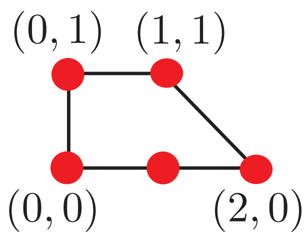
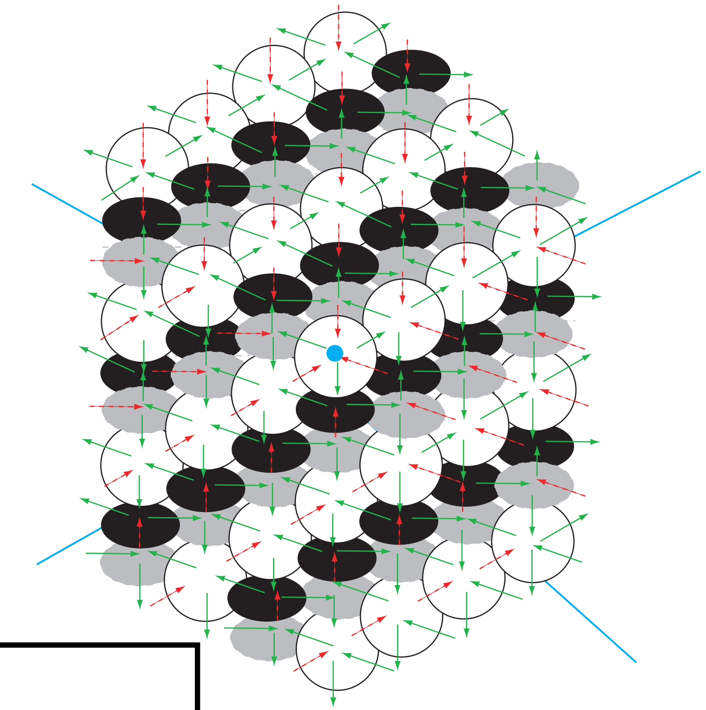


$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,, \\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\,, \\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\,, \\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,, \\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,, \\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,,
\end{aligned}
$$



Quiver Yangians and Crystal Melting

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IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

ICMP Geneva / online

Aug 5, 2021

Based on

Wei Li + MY

(2003.08909 [hep-th])

Dmitry Galakhov + MY

(2008.07006 [hep-th])

Dimitry Galakhov+Wei Li + MY

(2106.01230 [hep-th])

… and many works in the literature

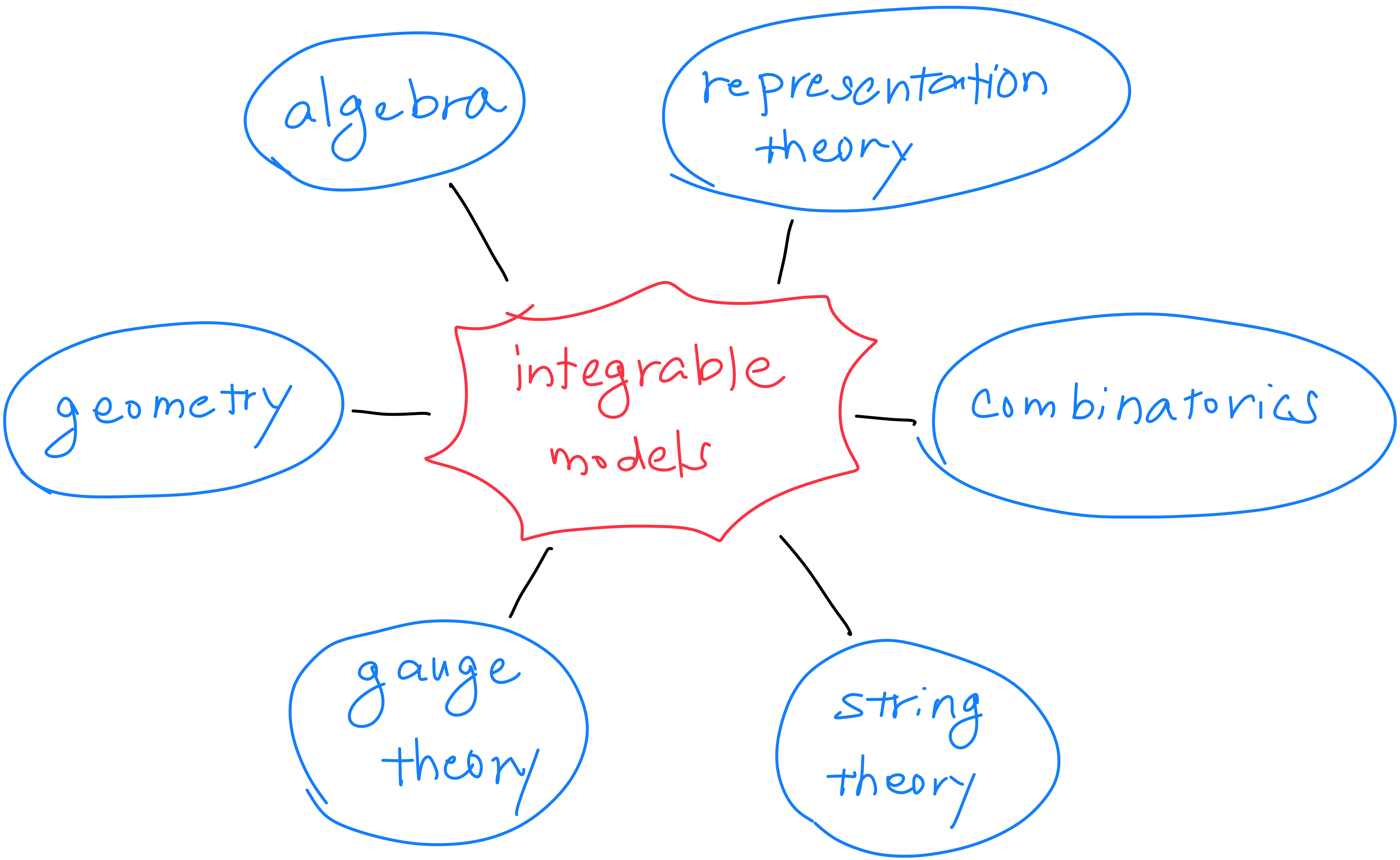
Also earlier works, e.g.

Hirosi Ooguri + MY (0811.2810 [hep-th])

MY (Ph.D. thesis, 1002.1709 [hep-th])

MY (Master thesis, 0803.4474 [hep-th])

Overview


integrable models
algebra
representation theory
geometry
combinatorics
gauge theory
string theory

