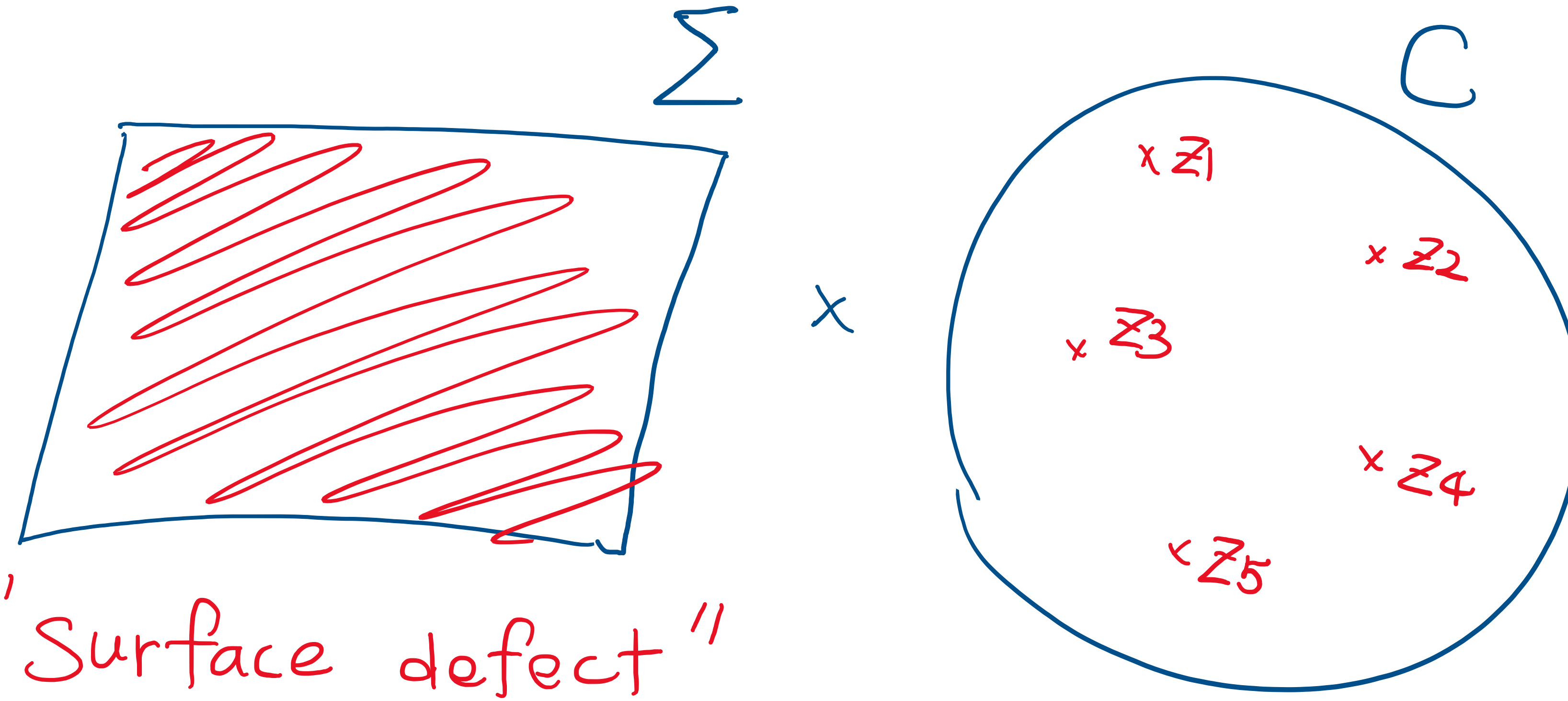


Σ
$\times$
C
z_1
z_2



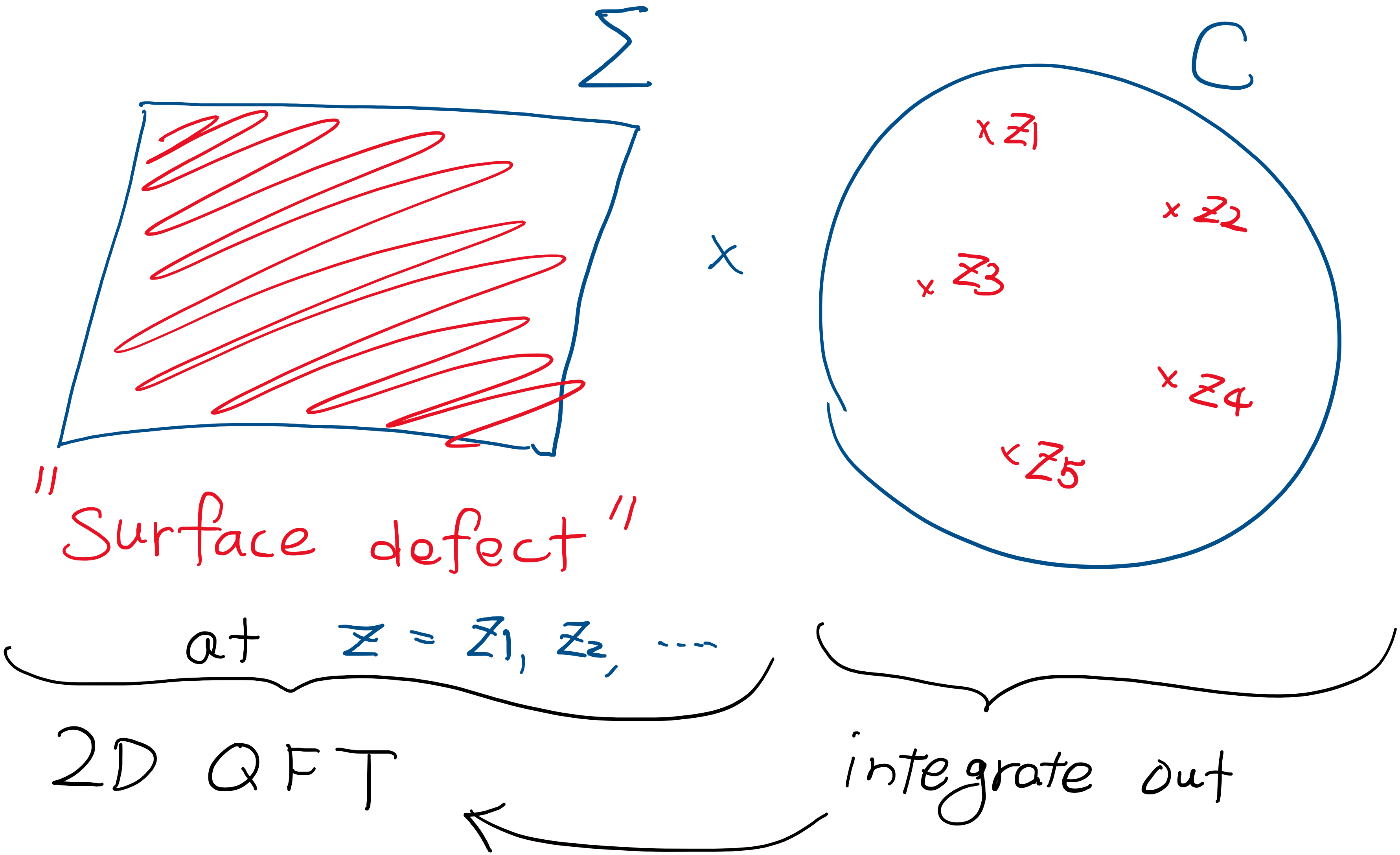
"Surface defect"

at $z = z_1, z_2$



"Surface defect"

at $z = z_1, z_2, \cdots$


Σ
×
C
× z1
× z2
× z3
× z4
× z5
"Surface defect"
at z = z1, z2, ...
2D QFT
integrate out

Why integrable?

$$\left(\Sigma_{w\bar{w}} \times C_{z\bar{z}} \right)$$

4D CS e.o.m.

$$\rightsquigarrow F_{w\bar{w}} = \partial_w A_{\bar{w}} - \partial_{\bar{w}} A_w + [A_w, A_{\bar{w}}] = 0$$

$$\left. \begin{aligned} F_{\bar{z}w} &= \partial_{\bar{z}} A_w = 0 \\ F_{\bar{z}\bar{w}} &= \partial_{\bar{z}} A_{\bar{w}} = 0 \end{aligned} \right) \quad (A_{\bar{z}} = 0 \text{ gauge})$$

Why integrable?

$$\left(\Sigma_{w\bar{w}} \times C_{z\bar{z}} \right)$$

4D CS e.o.m.

