Is $N=2$ Large?

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Problem
Today: Pure Yang-Mills Theory w/ θ-angle w/ G = SU(N)

Vacuum energy \( E(\theta, N) = ? \)

Expansion around \( \theta = 0 \)

\[
E(\theta) - E(0) = \frac{1}{2} \chi \theta^2 (1 + b_2 \theta^2 + b_4 \theta^4 + \cdots )
\]

\( \chi \) is the topological susceptibility (dimensionless)

(Also motivation from axionic inflation

\[ \text{[Nomura-Watari-Y, Nomura-Y (17)] } \] )
\[ E(\theta) \sim 1 - \cos \theta \Rightarrow b_2 = -\frac{1}{12}, \quad b_4 = \frac{1}{360}, \quad \ldots \]
\( \text{Instanton (DIGA) \ [t \ Hooft]} \)

\[ E(\theta) \sim 1 - \cos \theta \sim b_2 = -\frac{1}{12}, \quad b_4 = \frac{1}{360}, \ldots \]

\( \text{Large } N \ [t \ Hooft, \ Witten, \ldots] \)

\[ L \sim \frac{1}{N} \left( \frac{1}{g^2 N} \text{Tr} F \wedge F + \frac{\theta}{N} \text{Tr} F \wedge F \right) \]

\[ E(\theta) = N^2 f \left( \frac{\theta}{N} \right) = \frac{1}{2} \chi \theta^2 \left( 1 + b_2 \theta^2 + \ldots \right) \]

\[ \chi = \chi^{(\infty)} + O \left( \frac{1}{N^2} \right) \]

\[ b_{2n} = \frac{b_{2n}^{(\infty)}}{N^{2n}} + O \left( \frac{1}{N^{2n+2}} \right) \quad \text{NOT } 2\pi\text{-periodic} \]
Instanton

\[ Z \sim \int_{p_\sim 0}^{p_\sim \infty} \frac{dp}{p} \frac{1}{p^4} e^{-\frac{8\pi}{g^2(m)}} e^{i\theta} \left( \frac{\mu p}{\Lambda_{\text{QCD}}} \right)^{11N/3} \text{1-loop running} \]
Instanton

\[ Z \sim \int_{p_\sim 0}^{p_\sim \infty} \frac{dp}{p} \frac{1}{p^4} e^{-\frac{8\pi}{g^2(m)} + i\theta} \left( \mu p \right)^{11N/3} \] 1-loop running

\[ N > N_{\text{IR}}^{1-\text{loop}} = \frac{12}{11} : \text{IR divergence (IR problem)} \]
\[ \text{Instanton} \]

\[ Z \sim \int_{p \approx 0}^{p \approx \infty} \frac{dp}{p} \frac{1}{p^4} e^{-\frac{8\pi}{g^2(\mu)} + i\theta} \left( \frac{4\pi}{\mu} \right)^{11N/3} \]

\[ \Rightarrow N \gtrsim N_*^{1-\text{loop}} = \frac{12}{11} : \text{IR divergence (IR problem)} \]

\[ N \lesssim N_*^{1-\text{loop}} = \frac{12}{11} : \text{UV divergence} (p \gtrsim M^{-1}) \]

\[ \rightarrow Z \sim M^{4 - \frac{11N}{3}} \left( \frac{4\pi}{\mu} \right)^{\frac{11N}{3}} (1 - \cos\theta) \quad \text{dominates over other contributions} \]

[This happens for SU(2)_{EW} in SM \rightarrow \Lambda_c.o?] 

[Nomura-Watari-Yangida (00), \ldots, Ibe-Yangida-Y (18)]
\[ E = \frac{1}{2} N \theta^2 (1 + \frac{\theta^2}{N^2} + \frac{\theta^4}{N^4} + \cdots) \]

\[ E = N \hbar \sqrt{1 - \cos \theta} \]
$E = \frac{1}{2} \times \Theta \left(1 + \frac{B_2(\Theta)}{N^2} \Theta^2 + \frac{B_4(\Theta)}{N^4} \Theta^4 + \cdots \right)$

Large $N$

"classical"

$E \propto N^{\frac{N}{21}} \left(1 - \cos \Theta \right)$

Small $N$

"quantum"

$N_{\text{crit}}$

$115/12/11$

$\Theta$

$\pi$

$0$

$\frac{1}{N}$

[cf. $Z_N$-CP mixed anomaly Gaiotto Kapustin Komargodski Seiberg]
4d $\text{su}(N)$ YM

\[ \Theta \]

\[ \pi \]

CP broken

gapped ($\mathbb{Z}_N$ unbroken)

CP unbroken

gapless ($\mathbb{Z}_N$ broken)

\[ N_{cp} \]

2d $\text{CP}^{N-1}$ model

CP broken

gapped

CP unbroken

gapless

\[ \frac{1}{N} \]

\[ N=4 \quad N=3 \quad N_{cp} \quad N=2 \]

[Haldane, ...]
4d $SU(2)$ YM + $\Theta$-angle

$E(\theta) - E(0) = \frac{1}{2} \chi \theta^2 (1 + b_2 \theta^2 + b_4 \theta^4 + \cdots)$

\[\text{Compute}\]

Is $N=2$ large (large $N$) or small (inst.)?