new algebras

generalization of affine Yangian

Shifted Quiver Yangian

$Y(Q, W)$

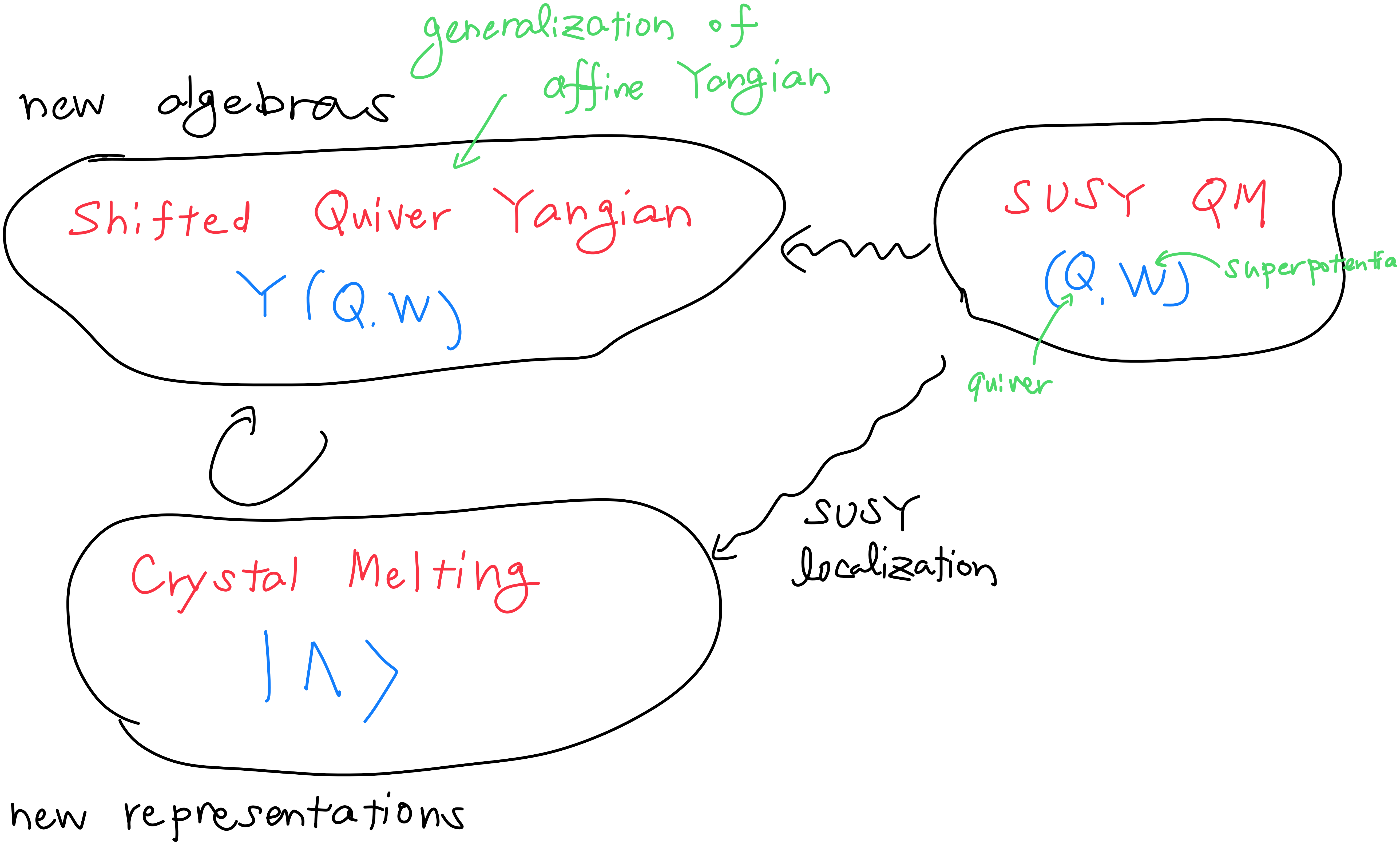
$\longleftarrow\!\!\!\sim\!\sim\!\sim$

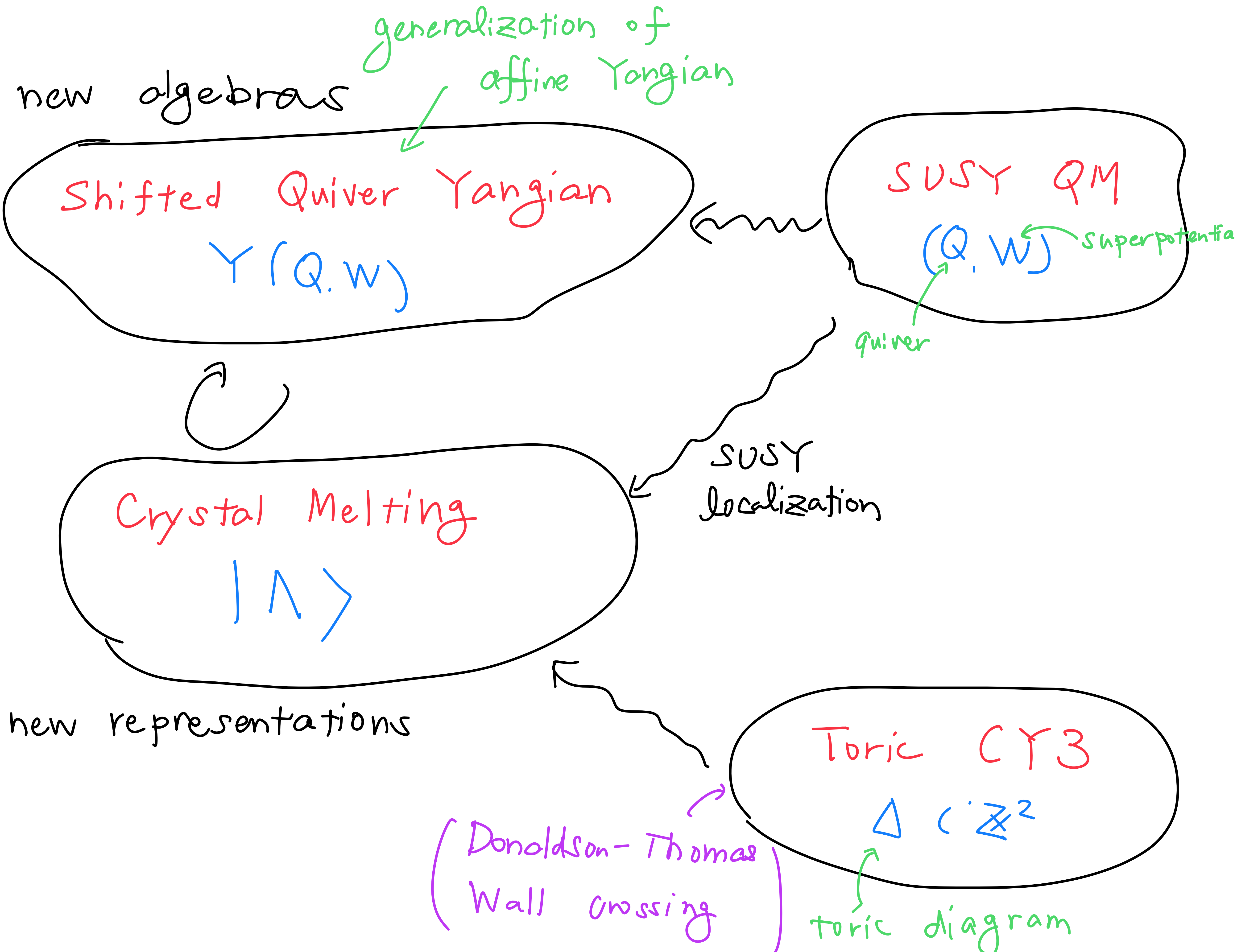
SUSY QM

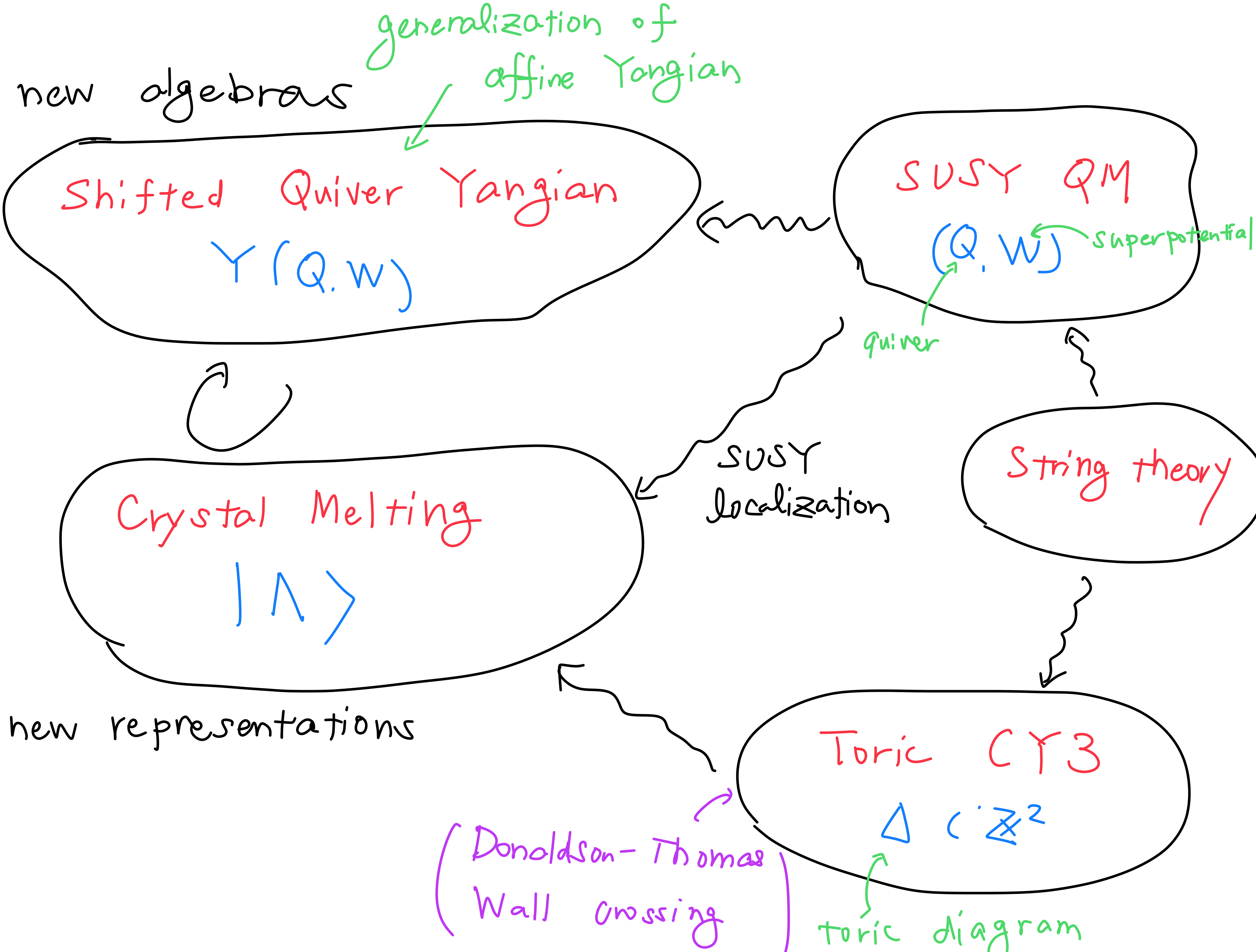
(Q, W)