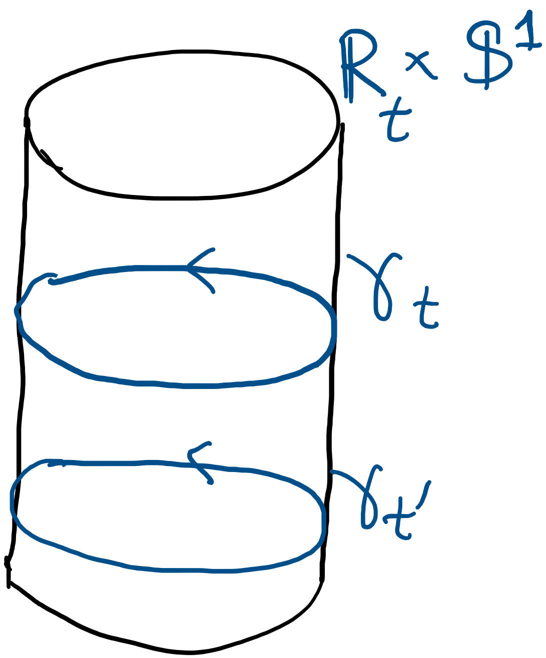
$$\rightsquigarrow \quad F_{w\bar{w}} = \partial_w A_{\bar{w}} - \partial_{\bar{w}} A_w + [A_w, A_{\bar{w}}] = 0$$

$$\left.\begin{aligned} F_{\bar{z}w} &= \partial_{\bar{z}} A_w = 0 \\ F_{\bar{z}\bar{w}} &= \partial_{\bar{z}} A_{\bar{w}} = 0 \end{aligned}\right) \quad (A_{\bar{z}} = 0 \text{ gauge})$$

$\rightsquigarrow$ Lax matrix

$$\mathcal{L}(z) \equiv A_w(z)\, dw + A_{\bar{w}}(z)\, d\bar{w}$$

is flat



monodromy / Wilson line

$$W(z) \equiv \left\langle \operatorname{Tr} P\exp \oint_{\gamma_t} \mathcal{L}(z) \right\rangle$$

$$\mathcal{L}(z) \text{ flat} \rightsquigarrow \partial_t \underbrace{W(z)}_{\exp\left(\sum Q_n z^n\right)} = 0$$

$$\rightsquigarrow \partial_t Q_n = 0$$

reproduce many known models, e.g.

