Large $N$ and Small $N$

in Yang-Mills Theory

Masahito Yamazaki (Kavli IPMU)

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Based on collaboration with

Ryuichiro Kitano
(KEK)

Nori kazu Yamada
(KEK)

To Appear
Motivations
Today:

Pure Yang-Mills Theory

with \( \theta \)-angle w/ \( G = SU(N) \)
Pure Yang-Mills Theory
w/ θ-angle w/ G = SU(N)

Vacuum energy $E(\theta, N) =$ ?

Expansion around $\theta = 0$

$E(\theta) - E(0) = \frac{1}{2} \chi \theta^2 (1 + b_2 \theta^2 + b_4 \theta^4 + \cdots)$

(Also motivation from axionic inflation
$[Nomura-Watari-Y, Nomura-Y (17)]$)
Instanton (DIGA) \([\text{t' Hoof}]\)

\[ E(\theta) \sim 1 - \cos \theta \sim b_2 = -\frac{1}{12}, \quad b_4 = \frac{1}{360}, \ldots \]
Instanton (DIGA) \[\text{[t Hooft]}\]

\[E(\theta) \sim 1 - \cos \theta \sim b_2 = -\frac{1}{12}, \quad b_4 = \frac{1}{360}, \ldots\]

Large \(N\) \[\text{[t Hooft, Witten, \ldots]}\]

\[L \sim \frac{1}{N^3} \left( \frac{1}{g^2 N} \text{Tr} F\wedge F + \frac{\theta}{N} \text{Tr} F \wedge F \right)\]

\[\downarrow\]

\[E(\theta) = N^2 f \left( \frac{\theta}{N} \right) = \frac{1}{2} \chi \theta^2 (1 + b_2 \theta^2 + \ldots)\]

\[\downarrow\]

\[\chi = \chi^{(\ast)} + O \left( \frac{1}{N^2} \right)\]

\[b_{2n} = \frac{b_{2n}^{(\ast)}}{N^{2n}} + O \left( \frac{1}{N^{2n+2}} \right)\]

\[\text{NOT ~ 2\Pi-periodic}\]
Expectation for $E(\theta)$

- Many branches
- Quadratic near $\Theta = \infty$
- Minimum $V$ discontinuous at $\Theta = \pi$

(CP broken)

[Figure from Nomura - Y (117)]
$\text{Instanton}$

$$Z \sim \int_{p_\infty}^{p_\sim \infty} \frac{dp}{p^4} \ e^{-\frac{8\pi}{g^2(\mu)} + i \theta} \ (\mu p)^{\frac{11N/3}{4\pi}} \ 1\text{-loop running}$$
Instanton

\[ Z \sim \int_{p_\sim 0}^{p_\sim \infty} \frac{dp}{p} \frac{1}{p^4} e^{-\frac{8\pi c}{g^2(\mu)} + i \theta} \left( \mu p \right)^{\frac{11N/3}{1-\text{loop running}}} \]

\[ N \gtrsim N_\text{*}^{1-\text{loop}} = \frac{12}{11} : \text{IR divergence (IR problem)} \]
Instanton

\[ Z \sim \int_{p \sim 0}^{p \sim \infty} \frac{dp}{p} \frac{1}{p^4} e^{-\frac{8\pi}{g^2(M^*)} + i\theta} \left( \frac{\ln N}{3} \right)^{1-\text{loop running}} \]

\begin{align*}
N \geq N_{*}^{1-\text{loop}} &= \frac{12}{11} : \text{IR divergence (IR problem)} \\
N \leq N_{*}^{1-\text{loop}} &= \frac{12}{11} : \text{UV divergence (} p \gtrsim M^{-1} \text{)} \\
\downarrow \quad Z \sim M^{4 - \frac{11N}{3}} \left( 1 - \cos \theta \right) \quad \text{dominates over other contributions}
\end{align*}

\[ \text{This happens for } SU(2)_{EW} \text{ in SM} \sim \Lambda_{\text{Planck}} \]

[Nomura-Watari-Yangilda (09), ..., Ibe-Yangilda-Y (18)]
E = \frac{1}{2} x \theta (1 + \frac{b_1^2}{N^2 \theta^2} + \frac{b_2^2}{N^4 \theta^4} + \cdots)

E = \frac{M}{N^2} (1 - \cos \theta)
[cf. $\mathbb{Z}_N$-CP mixed anomaly GKKKS ('17)]

$\theta$

CP broken

gapped ($\mathbb{Z}_N$ unbroken)

