Confinement as Analytic Continuation Beyond Infinity

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Holy Grail :

Analytic Proof of Confinement/Mass Gap in 4D YM

novel ideas ?

Today :

"analytic continuation beyond infinity"

Today : 2D O(N) model in large N limit

($\mathbb{C}P^{N-1}$)

— asymptotic free, mass gap

— "toy model" for 4D YM

cf. 2D $\mathbb{C}P^{N-1}$ model $\leftarrow$ 4D pure YM on T^2

(Yonekura - Y '17)

on ① $\mathbb{R} \times S^1_R$ weak coupling when $R \ll \Lambda$

(many recent studies, Bruckmann, Dunne, Unsal, Fujimori, Kamata Misumi, Nitta, Sakai, Ishikawa Morikawa Nakayama Shibata Suzuki Takaura ...)

O(N) model

- $$\mathcal{L} = \frac{1}{2g^2}\Big[\partial_\mu \vec{n}\,\partial^\mu \vec{n} + \underbrace{\alpha}_{\text{Lagrange multiplier}}(\vec{n}^2 - 1)\Big]$$

$$\sum_{i=1}^{N} n_i^2 = \vec{n}^2 = 1$$

- gap $$\Delta^2 = \langle \alpha \rangle = -\langle \partial_\mu \vec{n}\,\partial_\mu \vec{n} \rangle$$

(e.o.m. $-\partial^2 \vec{n} + \alpha \vec{n} = 0,\ \vec{n}^2 = 1$

$$\langle \alpha \rangle = \langle \vec{n}\cdot \partial^\mu \partial_\mu \vec{n} \rangle = -\langle \partial_\mu \vec{n}\,\partial^\mu \vec{n} \rangle$$)

by integrating out $\vec{n}$

$$\mathcal{L}_{eff}(\alpha) = \frac{N}{2} \underbrace{\mathrm{Tr}\log(-\partial^2+\alpha)} - \frac{1}{2g^2}\alpha$$

① $\mathbb{R}\times S^1$

$$\| $$

$$\frac{1}{2\pi R}\sum_{m\in\mathbb{Z}}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\log\left(k^2+\frac{m^2}{R^2}+\alpha\right)$$

$$\|$$

$$-\frac{2\Gamma\left(\frac{d-1}{2}\right)}{(4\pi)^{\frac{d+1}{2}}R^{d-1}}\sum_{m\in\mathbb{Z}}\left(\frac{m^2}{R^2}+\alpha\right)^{\frac{d-1}{2}}$$

't Hooft coupling

$$\lambda = \frac{Ng^2}{4\pi}$$

$$\frac{\partial\mathcal{L}_{eff}(\alpha)}{\partial\alpha}=0 \rightsquigarrow \underbrace{\sum_{m\in\mathbb{Z}}\frac{1}{\left(m^2+R^2\Delta^2\right)^{\frac{d-3}{2}}}}_{\| \atop F(R\Delta)+2\zeta(3-d)} = \frac{(4\pi)^{\frac{d-1}{2}}R^{d-2}}{2\Gamma\left(\frac{3-d}{2}\right)}\frac{1}{\lambda}$$

gap

$\alpha = \Delta^2$

$$\underbrace{F(R\Delta)}_{\|}$$

$$\frac{1}{(R\Delta)^{3-d}}+2\sum_{m=1}^{\infty}\left[\frac{1}{\left(m^2+(R\Delta)^2\right)^{\frac{3-d}{2}}}-\frac{1}{m^{3-d}}\right]$$

Now consider renormalized coupling λ_μ ← ②!

$$\frac{1}{\lambda} = \mu^{d-2}\left(\frac{1}{\lambda_\mu} + \underbrace{\frac{2}{2-d}}_{\text{pole}} + C\right)$$

s.t.

$$\frac{(4\pi)^{\frac{d-1}{2}} R^{d-2}}{2\Gamma\left(\frac{3-d}{2}\right)} \frac{1}{\lambda} - 2\underbrace{\zeta(3-d)}_{\text{pole}} \xrightarrow{d\to 2} \frac{1}{\lambda_\mu} - 2\log(R\mu) = \frac{1}{\lambda_{R^{-1}}}$$

therefore

$$\boxed{(\lambda_{R^{-1}})^{-1} = F(R\Delta) = \frac{1}{R\Delta} + 2\sum_{m=1}^{\infty}\left[\frac{1}{\sqrt{m^2+(R\Delta)^2}} - \frac{1}{m}\right]}$$

We find

$$\Delta \sim \frac{\lambda_{R^{-1}}}{R} \qquad \text{when} \quad R: \text{small}$$

So for $\Delta \simeq \frac{\lambda_{R^{-1}}}{R} + \cdots$ when R small

analytic continuation to large R ?