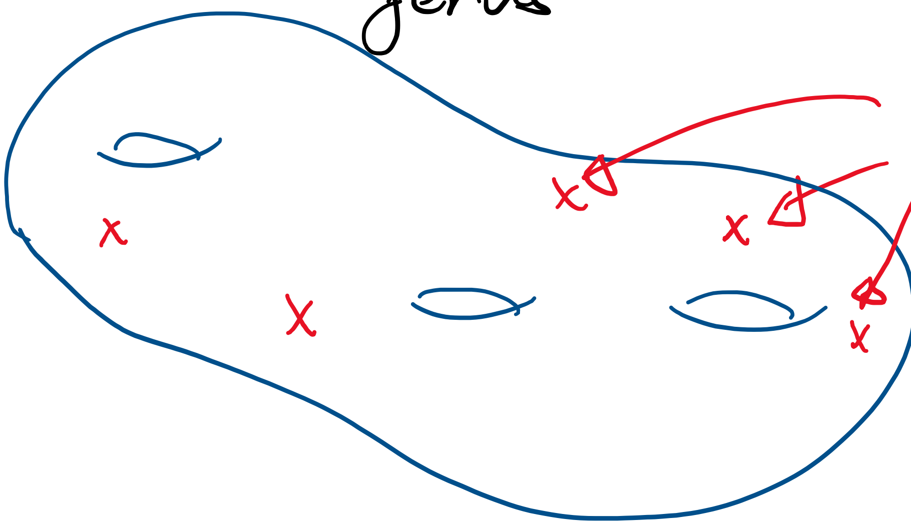
* massive Thirring
* Gross - Neveu
* S^2, S^3 sausage
* conformal / non-conformal WZW
* $\mathbb{Z}_n$ - graded coset G/H

$\vdots$

a zoo of new integrable field theories

$C = \Sigma_g$ ← higher genus

D_1



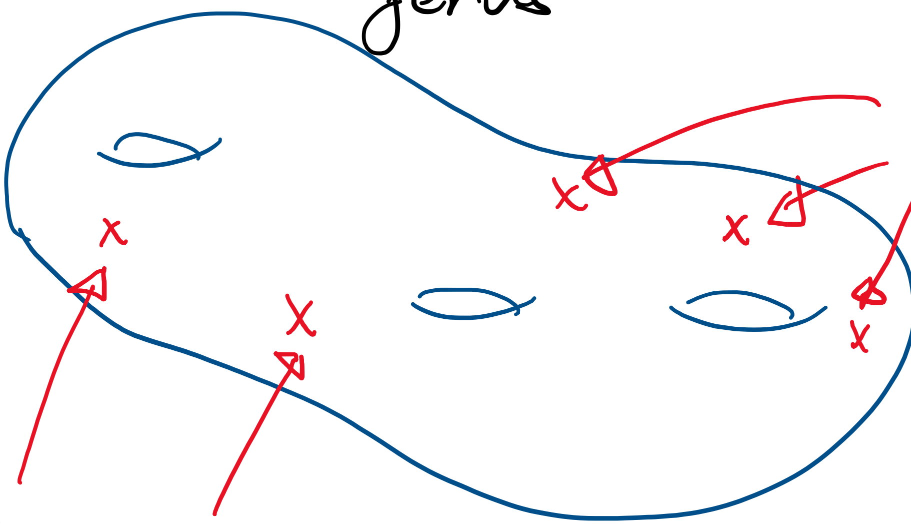
order defect

(coupling to 2d theory)

(e.g. $\mathcal{L} \supset \bar{\psi}(\partial + A)\psi$)

a zoo of new integrable field theories

$C = \Sigma_g$ ← higher genus



D_1

order defect
(coupling to 2d theory)

(e.g. $\mathcal{L} \supset \bar{\psi}(\partial + A)\psi$)

(disorder defect
singularities of A)