$\pi$

CP unbroken

gapless ($\mathbb{Z}_N$ broken)

$E = \frac{1}{2} \times \theta^2 \left( 1 + \frac{b_2^{(1)}}{N^2} \theta^2 + \frac{b_4^{(1)}}{N^4} \theta^4 + \cdots \right)$

$E = \mathcal{M}^{\#} (1 - \cos \theta)$

$0$

Large $N$

"classical"

$\frac{1}{N}$

Small $N$

"quantum"

$N_{\text{cr}}^+$
4d $SU(N)$ YM

CP broken

CP unbroken

gapped ($\mathbb{Z}_N$ unbroken)
gapless ($\mathbb{Z}_N$ broken)

2d $\mathbb{CP}^{N-1}$ model

CP broken
gapped

CP unbroken
gapless

$\frac{1}{N}$

$N = 4$, $N = 3$, $N_{\text{crit}}$, $N = 2$

[Haldane, ...]
4d $SU(2)$ YM + $\Theta$-angle

$E(\Theta) - E(0) = \frac{1}{2} \chi \Theta^2 \left( 1 + b_2 \Theta^2 + b_4 \Theta^4 + \ldots \right)$

Compute

Is $N=2$ large (large $N$) or small (inst.)?

Is CP preserved/broken $\Theta = \pi$?

Gapped / gapless
Lattice

[many relevant references
see the forthcoming paper for refs]
In this work we explore the $\theta$ dependence of the vacuum energy of the 4d SU(2) pure Yang-Mills gauge theory. In sec. II, we perform lattices numerical calculations to determine the first two coefficients in the $\theta$ expansion of the vacuum energy. The response of topological excitations to the smearing procedure is investigated in detail in order to efficiently extract physical information from lattice configurations. The coefficients determined at $N = 2$ are compared to those previously obtained for $N \geq 3$ to see how the results for $N = 2$ can be seen as a natural extrapolation of those for $N \geq 3$. In sec. III, we revisit CP$_{N-1}$ model. After discussing characteristic features specific to CP$_1$, a plausible argument about the origin of the features is given. By applying the argument found in 2d CP$_{N-1}$ model to 4d SU($N$) theory, we conclude that SU(2) Yang-Mills theory at $\theta = \pi$ is gapped with spontaneous broken CP symmetry. The argument is made confident through a test using available numerical data.

II. LATTICE SIMULATIONS

The vacuum energy can be expanded around $\theta = 0$ as

$$E(\theta) - E(0) = \chi_2 \theta^2 (1 + b_2^2 \theta^2 + b_4^4 \theta^4 + \cdots),$$

where $\chi$ is the topological susceptibility, and $b_i$ ($i = 1, 2, 3, \cdots$) are dimensionless coefficients describing the deviation of topological charge distribution from the Gaussian. These quantities can be determined on the lattice from configurations generated at $\theta = 0$ as

$$\chi = \frac{\langle Q^2 \rangle_{\theta=0}}{V},$$

$$b_2 = -\frac{\langle Q^4 \rangle_{\theta=0} - 3 \langle Q^2 \rangle_{\theta=0}^2}{12 \langle Q^2 \rangle_{\theta=0}},$$

$$b_4 = \frac{\langle Q^6 \rangle_{\theta=0} - 15 \langle Q^2 \rangle_{\theta=0} \langle Q^4 \rangle_{\theta=0} + 30 \langle Q^2 \rangle_{\theta=0}^3}{360 \langle Q^2 \rangle_{\theta=0}^3}.$$

* conceptually "simple"

generate gauge conf. at $\theta = 0$ < no sign problem

measure top. charge $Q$

\[
\chi = \frac{\langle Q^2 \rangle_{\theta=0}}{V},
\]

\[
b_2 = -\frac{\langle Q^4 \rangle_{\theta=0} - 3 \langle Q^2 \rangle_{\theta=0}^2}{12 \langle Q^2 \rangle_{\theta=0}},
\]

\[
b_4 = \frac{\langle Q^6 \rangle_{\theta=0} - 15 \langle Q^2 \rangle_{\theta=0} \langle Q^4 \rangle_{\theta=0} + 30 \langle Q^2 \rangle_{\theta=0}^3}{360 \langle Q^2 \rangle_{\theta=0}^3},
\]