superpotential

quiver


new algebras
generalization of affine Yangian
Shifted Quiver Yangian
Y(Q, W)
SUSY QM
(Q, W)
superpotential
quiver
SUSY localization
Crystal Melting
|Λ⟩
new representations


new algebras
generalization of affine Yangian
Shifted Quiver Yangian
$Y(Q, W)$
SUSY QM
(Q, W)
superpotential
quiver
SUSY localization
Crystal Melting
$|\Lambda\rangle$
new representations
Toric CY3
$\Delta \subset \mathbb{Z}^2$
toric diagram
Donaldson-Thomas
Wall crossing


new algebras
generalization of affine Yangian
Shifted Quiver Yangian
$Y(Q,W)$
SUSY QM
(Q,W)
superpotential
quiver
SUSY localization
String theory
Crystal Melting
$|\Lambda\rangle$
new representations
Toric CY3
$\Delta \subset \mathbb{Z}^2$
toric diagram
(Donaldson-Thomas Wall crossing)

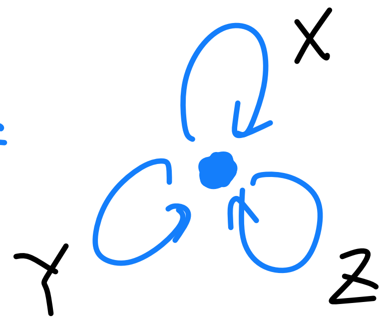
Shifted

Quiver Yangian

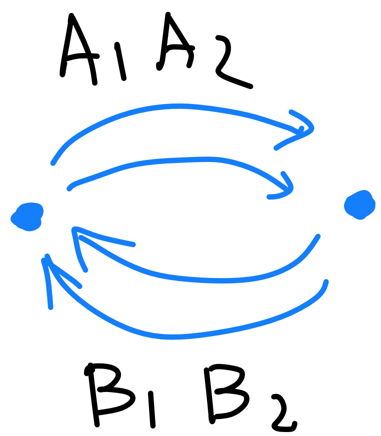
Wei Li + MY (20)

Dimitry Galakhov + Wei Li + MY (21)

Quiver Q & Superpotential W ⟵ toric CY_3

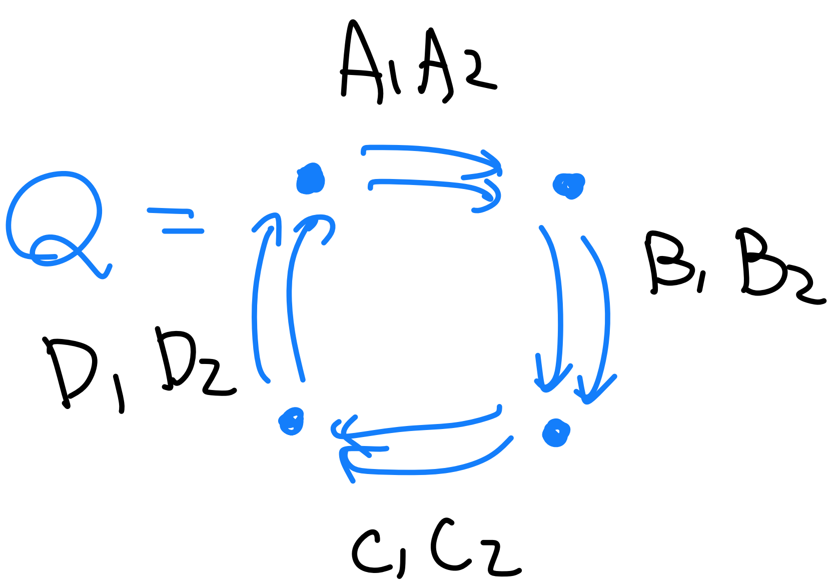
* $Q =$ 

$W = \mathrm{Tr}(XYZ - XZY)$ $\quad (CY_3 = \mathbb{C}^3)$

* $Q =$ 

$W = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$

$(CY_3 = \text{conifold})$

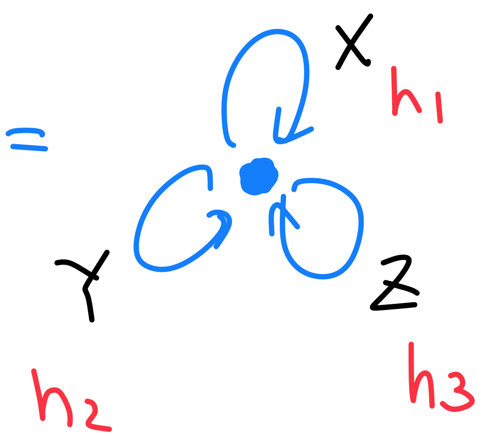
* $Q =$ 

$W = \mathrm{Tr}(A_1 B_1 C_1 D_1 - A_1 B_2 C_1 D_2$

$- A_2 B_1 C_2 D_1 + A_2 B_2 C_2 D_2)$

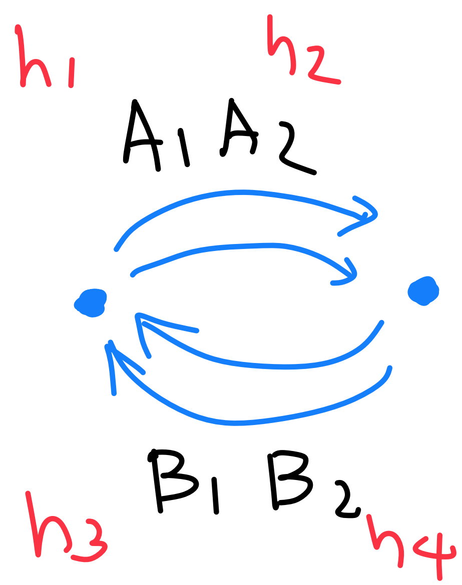
$(CY_3 = K_{\mathbb{P}^1 \times \mathbb{P}^1})$

Quiver Q & Superpotential W ⟵ toric CY_3

* $Q =$ 

$W = \mathrm{Tr}(XYZ - XZY)$ $\quad (CY_3 = \mathbb{C}^3)$

$h_1 + h_2 + h_3 = 0$

* $Q =$ 

$W = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$

$(CY_3 = \text{conifold})$

$h_1 + h_2 + h_3 + h_4 = 0$

* Assign equivariant parameters h_I consistent w/ W

edge

Generators

(z : spectral parameter)

$$e^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{e_n^{(a)}}{z^{n+1}}\,, \qquad \psi^{(a)}(z) \equiv \sum_{n=-\infty}^{+\infty} \frac{\psi_n^{(a)}}{z^{n+1}}\,, \qquad f^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{f_n^{(a)}}{z^{n+1}}\,,$$

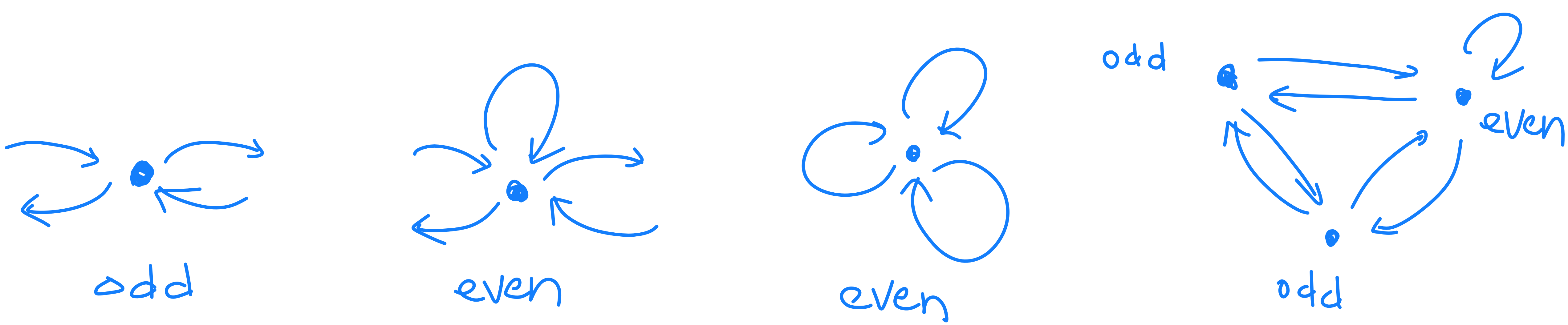
$n = -k$

"k-shifted Quiver Yangian"

a : quiver vertex

$\mathbb{Z}_2$-grading

$$|a| = \begin{cases} 0 & (\exists \text{ edge } I \text{ s.t. } I \text{ starts and ends at } a) \\ 1 & (\text{otherwise}) \end{cases}$$