D_2

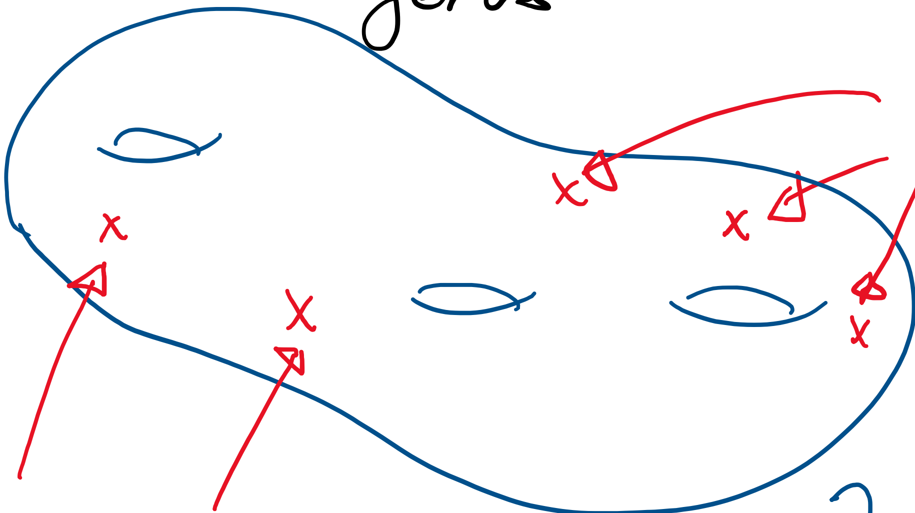
(e.g. $A_w \sim \mathcal{O}\left(\frac{1}{z - z_0}\right)$)

a zoo of new integrable field theories

$C = \Sigma_g \leftarrow$ higher genus

D_1

order defect
(coupling to 2d theory)



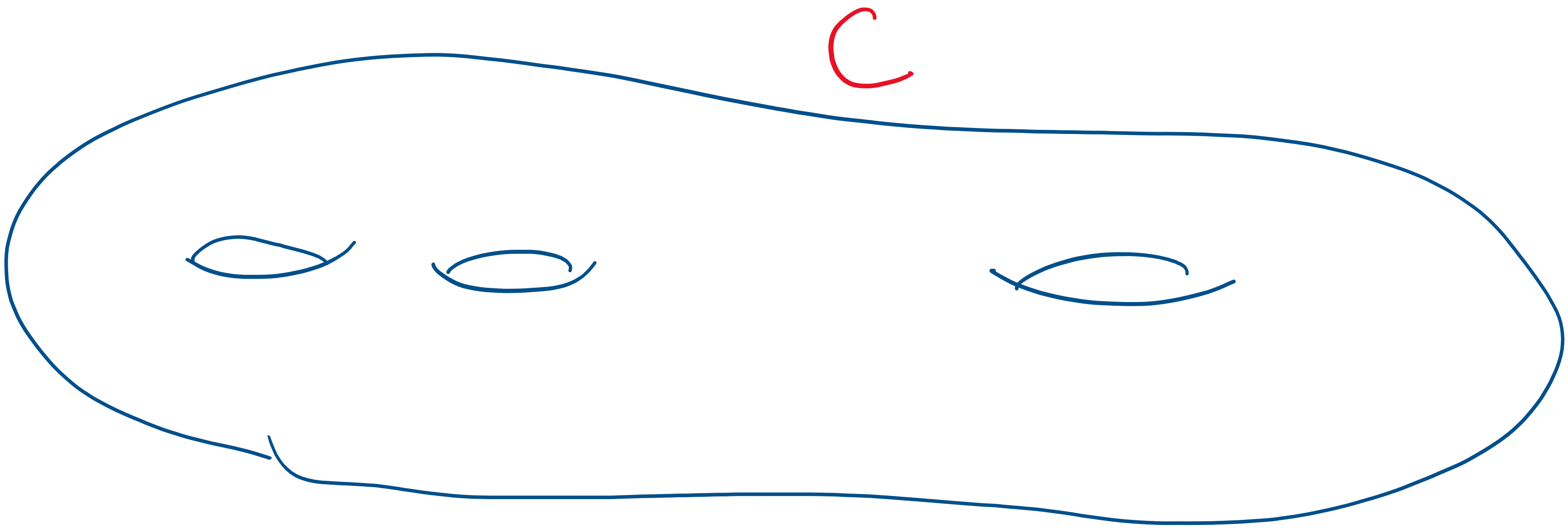
(disorder defect
singularities of A)
D_2