* in practice several subtleties / difficulties
Need statistics \[ \leftarrow \text{ deviation from Gaussian } \]

\[
e^{-\frac{1}{2} x^2} \sim z(\theta) = \sum_Q z_Q e^{i Q \theta} \]

\[
\downarrow \quad -Q^{1/2} \chi \quad z_Q \sim \mathcal{C}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\beta & N_S & N_{T_c} & (a T_c)^2 & L \sigma_{\text{str}}^{1/2} & \text{statistics} \\
\hline
1.750 & 16 & 4.65 & 0.0462 & 4.9 & 80,100 \\
1.850 & 16 & 6.50 & 0.0237 & 3.5 & 71,040 \\
1.975 & 16 & 9.50 & 0.0111 & 2.4 & 30,490 \\
1.975 & 24 & 9.50 & 0.0111 & 3.6 & 131,830 \\
\hline
\end{array}
\]

\[
(\text{symanzik action, HMC, Bridge++ })
\]
Short-Distance Fluctuations

We have fluctuations of size \( \sim 0(a) \)

\( \rightarrow \) removed by "smearing"

[many different methods, see Alexandrou et al. (17) for comparison]

We use APE smearing (& gradient flow)

[Albanese et al. (87)]

\[
U^{(\text{new})}_\mu = \text{Proj} \left[ (1 - \rho)U^{(\text{old})}_\mu(x) + \rho X_\mu(x) \right],
\]

\[
X_\mu(x) = \sum_{\nu \neq \mu} \left[ U^{(\text{old})}_\nu(x) U^{(\text{old})}_\mu(x + \hat{\nu}) U^{(\text{old})\dagger}_\nu(x + \hat{\mu}) + U^{(\text{old})\dagger}_\nu(x - \hat{\nu}) U^{(\text{old})}_\mu(x - \hat{\nu}) U^{(\text{old})}_\nu(x - \hat{\nu} + \hat{\mu}) \right],
\]
FIG. 7: Histogram of $Q$ for four ensembles at $n_{\text{APE}} = 0, 20, 100$. Figure 8 shows the topological susceptibility in lattice unit, $a^4 \chi (n_{\text{APE}}) = \langle Q^2 \rangle / N_{\text{site}}$, as a function of $n_{\text{APE}}$. A mild decrease is seen for $n_{\text{APE}} \geq 20$ as expected from a negative constant observed in Fig. 5. We determine topological susceptibility at each lattice by extrapolating the smeared data in the second phase to $n_{\text{APE}} \rightarrow 0$ because the "falling" issue is supposed to take place even in the first phase. The data points in $n_{\text{APE}} \in [20, 40]$ are well described by a linear function, $a^4 \chi (n_{\text{APE}}) = a^4 \chi (0) + c_1 n_{\text{APE}}$.

The fit results are tabulated in Tab. II.

Figure 9 shows $n_{\text{APE}}$ dependence of $b_2$. Since $b_2$ is found to be constant for $n_{\text{APE}} \geq 20$, $\beta N S a^4 \chi (0) \times 10^4 c_1 \times 10^7 b_2 (0) \times 10^2$.

<table>
<thead>
<tr>
<th>$n_{\text{APE}} = 0$</th>
<th>$n_{\text{APE}} = 20$</th>
<th>$n_{\text{APE}} = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.750 \pm 0.160$</td>
<td>$1.850 \pm 0.101$</td>
<td>$1.975 \pm 0.269$</td>
</tr>
<tr>
<td>$3.08(2)$</td>
<td>$1.50(1)$</td>
<td>$0.22(2)$</td>
</tr>
<tr>
<td>$3.49(3)$</td>
<td>$5.80(3)$</td>
<td>$4.22(4)$</td>
</tr>
</tbody>
</table>

TABLE II: Fit results.
Smearing

\[
\begin{cases}
\text{successfully removes} \\
\text{short-distance fluctuations}
\end{cases}
\]

but also changes physical instanton configurations

"falling through lattice"

[ Bilson-Thompson, Leinweber, Williams, Dunne ('03)]
occurs also in the first phase, but it is hidden by changes originating from other reasons. Instanton and anti-instantons will "fall" at an equal rate. In configurations with $Q > 0$, more instantons exist than anti-instantons and vice versa. Then, it is expected that the "decreasing" would happen more frequently than "increasing" in the second phase. To see if this is the case, we calculate the ensemble average of 

$$
\Delta Q(n_{APE}) = \begin{cases} 
Q(n_{APE} + 1) - Q(n_{APE}) & \text{for } Q(n_{APE}) > 0 \\
Q(n_{APE}) - Q(n_{APE} + 1) & \text{for } Q(n_{APE}) < 0
\end{cases}
$$