Relations

$Y(Q,W)$

$$\psi^{(a)}(z)\,\psi^{(b)}(w) = \psi^{(b)}(w)\,\psi^{(a)}(z)\;,$$

$$\psi^{(a)}(z)\,e^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\;,$$

$$e^{(a)}(z)\,e^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\;,$$

$$\psi^{(a)}(z)\,f^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\;,$$

$$f^{(a)}(z)\,f^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\;,$$

$$\left[e^{(a)}(z), f^{(b)}(w)\right\} \sim -\delta^{a,b}\,\frac{\psi^{(a)}(z) - \psi^{(b)}(w)}{z-w}\;,$$

"$\simeq$" means equality up to $z^n w^{m\geq 0}$ terms

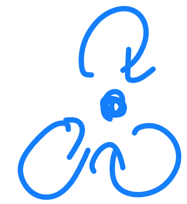
"$\sim$" means equality up to $z^{n\geq 0} w^m$ and $z^n w^{m\geq 0}$ terms

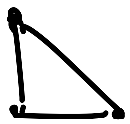
bonding factor

$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

equivariant weight

edge

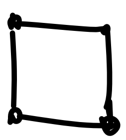
* $\mathbb{C}^3 \rightsquigarrow Q =$  $\rightsquigarrow Y(\widehat{gl}_1)$



$W = \mathrm{Tr}(XYZ - XZY)$

[Miki; Ding-Iohara;⋯
Tsymbaulik; Prochazka;
Gaberdiel, Gopakumar, Li, Peng,⋯]

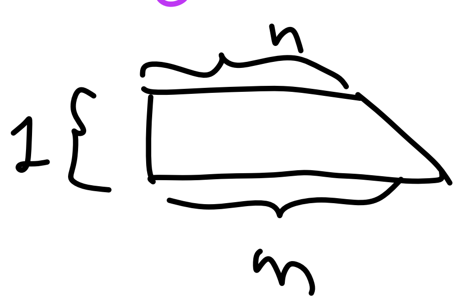
* conifold $\rightsquigarrow Q =$  $\rightsquigarrow Y(\widehat{gl}_{1|1})$



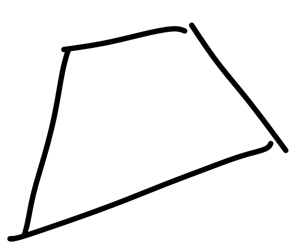
$W = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$

* $xy = z^n w^m \rightsquigarrow Y(\widehat{gl}_{m|n})$

[Rapcak; Bezerra-Mukhin]
cf. [Nagao-MY '10]



* general toric $CY_3 \rightsquigarrow Y(Q, W)$



$\Delta \subset \mathbb{Z}^2$

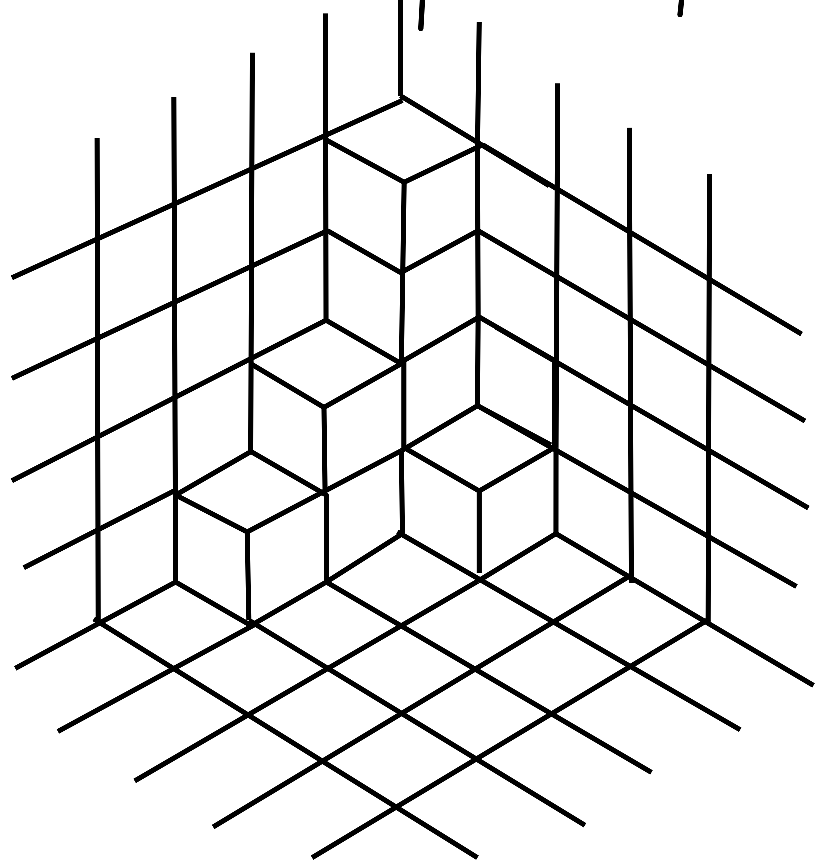
has no "$\mathfrak{g}$"