$\rightsquigarrow$

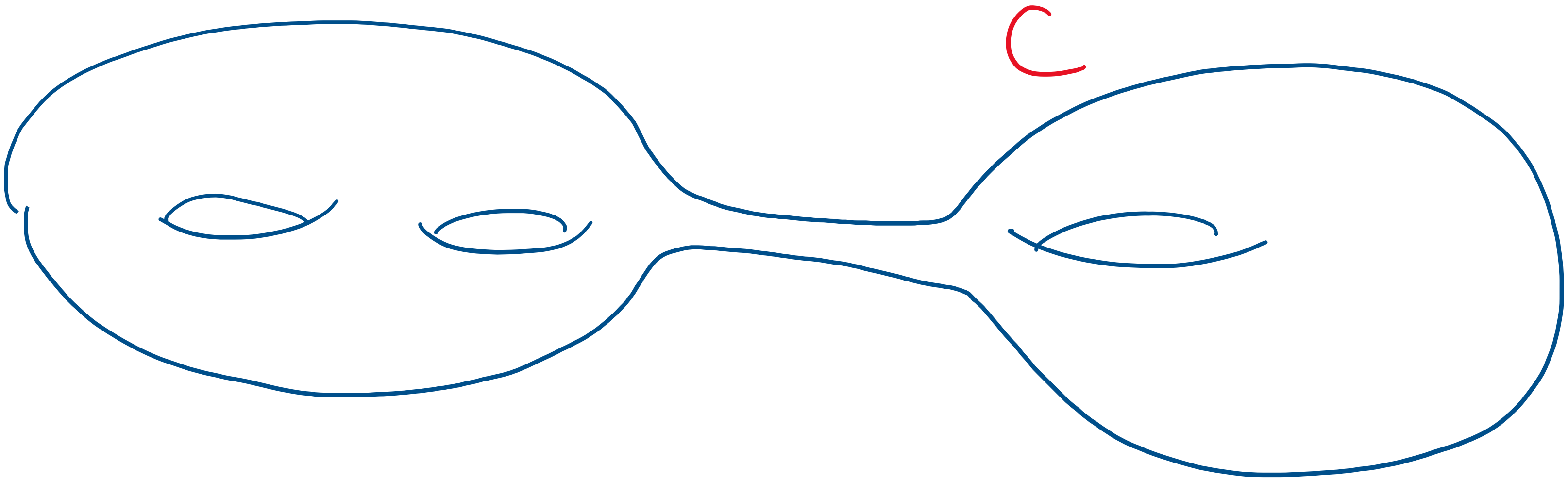
integrable field theory

$\mathrm{IFT}[C, D_1, D_2]$

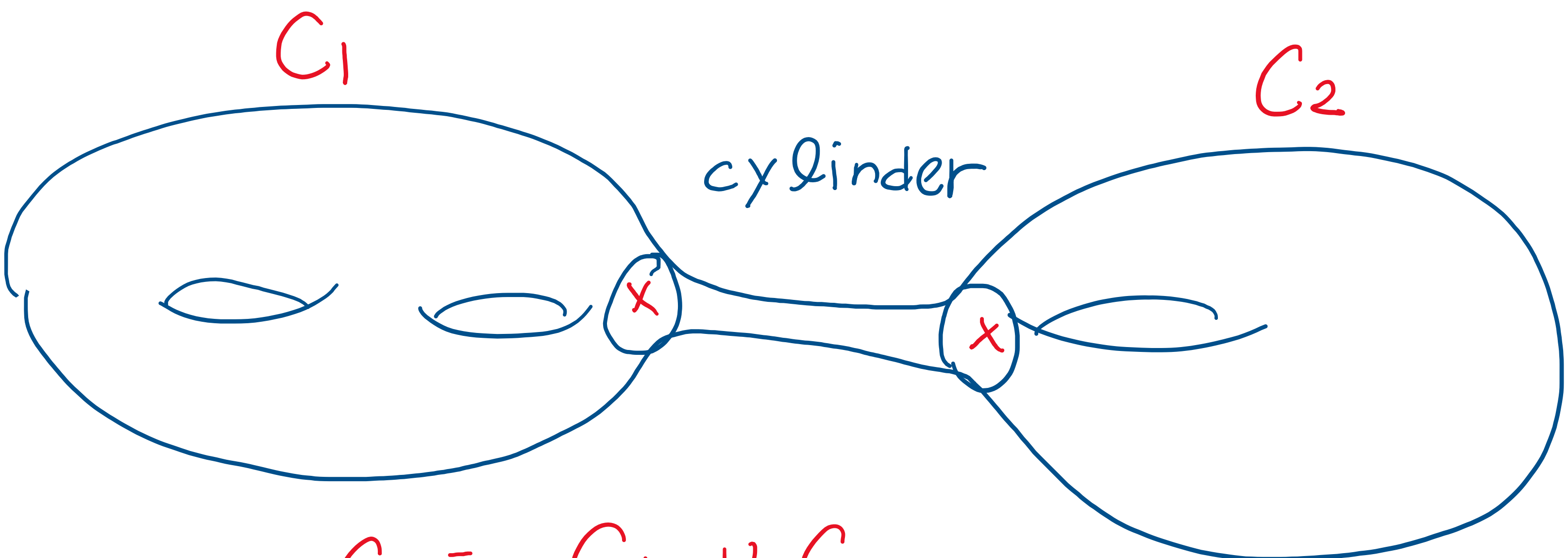
Cutting / Gluing of integrable field theories