RG eqn $\mu \frac{\partial}{\partial \mu}\left(\frac{1}{\lambda_\mu}\right) = 2$

R small		R large
$\frac{1}{\lambda} > 0$	$\longrightarrow$	$\frac{1}{\lambda} < 0$
$\Delta \sim \frac{\lambda_{R^{-1}}}{R}$		$\Delta \sim e^{-\frac{1}{2\lambda_{R^{-1}}}} / R$

goes through

$\lambda = \infty$: infinite coupling

How is this possible?

$$(\lambda_{R^{-1}})^{-1} = F(R\Delta) = \frac{1}{R\Delta} + 2\sum_{m=1}^{\infty}\left[\frac{1}{\underbrace{\sqrt{m^2+R^2\Delta^2}}} - \frac{1}{m}\right]$$

pole at $R\Delta = \pm m i$

R small

R large

$$\frac{1}{R\Delta}$$

$$-2\left(\log R\Delta - \log 2 + \gamma\right) + \mathcal{O}\left(\frac{1}{R\Delta}\right)$$

$$\Delta \sim \frac{\lambda_{R^{-1}}}{R}$$

$$\Delta \sim \frac{1}{R} e^{-\frac{1}{2\lambda_{R^{-1}}}} = \Lambda$$

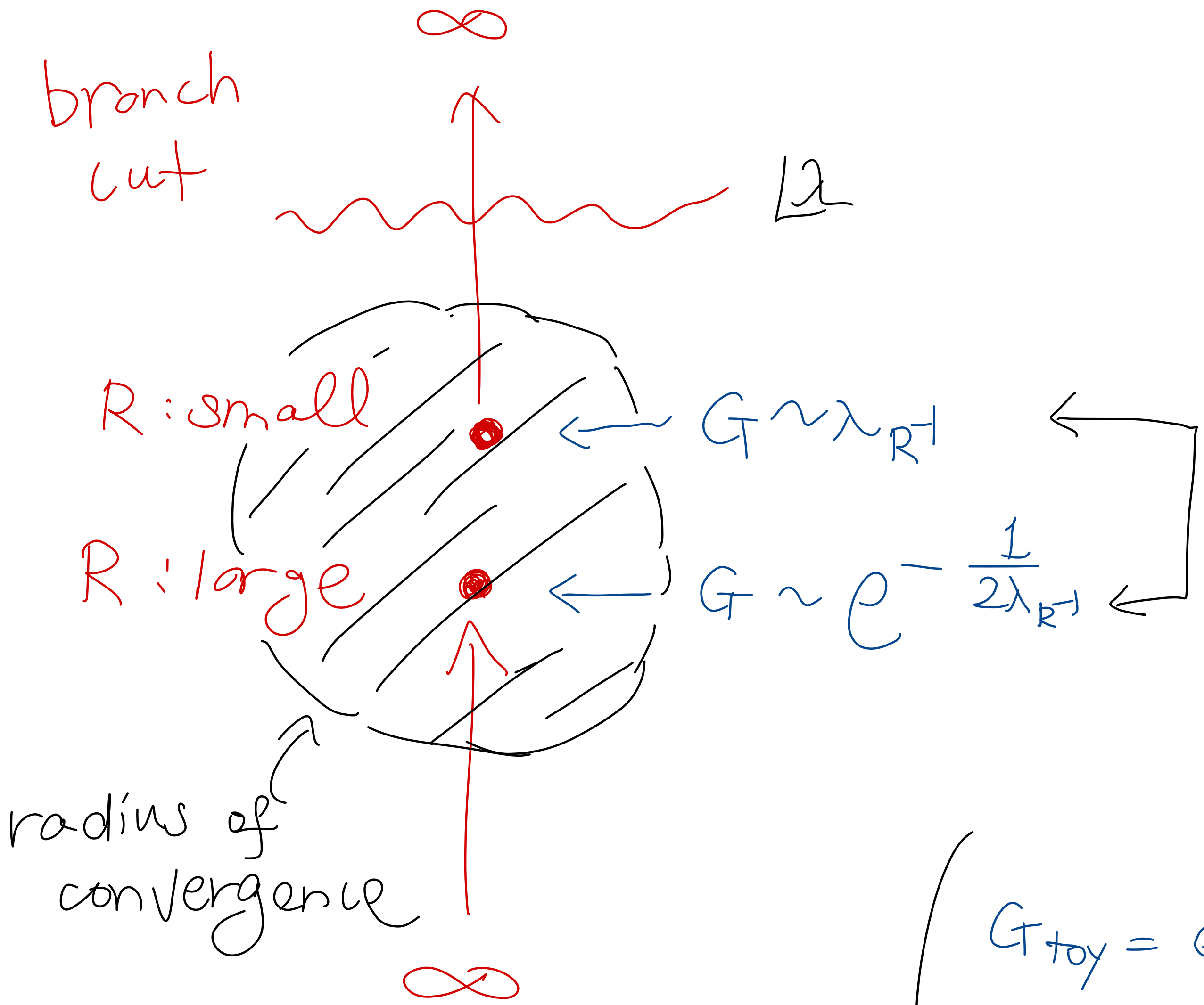
dynamical scale

$$R\Delta = G\left((\lambda_{R^{-1}})^{-1}\right)$$

inverse of F : finite radius of convergence

branch cuts in complex plane

$$R\Delta = G(\lambda_{R^{-1}})$$



different sheets after analytic continuation

③ "analytic continuation beyond infinity"

$$\left(G_{toy} = \exp\left(\frac{\sqrt{1+8\lambda_{R^{-1}}^{2}} - 1}{4\lambda_{R^{-1}}} \right) - 1 \right)$$