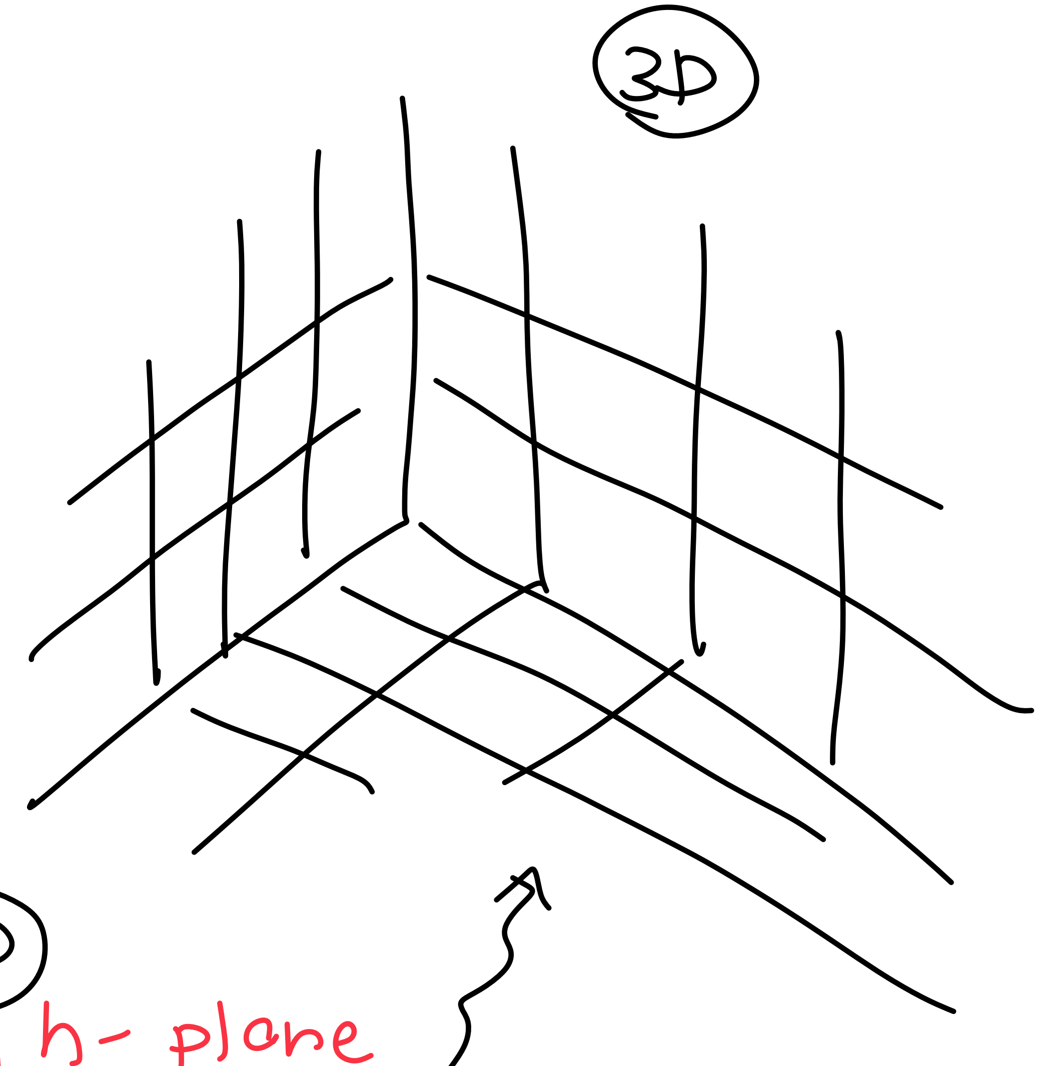
Representations from Crystal Melting

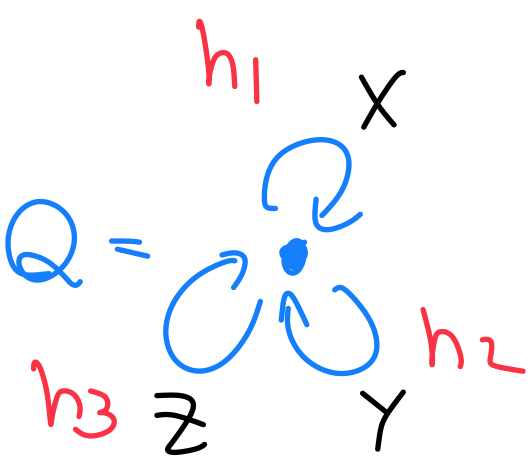
cf. earlier developments on quantum toroidal algebras (Ding-Iohara-Miki) and affine Yangians by [Feigin, Jimbo, Miwa, Mukhin; Tsymbaulik; Prochazka; Rapcak; Gaberdiel, Gopakumar; Li, Peng,⋯]

$\mathbb{C}^3$: crystal melting [Okounkov-Reshetikhin-Vafa; Iqbal, Nekrasov,...]

plane partition

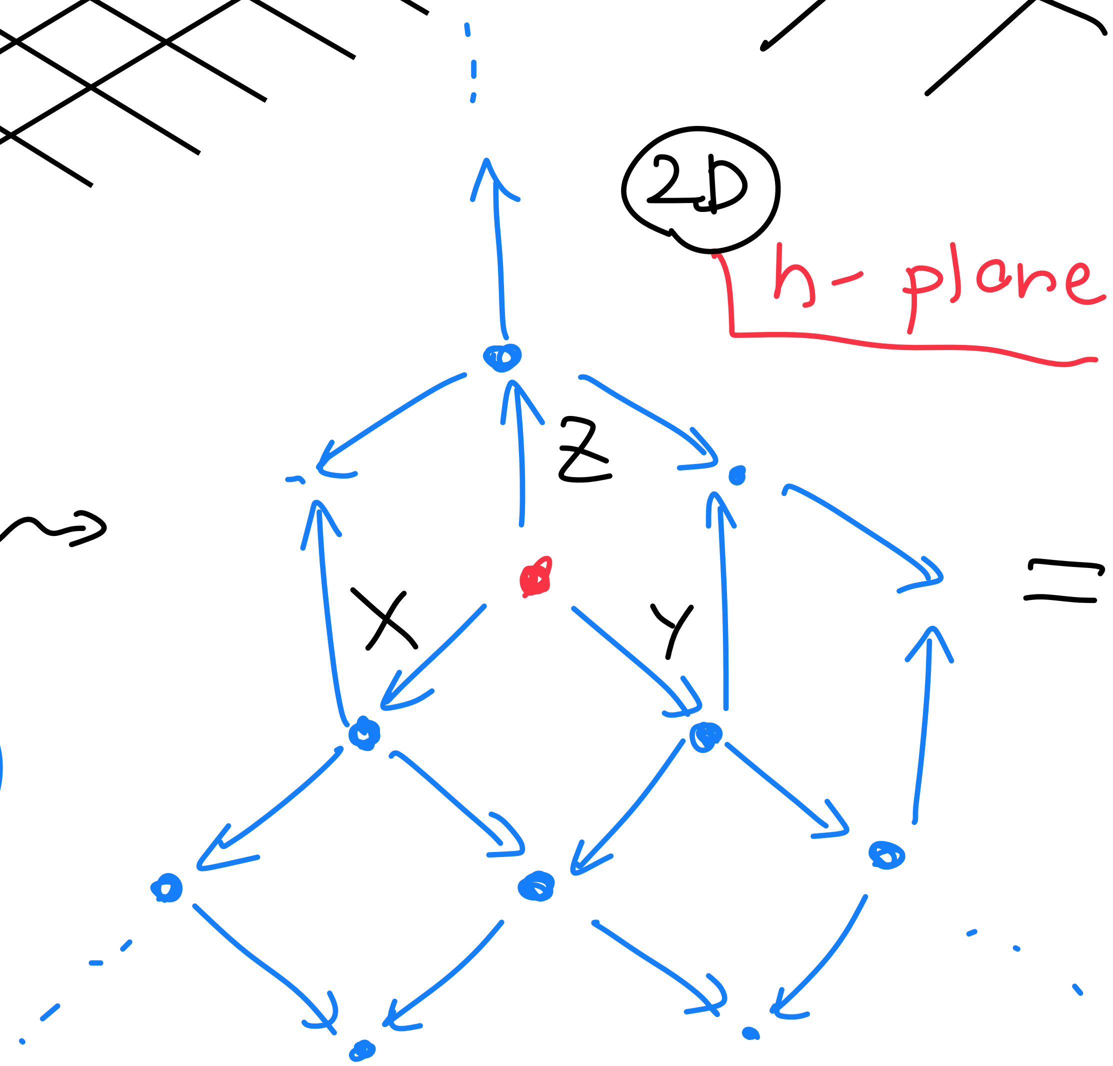




* $Q =$ 

$W = \mathrm{Tr}(XYZ - XZY)$

$h_1 + h_2 + h_3 = 0$

$\rightsquigarrow$ 

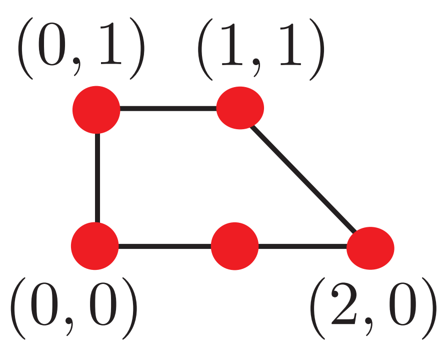
$= \left(\text{paths from } \bullet \right) \Big/ \left(\text{relations} \right)$

The story generalizes to an arbitrary toric CY3

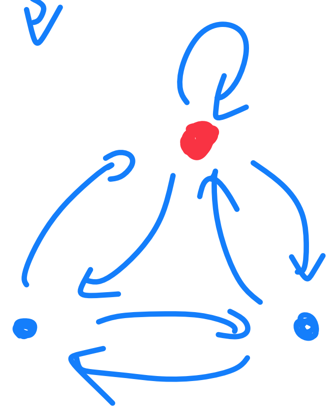
[Ooguri-MY '08'09]

See also [Szendroi; Mozgovoy, Reineke; Nagao, Nakajima; Ooguri, MY; Jafferis, Chuang, Moore; Sulkowski; Aganagic, Vafa; ⋯]