Is CP preserved/broken $\Theta$ \theta = $\pi$?

gapped / gapless
Lattice
In this work we explore the $\theta$ dependence of the vacuum energy of the 4d SU(2) pure Yang-Mills gauge theory. In sec. II, we perform lattices numerical calculations to determine the first two coefficients in the $\theta$ expansion of the vacuum energy. The response of topological excitations to the smearing procedure is investigated in detail in order to efficiently extract physical information from lattice configurations. The coefficients determined at $N=2$ are compared to those previously obtained for $N\geq 3$ so that the latter can be seen as a natural extrapolation of those for $N=2$.

In sec. III, we revisit CP$^{N-1}$ model. After discussing characteristic features specific to CP$^1$, a plausible argument about the origin of the features is given. By applying the argument found in 2d CP$^N$ model to 4d SU($N$) theory, we conclude that SU(2) Yang-Mills theory at $\theta=\pi$ is gapped with spontaneous broken CP symmetry. The argument is made confident through a test using available numerical data.

II. LATTICE SIMULATIONS

The vacuum energy can be expanded around $\theta=0$ as

$$E(\theta) - E(0) = \chi_2 \theta^2 (1 + b_2^2 \theta^2 + b_2^4 \theta^4 + \cdots),$$

where $\chi$ is the topological susceptibility, and $b_2^i (i=1, 2, 3, \cdots)$ are dimensionless coefficients describing the deviation of topological charge distribution from the Gaussian. These quantities can be determined on the lattice from configurations generated at $\theta=0$ as

$$\chi = \langle Q^2 \rangle_{\theta=0} V,$$

$$b_2 = -\frac{\langle Q^4 \rangle_{\theta=0} - 3 \langle Q^2 \rangle_{\theta=0}^2}{12 \langle Q^2 \rangle_{\theta=0}},$$

$$b_4 = \frac{\langle Q^6 \rangle_{\theta=0} - 15 \langle Q^2 \rangle_{\theta=0} \langle Q^4 \rangle_{\theta=0} + 30 \langle Q^2 \rangle_{\theta=0}^3}{360 \langle Q^2 \rangle_{\theta=0}},$$

* conceptually "simple"

generate gauge conf. at $\theta=0$ < no sign problem

measure top. charge $Q$

* in practice several subtleties / difficulties
Need statistics \leftarrow \text{deviation from Gaussian}

\[
e^{-\frac{1}{2}x^2} \sim z(\theta) = \sum_{\theta} z_{\theta} e^{iQ\theta} \\
\downarrow \quad -\frac{Q^2}{2x}
\]

\[
z_{\theta} \sim C
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\beta & N_S & N_{Tc} & (aT_c)^2 & L\sigma_{str}^{1/2} \text{ statistics} \\
\hline
1.75 & 16 & 4.65 & 0.0462 & 4.9 & 80,100 \\
1.85 & 16 & 6.50 & 0.0237 & 3.5 & 71,040 \\
1.975 & 16 & 9.50 & 0.0111 & 2.4 & 30,490 \\
1.975 & 24 & 9.50 & 0.0111 & 3.6 & 131,830 \\
\hline
\end{array}
\]

(symanzik action, HMC, Bridge++ )
Short-Distance Fluctuations

We have fluctuations of size $\sim O(a)$ removed by "smearing"

[many different methods, see Alexandrou et al. (1977) for comparison]

We use APE smearing (& gradient flow)

[Albanese et al. (1977)]

\[ U^{(\text{new})}_\mu = \text{Proj} \left[ (1 - \rho) U^{(\text{old})}_\mu (x) + \rho X_\mu (x) \right] , \]

\[ X_\mu (x) = \sum_{\nu \neq \mu} \left[ U^{(\text{old})}_\nu (x) U^{(\text{old})}_\mu (x + \hat{\nu}) U^{(\text{old})}_{\nu} (x + \hat{\mu}) \right. \]

\[ + U^{(\text{old})}_{\nu} (x - \hat{\nu}) U^{(\text{old})}_\mu (x - \hat{\nu}) U^{(\text{old})}_{\nu} (x - \hat{\nu} + \hat{\mu}) \] ,
FIG. 7: Histogram of $Q$ for four ensembles at $n_{APE} = 0, 20, 100$. Figure 8 shows the topological susceptibility in lattice unit, $\chi(n_{APE}) = \langle Q^2 \rangle / N_{\text{site}}$, as a function of $n_{APE}$. A mild decrease is seen for $n_{APE} \geq 20$ as expected from a negative constant observed in Fig. 5. We determine topological susceptibility at each lattice by extrapolating the smeared data in the second phase to $n_{APE} \to 0$ because the "falling" is supposed to take place even in the first phase. The data points in $n_{APE} \in [20, 40]$ are well described by a linear function,