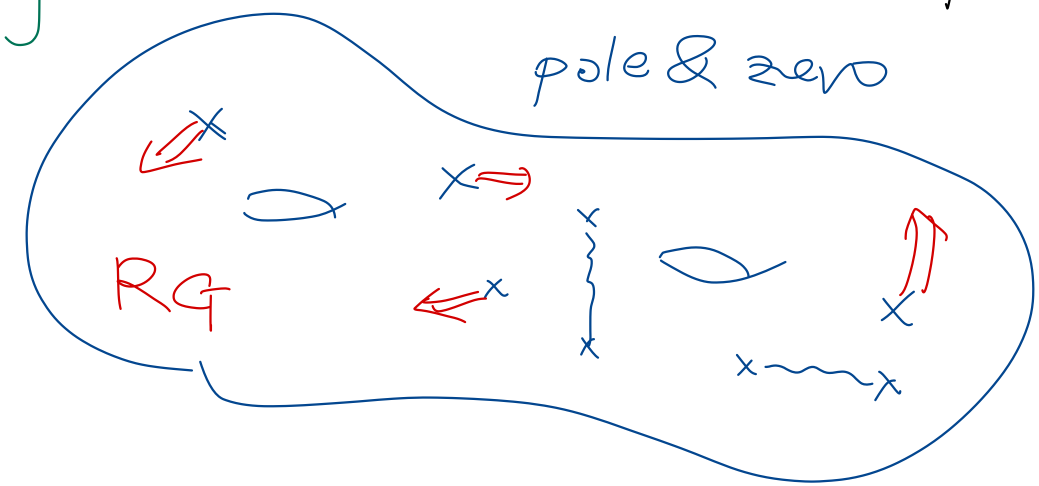
choice of branch cut

More generally, one should regard coupling constant as a point in some non-trivial manifold $\mathcal{M}$

cf. 2D integrable field theories

$\leftsquigarrow$ 4D Chern-Simons on C spectral curve

[Costello-Y '19]



(choice of scheme $\doteqdot$ choice of coordinate)

Summarizing

2D $O(N)$ model in large N limit

- on $\mathbb{R} \times S^1_R$: small R $\rightsquigarrow$ large R
  (weak coupling)

- RG running
  $\lambda = (Ng^2)/4\pi \rightsquigarrow \lambda_{R^{-1}}$ running coupling

- analytic continuation in $\lambda_{R^{-1}}$
  beyond infinite coupling

Conjecture: This applies to 4D Yang-Mills implications for the renormalon problem

traditional argument

operator w/ dim = $D_{\mathcal{O}}$

OPE $\langle J(x) J(0) \rangle = C_0(x) + C_1(x) \langle \mathcal{O} \rangle + \cdots$

$$\Lambda^{D_{\mathcal{O}}} \sim \mu^{D_{\mathcal{O}}} \exp\Big(- \underbrace{\frac{D_{\mathcal{O}}\, 8\pi^2}{b_0\, g^2}}\Big)$$

b_0: 1-loop β-function coefficient

$$e^{-\underbrace{\frac{8\pi^2}{g^2}}_{S_{\text{inst.}}}} \ll e^{-\frac{D_{\mathcal{O}}\, 8\pi^2}{b_0 N g^2}}$$

$S_{\text{something}}$?

Many papers: there should exist classical saddle point config w/ $S \sim \frac{D_{\mathcal{O}}}{\beta_0 g^2}$

[Dunne, Unsal, Argyres, Fujimori, Kamata, Misumi, Sakai, Chermon, Dorigoni, ...]

at least for controlled setup

(e.g. 2D $O(N)$ model on S^1, 4D YM on S^1 ...)

$$\langle O \rangle = \int \mathcal{D}A \, O \, e^{-S[A]/g^2}$$

$$= \underbrace{\sum}_{\text{classical saddle}} \underbrace{e^{-S[A_c]/g^2}}$$ (perturbative power series in g)

classical saddle point, even for $S \sim \frac{8\pi^2}{b_0 N g^2} \ll S_{inst.}$

"renormalon"

However, our analysis showes that

- classical saddle w/ $S \sim \frac{1}{Ng^2}$ ~~not necessary~~

[cf. Anber - Sulejmonpasić '14
Ishikawa - Morikawa - (Nakayama) - Shibata - Takaura - Suzuki '19]

- Such terms can arise from
analytic continuation into different sheets

※ consistent w/ factorial growth

$$\beta \sim \sum n! \left(\frac{\beta_0 \lambda}{D}\right)^n$$

& poles in the Borel plane