Cutting / Gluing of integrable field theories

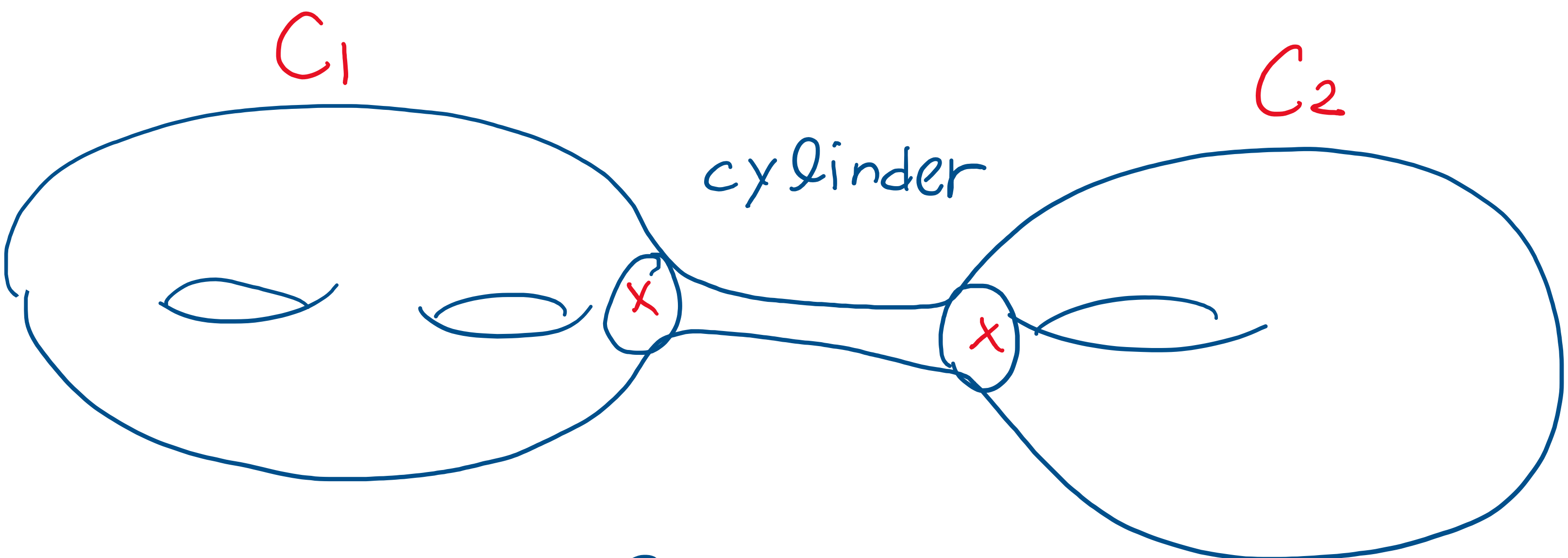


Cutting/Gluing of integrable field theories



$$C = C_1 \cup C_2$$

Cutting / Gluing of integrable field theories



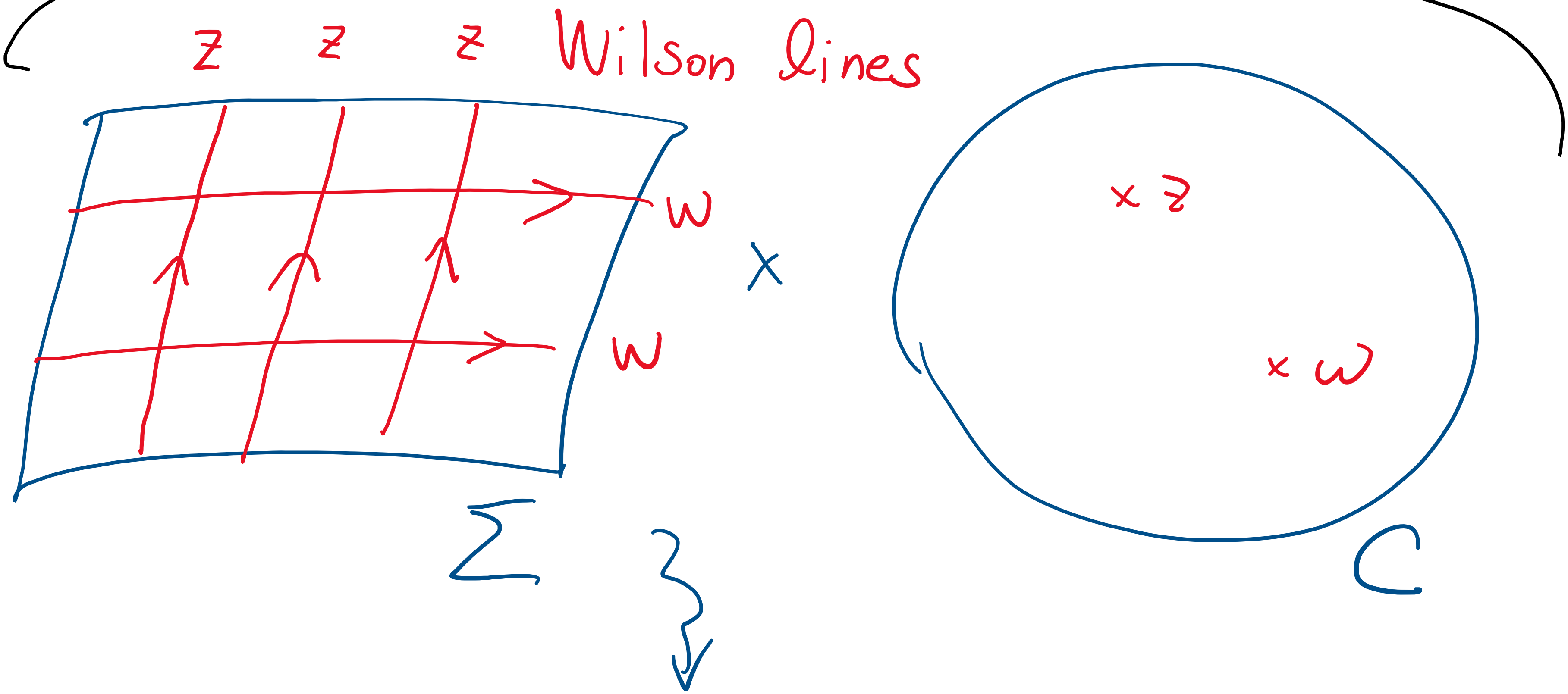
$$C = C_1 \cup C_2$$

$$\rightsquigarrow$$

$$\text{“} IFT[C] = IFT[C_1] \times IFT[C_2] /\!\!/ G \text{”}$$

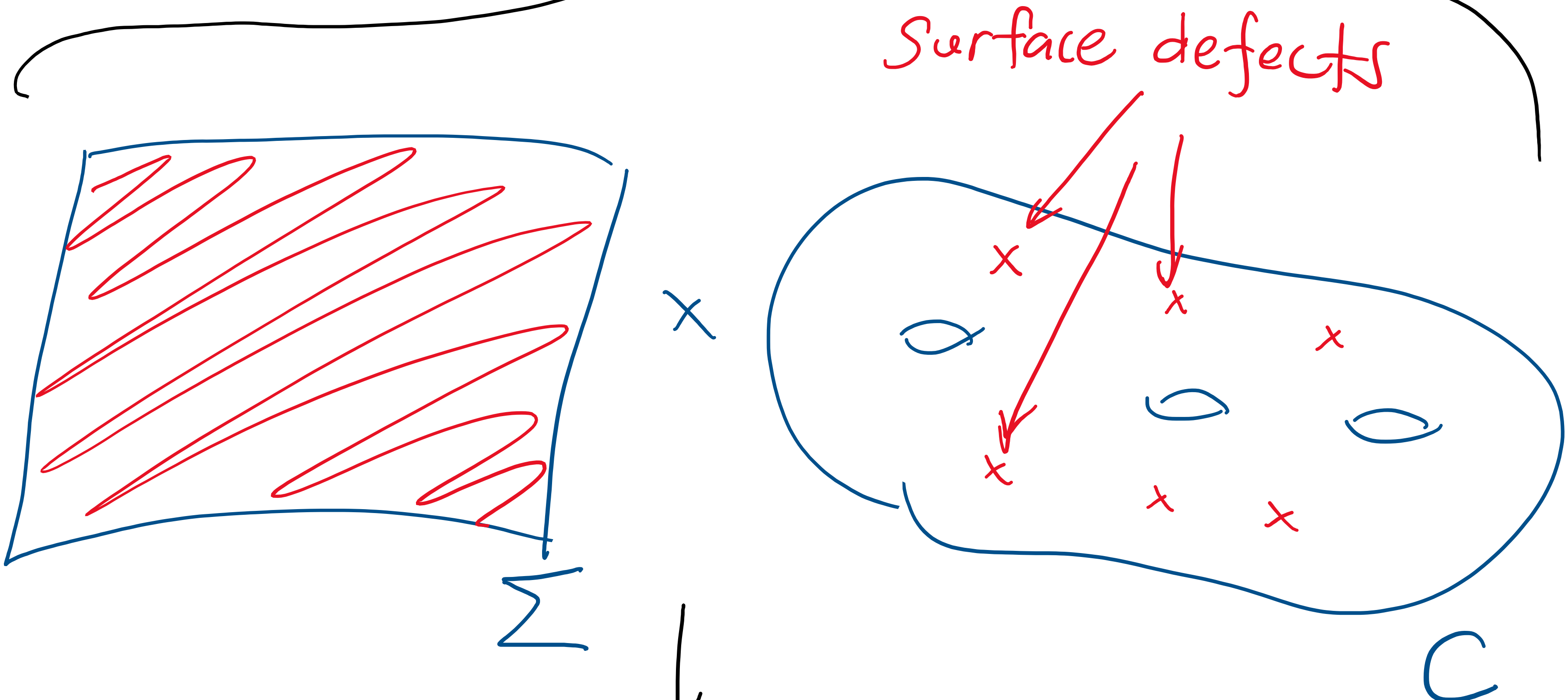