(16)

The sign of $\Delta Q(n_{APE})$ tells us which of "increase" or "decrease" happens when going from $n_{APE}$ to $n_{APE} + 1$. Figure 5 shows the $n_{APE}$ dependence of $|\langle \Delta Q(n_{APE}) \rangle|$, where the symbols are filled when its original value is negative. The results from four ensembles show exponential fall with approximately a common exponent for $n_{APE} < \sim 10$, while they take almost constant negative values for $n_{APE} > \sim 20$. The result for $\beta = 1.975$ and $N_S = 16$ (triangle-up) shows slightly different behavior probably because of small physical volume. At any rate, this plot clearly shows that the boundary separating the two phases is located $n_{APE} \sim 20$.

In the following analysis, we only deal with the data for $n_{APE} \geq 20$, where the short distance fluctuations are gone.
\[ P(n_{\text{APE}}) = \frac{1}{N_{\text{site}}} \left( \sum_x q(x, n_{\text{APE}})^2 \right)^2 \]

\[ Q(n_{\text{APE}}) = \frac{1}{N_{\text{site}}} \left( \sum_x q(x, n_{\text{APE}})^4 \right) \]

\[ \max P = 1 \quad \text{(uniform)} \]

\[ \min P = \frac{1}{N_{\text{site}}} \quad \text{(G function)} \]

![Graph showing the smearing history of the topological charge density for one particular configuration at different times.](image)
FIG. 4: Distribution of topological charge projected onto $z$-$t$ plane at $n_{APE} = 50, 60, 100, 200, 450, \text{and} 470$. Between $n_{APE} = 100$ and 200, positive peaks seem to be a bit smeared but not suddenly disappear. We guess that a pair annihilation or something complicated happens in the latter case. From these observations, we conclude that the changes of $Q$ occurring in the second phase are dominated by the "falling" of instantons or anti-instantons. The "falling" probably
Result
Before closing this subsection, let us add one comment. In Ref. [42], the shape of topological objects in SU(3) gauge theory is examined, and the low dimensional long range structure rather than local lump is discovered. Note that the analysis presented above does not indicate anything about the shape because the smearing changes it. Clearly, it is interesting to perform the similar study in the SU(2) case because the analysis performed in Refs. [43, 44] suggests that the structure could be more local for SU(2) than for SU(3).

D. results

Figure 6 shows the Monte Carlo history of $Q$ over thousand configurations in four ensembles obtained at $n_{\text{APE}}=800$. It is seen that the fluctuation of $Q$ is frequent enough, and the amplitudes depend on $\beta$ and $N_{\text{site}}$. In the following analysis, all the measurements are binned with the bin size of 100 configurations, and a single elimination jackknife method is used to estimate uncertainties.

Figure 7 shows the histogram of $Q$ for four ensembles at $n_{\text{APE}}=0, 20, 100$. Approximate Gaussian shape is seen in all ensembles.
FIG. 7: Histogram of $Q$ for four ensembles at $n_{\text{APE}} = 0, 20, 100$.

Figure 8 shows the topological susceptibility in lattice unit, $a_4 \chi_n(\text{APE}) = \langle Q^2 \rangle / N_{\text{site}}$, as a function of $n_{\text{APE}}$. A mild decrease is seen for $n_{\text{APE}} \geq 20$ as expected from a negative constant observed in Fig. 5. We determine topological susceptibility at each lattice by extrapolating the smeared data in the second phase to $n_{\text{APE}} \rightarrow 0$ because the "falling" issues are supposed to take place even in the first phase. The data points in $n_{\text{APE}} \in [20, 40]$ are well described by a linear function,$a_4 \chi_n(\text{APE}) = a_4 \chi_n(0) + c_1 n_{\text{APE}}$. (17)

The fit results are tabulated in Tab. II.

Table II: Fit results.

<table>
<thead>
<tr>
<th>$n_{\text{APE}}$</th>
<th>$\beta N^2$</th>
<th>$a_4 \chi_n(0) \times 10^4$</th>
<th>$c_1 \times 10^7$</th>
<th>$b_2(0) \times 10^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.750 16</td>
<td>1.08(2)</td>
<td>-9.4(3)</td>
<td>-5(5)</td>
</tr>
<tr>
<td>20</td>
<td>1.850 16</td>
<td>1.10(1)</td>
<td>-10.8(1)</td>
<td>-6(3)</td>
</tr>
<tr>
<td>100</td>
<td>1.975 24</td>
<td>1.269(8)</td>
<td>-0.22(2)</td>
<td>-4(2)</td>
</tr>
</tbody>
</table>
Next we discuss the continuum limit. Figure 10 shows the extrapolation of $a^4 \chi$ and $b^2$ to the continuum. The limit for both quantities is examined by applying two functional forms.