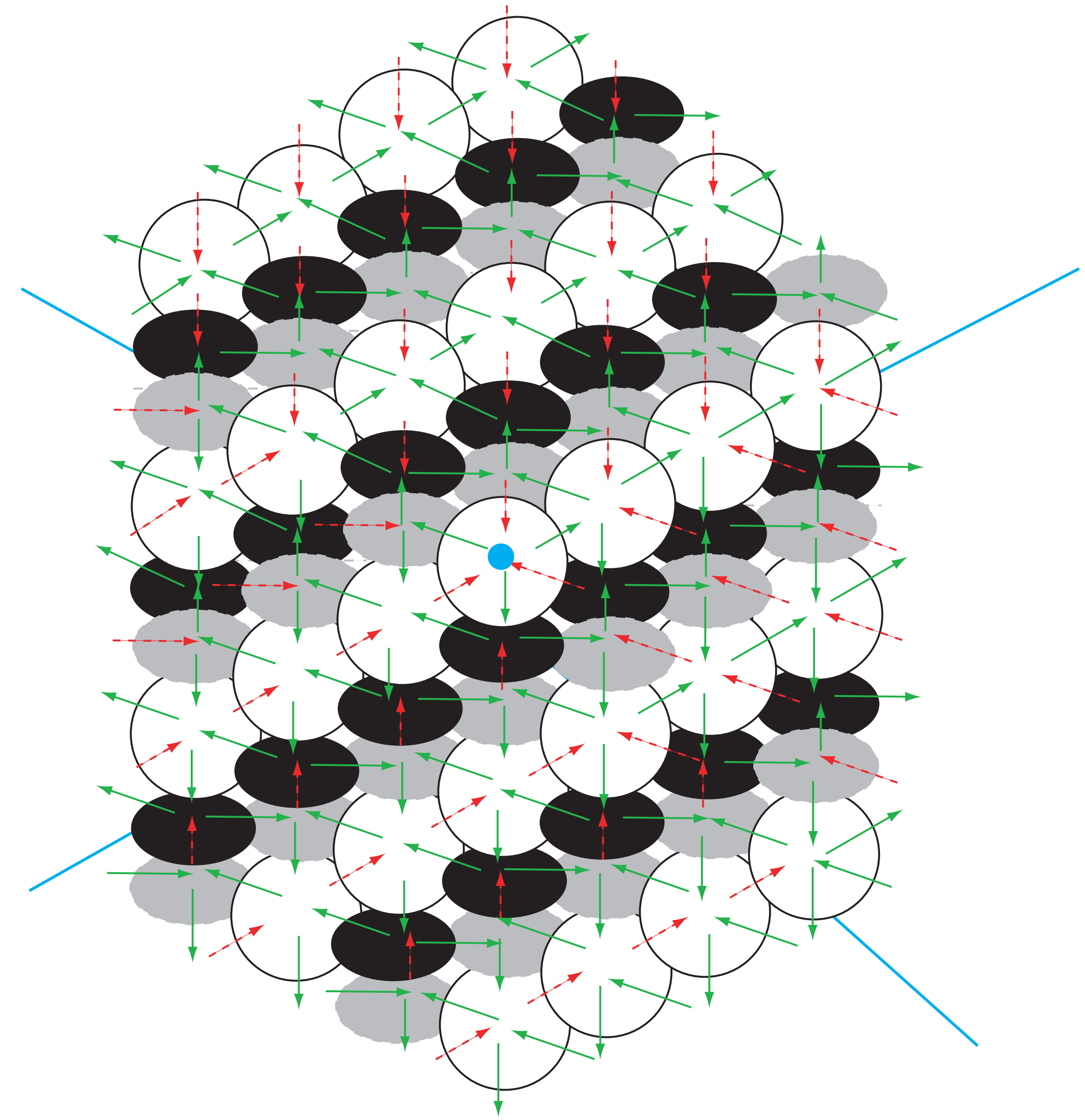
$CY_3 \quad xy = zw^2$



$Q \simeq$



$W = \mathrm{Tr}(\cdots)$



[Ooguri-MY '08]

$\left(\text{paths from a vertex}\right) / (\partial W)$

$\shortparallel$

$\left(\text{atoms in crystal}\right)$

path algebra

$\mathbb{C}Q / (\partial W)$

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle\ ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in\mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle\ ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in\mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle\ ,$$

add/remove an atom

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle ,$$

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poles for atom $\boxed{a}$

add/remove an atom

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle ,$$

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$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to \mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle ,$$

poles for atom $\boxed{a}$

$$h(\boxed{a}) \equiv \sum_{I\in \mathrm{path}[\mathfrak{o}\to \boxed{a}]} h_I .$$

$\Psi_K^{(a)}$: $$\Psi^{(a)}_{\mathrm{K}}(u) = \psi_0^{(a)}(z) \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{b\Rightarrow a}(u-h(\boxed{b})) ,$$

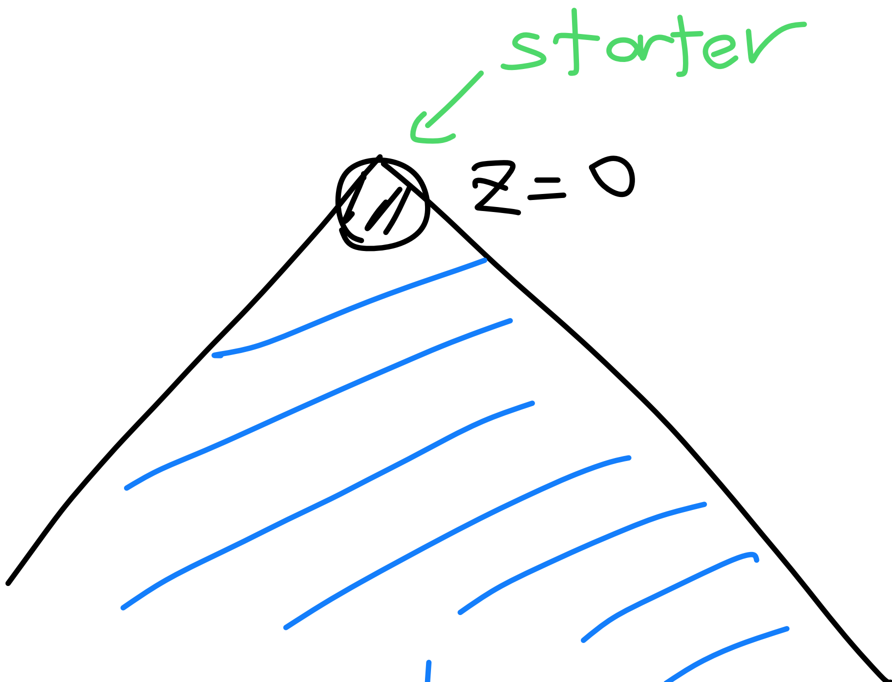
$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

$E^{(a)}/F^{(a)}$: $$E^{(a)}/F^{(a)} = \sqrt{\pm \operatorname*{Res}_{u=h(\boxed{a})} \Psi_K^{(a)}(u)}$$

$$\psi^{(a)}(z)|\emptyset\rangle = \psi_0^{(a)}(z)|\emptyset\rangle$$

[Galakhov-Li-MY '21]