Summary

4d Chern-Simons



integrable lattice model

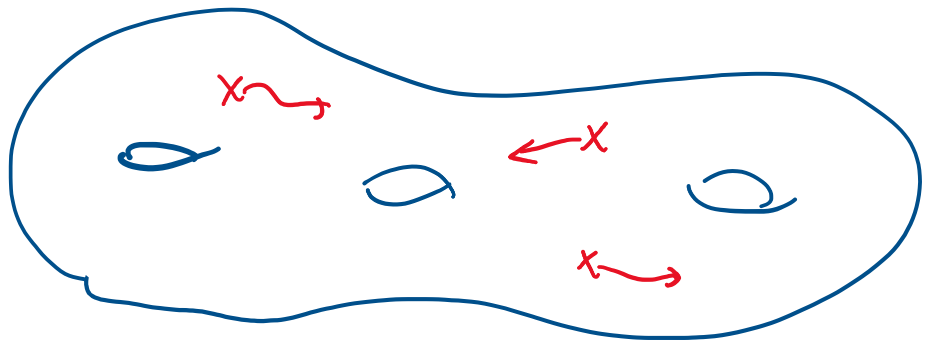
4d Chern-Simons



(many new) integrable field theories

Many question for quantum case

e.g. Renormalization-group flow



[cf. "confinement as analytic continuation"
[Yonekura-Y '19]

large N $O(N)$ model on $\mathbb{R} \times S^1_R$

$(R \ll 1 \rightsquigarrow R \gg 1)$]