1. constant excluding the coarsest lattice

We perform the constant fit to extract $b^2$ at $n_{APE} = 0$. The results are shown in Table. II. The values of $b^2$ obtained at $\beta = 1.975$ with two lattice volumes turns out to be consistent with each other due to the large statistical uncertainty, while $1.8 \sigma$ difference is observed for $\chi$. In Ref. [16], these quantities are calculated with several different volumes for $SU(N)$ with $N = 3, 4, 6$ down to $L_{\sigma} \sim 2.7$, and no finite volume effect is observed. Our lattice with $\beta = 1.975$ and $N_S = 16$ correspond to $L_{\sigma} \sim 2.4$ (see Table I), which is smaller than but close to 2.7 and hence finite volume effects, if any, should not be significant. Thus, $1.8 \sigma$ difference observed at $\beta = 1.975$ is considered as a statistical fluctuation, and we include both results in the following analysis.
These two are chosen because they turn out to yield the smallest and largest value for $\chi/T^4$ among other reasonable choices. In either quantities, the constant fit is taken as the central value, and the difference between two methods is taken as the systematic uncertainty in the final result.

The continuum limit of $\chi/T^4$ turns out to strongly depend on the functional form, and as a result, the error is dominated by the systematic uncertainty. On the other hand, thanks to the constant behavior for $b_2$, the inclusion of the linear term into the functional form does not alter the limit for the constant fit by much. The final results thus obtained are

$$\chi/T^4 = 0.200(39), \quad \chi_1/T^4 = 0.674(31), \quad b_2 = -0.049(20), \quad (18)$$

where the errors are summed in quadrature.

In Refs. [14–16], the topological susceptibility $\chi$ is calculated in SU($N$) gauge theory with several values of $N$ to study the large $N$ behavior. In Refs. [14, 39, 45–47], $\chi$ is estimated for $N = 15$.
\[ \frac{\chi}{T_c^4} = 0.200(39), \quad \frac{\chi^{1/4}}{T_c} = 0.674(31), \quad b_2 = -0.049(20), \]

seems to be the first determination of \( b_2 \)

[cf. Bonanno, Bonati, D'Elia (18) \( b_4 = 6(2) \cdot 10^{-4} \)]
FIG. 11: The $N$ dependence of $\chi/\sigma^2$ and $b^2$. Each data point is slightly shifted horizontally to make it easier to see. The horizontal dashed line in the $b^2$ plot represents the dilute instanton gas approximation (DIGA).
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\[ \left[ \text{cf. Lüscher (182) for 2d CP}^{N-1} \text{-model} \right] \left( \frac{\chi}{\sigma} \right)_{N=\text{N$_{\text{crit}}$}} \xrightarrow{N \to \infty} \frac{\chi}{\sigma} = \left( \frac{\chi}{\sigma} \right)_{N=\infty} \frac{N^2}{N^2 - N_{\text{crit}}^2} \]
FIG. 11: The $N$ dependence of $\chi/\sigma^2_{\text{str}}$ and $b_2$. Each data point is slightly shifted horizontally to make it easier to see. The horizontal dashed line in the $b_2$ plot represents the dilute instanton gas approximation (DIGA).

\[ b_2 = -\frac{L}{12} \] (instanton)
FIG. 11: The $N$ dependence of $\chi/\sigma^2$ str and $b_2$. Each data point is slightly shifted horizontally to make it easier to see. The horizontal dashed line in the $b_2$ plot represents the dilute instanton gas approximation (DIGA).

$\chi/\sigma^2$ str

$1/N^2$

$-0.1$

$-0.05$

$0$

$0.1$

$0.2$

$0.3$

$\text{DIGA}$

Del Debbio

Bonati

this work

ph fit

$b_2 = -0.083$

$\text{instanton}$

$b_2 (N_{\text{crit}})$

$-0.087 (5)$
Summary

* 4d SU(2) YM: still "large N"

spontaneous CP breaking, mass gap
\[ \Theta = \pi \]

\[ \frac{\chi^{1/4}}{T_c} = 0.674(31), \quad b_2 = -0.049(20) \]

[Quantitatively different from 2d CP^{N-1} model]

* Transition to "small N" happens at

\[ N_{crit} \approx 1.52 \]