vacuum charge function $\longleftrightarrow$ representation

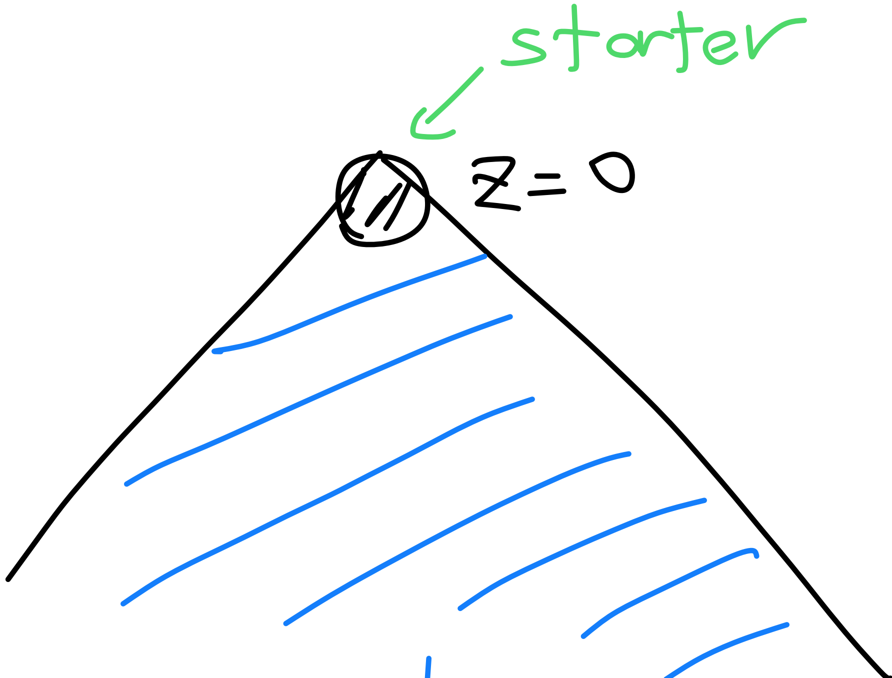


$\psi_0 \sim \frac{1}{z}$

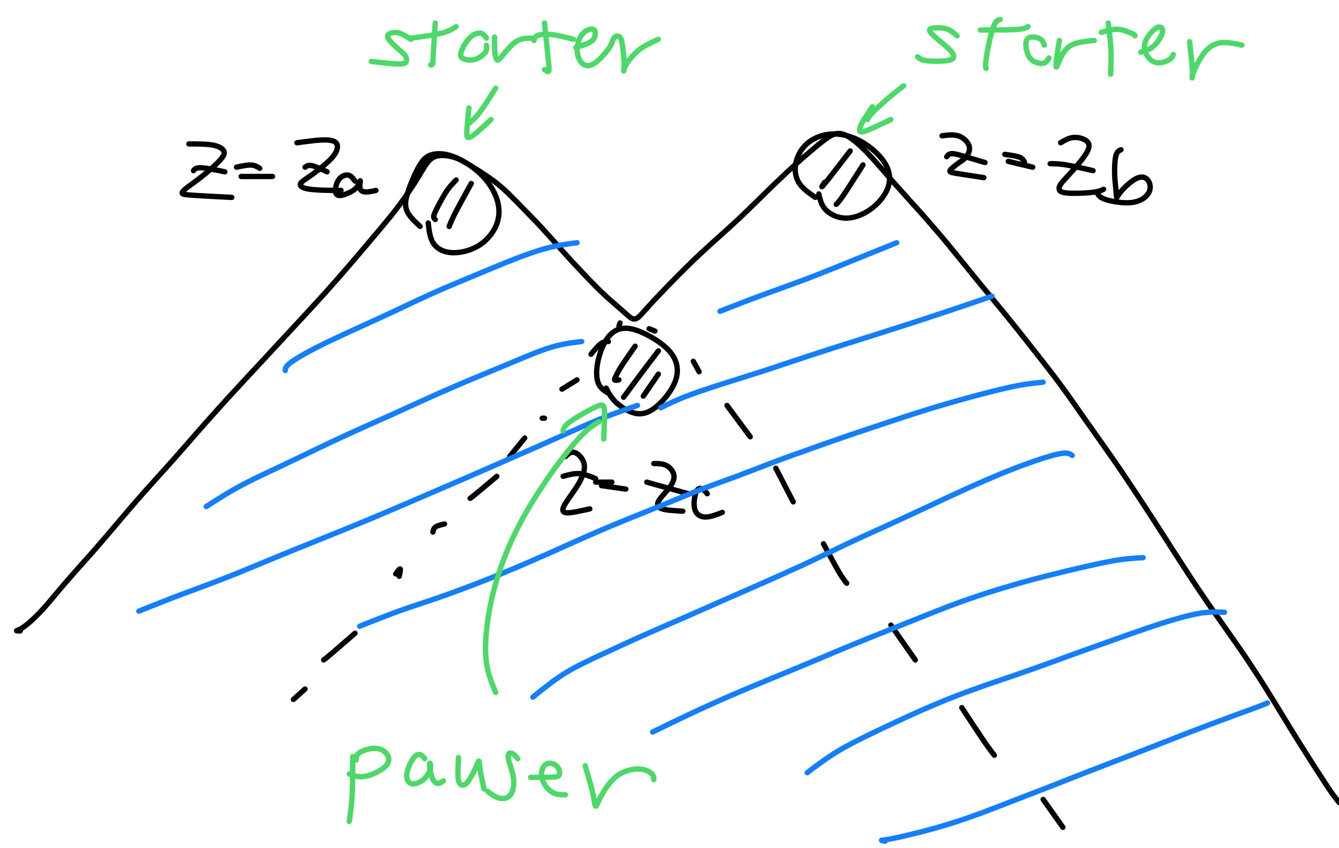
$$\psi^{(a)}(z)\,|\emptyset\rangle = \underbrace{\psi_0^{(a)}(z)}\,|\emptyset\rangle$$

[Galakhov-Li-MY '21]

vacuum charge function $\longleftrightarrow$ representation



$\psi_0 \sim \frac{1}{z}$

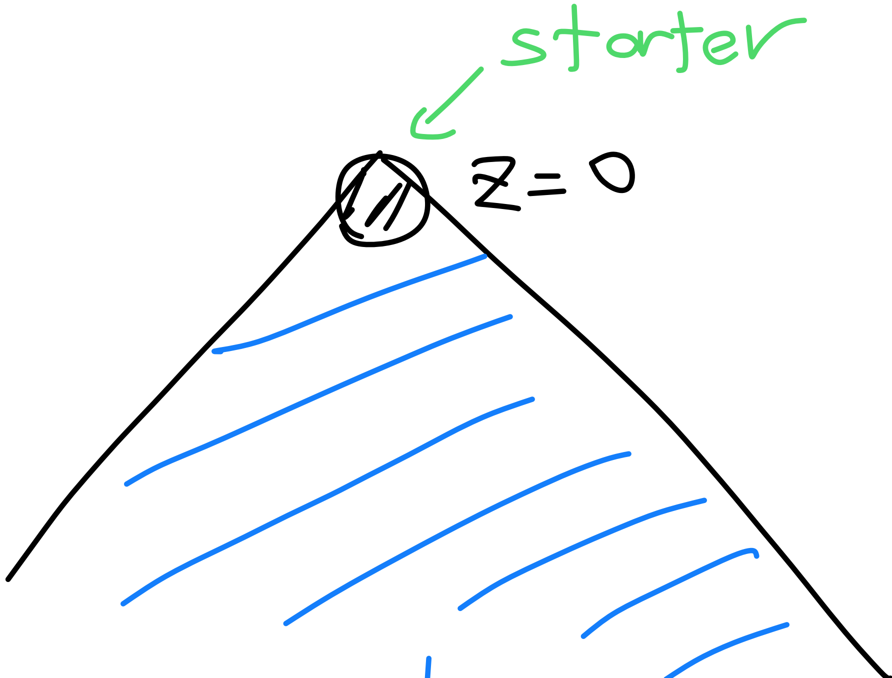


$$\psi_0 \sim \frac{z - z_c}{(z - z_a)(z - z_b)}$$

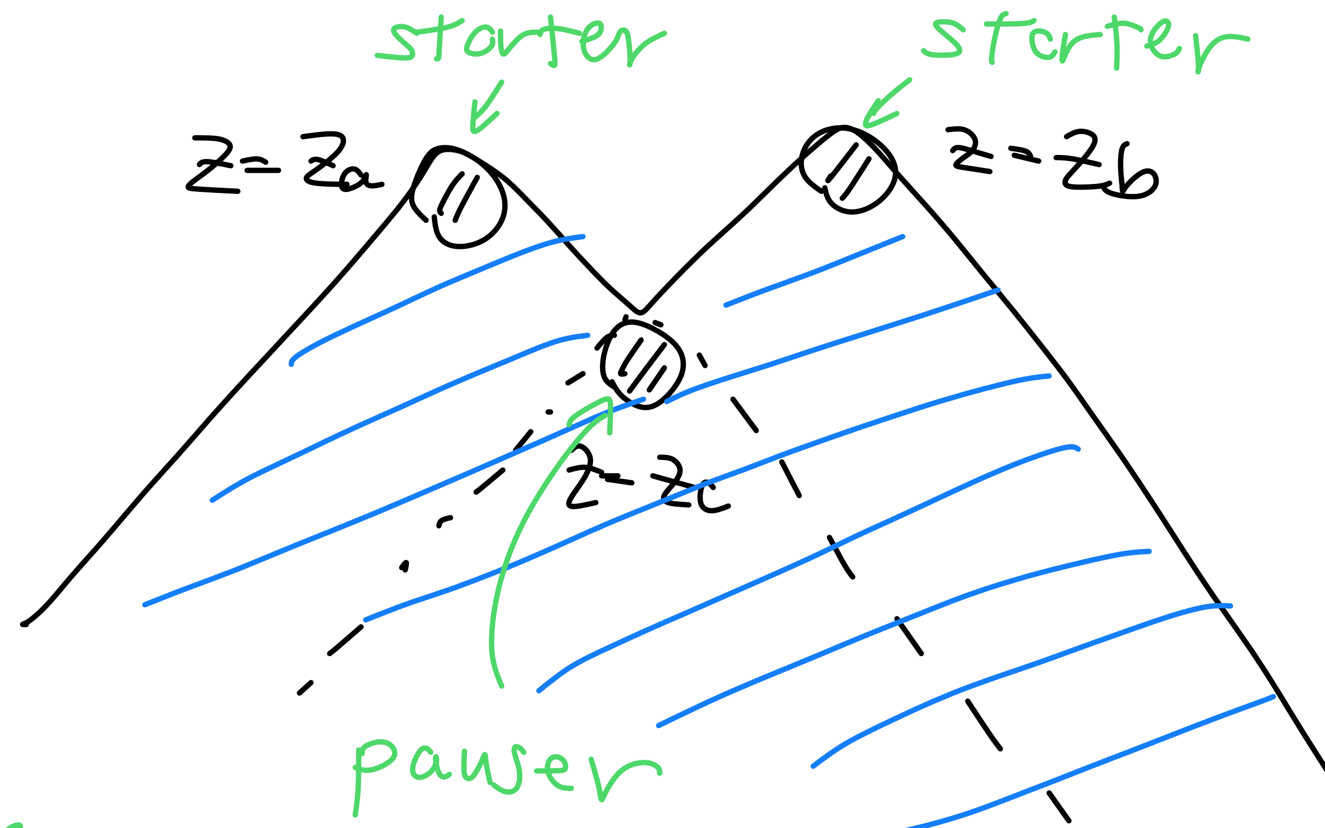
$$\psi^{(a)}(z)|\emptyset\rangle = \underbrace{\psi_0^{(a)}(z)}|\emptyset\rangle$$