2D integrable model Σ

4D Chern-Simons $\Sigma \times C$

2D integrable model Σ

4D Chern-Simons $\Sigma \times C$

[Ashwinkumar-Tan-Zhao '18]

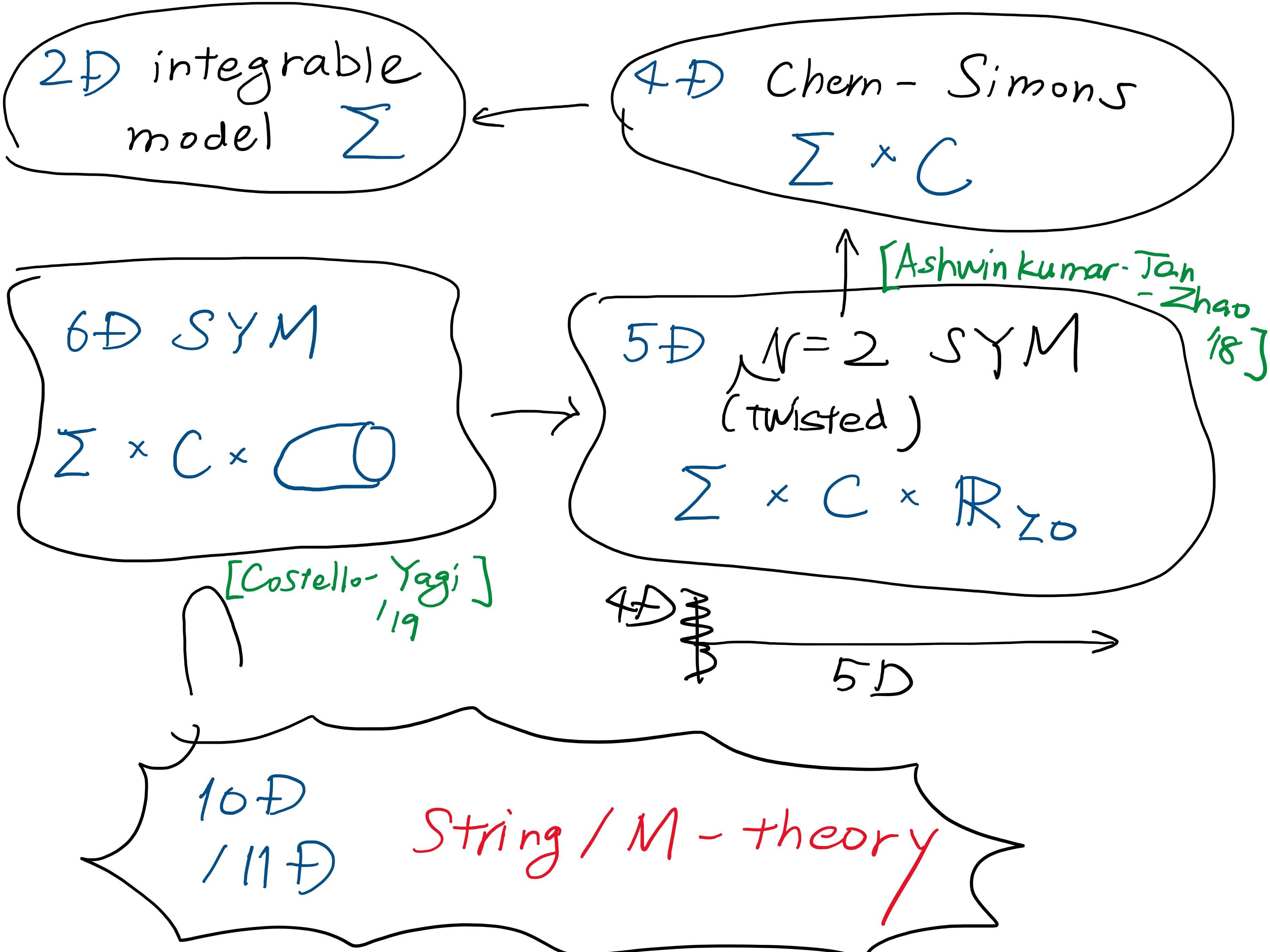
6D SYM $\Sigma \times C \times$ (cylinder)

[Costello-Yagi '19]

5D $\mathcal{N}=2$ SYM (twisted) $\Sigma \times C \times \mathbb{R}_{\geq 0}$

4D

5D


2D integrable model Σ
4D Chern-Simons Σ × C
[Ashwinkumar-Tan-Zhao '18]
6D SYM Σ × C ×
5D N=2 SYM (twisted) Σ × C × R≥0
[Costello-Yagi '19]
4D
5D
10D /11D
String / M-theory