$$\chi(n_{APE}) = \chi(0) + c_1 n_{APE}.$$  

(17) The fit results are tabulated in Tab. II. Figure 9 shows $n_{APE}$ dependence of $b_2$. Since $b_2$ is found to be constant for $n_{APE} \geq 20$,

$$\beta N_S a_4 \chi(0) \times 10^4 c_1 \times 10^7 b_2(0) \times 10^2$$

$\begin{align*}
1.750 & \pm 0.08(2) & \pm 0.4(3) & \pm 0.5(5) \\
1.850 & \pm 0.10(1) & \pm 0.8(1) & \pm 0.6(3) \\
1.975 & \pm 0.269(8) & \pm 0.22(2) & \pm 0.22(2) \\
1.975 & \pm 0.254(3) & \pm 0.20(1) & \pm 0.20(1) \\
\end{align*}$

TABLE II: Fit results.
we perform the constant fit to extract $b^2$ at $n_{\text{APE}} = 0$. The results are shown in Table.

The values of $b^2$ obtained at $\beta = 1.975$ with two lattice volumes turns out to be consistent with each other due to the large statistical uncertainty, while $1.8\sigma$ difference is observed for $\chi$. In Ref. [16], these quantities are calculated with several different volumes for $\text{SU}(N)$ with $N = 3, 4, 6$ down to $L_{\sigma} \sim 2.7$, and no finite volume effect is observed. Our lattice with $\beta = 1.975$ and $N_S = 16$ corresponds to $L_{\sigma} = 2.4$ (see Table I), which is smaller than but close to 2.7 and hence finite volume effects, if any, should not be significant. Thus, $1.8\sigma$ difference observed at $\beta = 1.975$ is considered as a statistical fluctuation, and we include both results in the following analysis.

Next we discuss the continuum limit. Figure 10 shows the extrapolation of $\chi/T^4$ and $b^2$ to the continuum. The limit for both quantities is examined by applying two functional forms.
These two are chosen because they turn out to yield the smallest and largest value for $\chi / T^4$ among other reasonable choices. In either quantities, the constant fit is taken as the central value, and the difference between two methods is taken as the systematic uncertainty in the final result.

The continuum limit of $\chi / T^4$ turns out to strongly depend on the functional form, and as are results dominate by the systematic uncertainty. On the other hand, thanks to the constant behavior for $b_2$, the inclusion of the linear term into the functional form does not alter the limit for the constant fit by much. The final results thus obtained are $\chi / T^4 = 0.200(39)$, $\chi / T^4 = 0.674(31)$, $b_2 = -0.049(20)$, (18) where the errors are summed in quadrature.

In Refs. [14–16], the topological susceptibility $\chi$ is calculated in SU($N$) gauge theory with several values of $N$ to study the large $N$ behavior. In Refs. [14, 39, 45–47], $\chi$ is estimated for $15$
\[
\frac{\chi}{T_c^4} = 0.200(39), \quad \frac{\chi^{1/4}}{T_c} = 0.674(31), \quad b_2 = -0.049(20),
\]

seems to be the first determination of \(b_2\)

\[
\text{cf. Bonanno, Bonati, D'Elia (18) } b_4 = 6(2) \cdot 10^{-4}
\]
FIG. 11: The $N$ dependence of $\chi/\sigma^2$ and $b^2$. Each data point is slightly shifted horizontally to make it easier to see. The horizontal dashed line in the $b^2$ plot represents the dilute instanton gas approximation (DIGA).

when fitted with

\[
\frac{\chi}{\sigma} = \left( \frac{\chi}{\sigma} \right)_{N=\infty} \quad \text{when } N_{\text{crit}} = 1.52(2)
\]

\[
\left( \frac{\chi}{\sigma} \right)_{N=\infty} \approx \frac{\chi}{\sigma} = \frac{N^2}{N^2 - N_{\text{crit}}^2}
\]

[cf. Lüscher (1982) for 2d $\mathbb{C}P^n$ model]
FIG. 11: The $N$ dependence of $\chi/\sigma^2_{\text{str}}$ and $b_2$. Each data point is slightly shifted horizontally to make it easier to see. The horizontal dashed line in the $b_2$ plot represents the dilute instanton gas approximation (DIGA).

$b_2 = -0.083$ (instanton) \[ N_{\text{crit}} \approx 1.5 \]

$b_2 (N_{\text{crit}}) \approx -0.087 (5)$
Summary

* 4d \( SU(2) \) YM: still "large \( N \)"

spontaneous CP breaking, mass gap

\[ \frac{\chi^{1/4}}{T_c} = 0.674(31), \quad b_2 = -0.049(20) \]

[Quantitatively different from 2d \( CP^{N-1} \) model]

Diagram:

CP broken

\( \text{gapped} \)

\( \begin{array}{ccc}
N=4 & N=3 & N=2 \\
\cdot & \cdot & \cdot \\
\end{array} \)

CP unbroken

\( \text{gapless} \)

\( N_{CP} \approx N_{\text{crit}} \)

\( N=1 \quad \frac{1}{N} \)