[Galakhov-Li-MY '21]

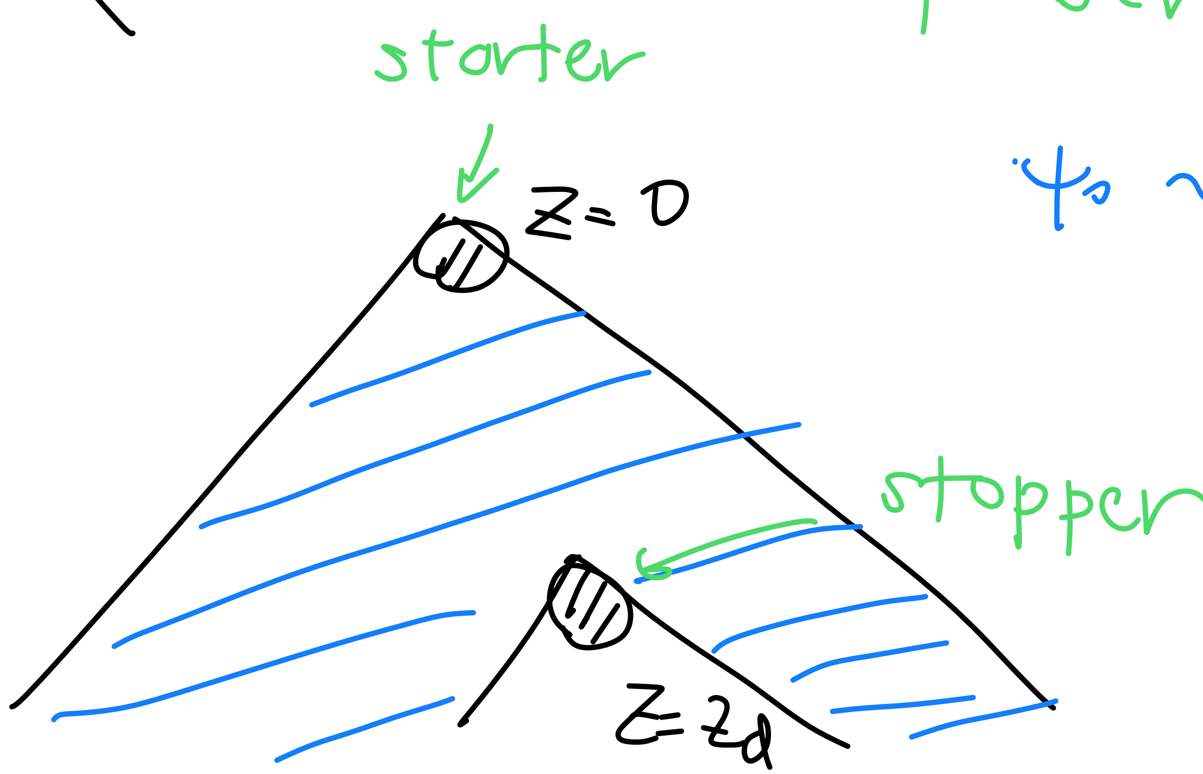
vacuum charge function $\longleftrightarrow$ representation



$$\psi_0 \sim \frac{1}{z}$$



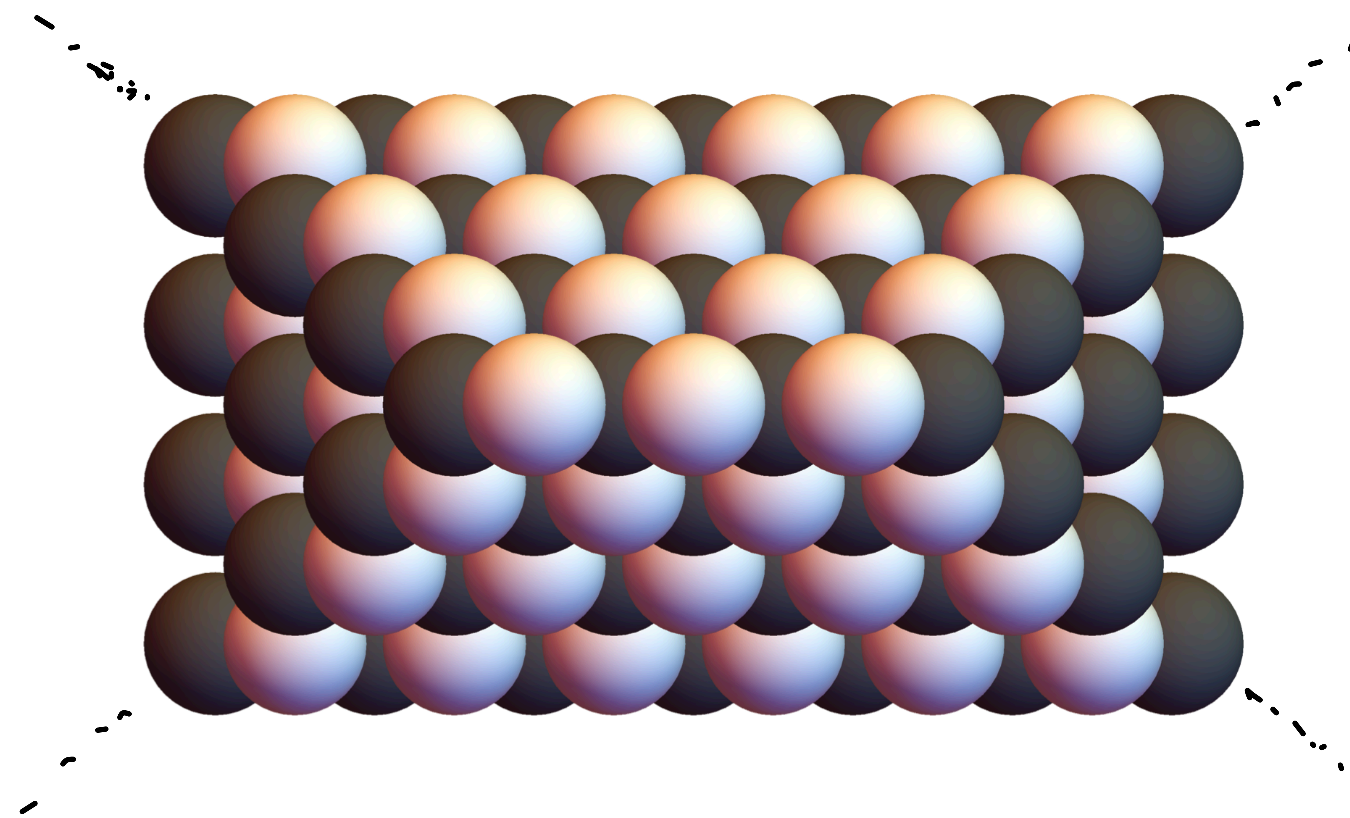
$$\psi_0 \sim \frac{z-z_c}{(z-z_a)(z-z_b)}$$



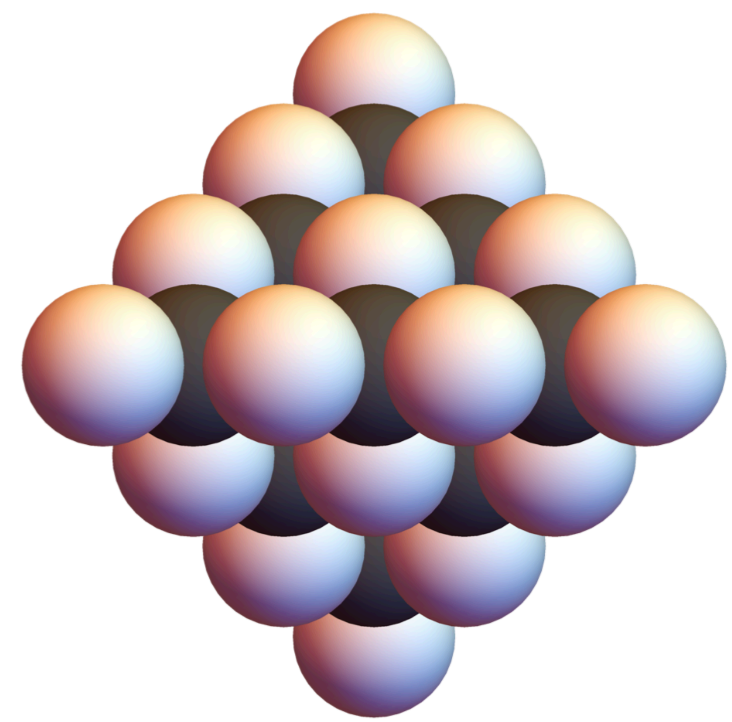
$$\psi_0 \sim \frac{z-z_d}{z}$$

We can obtain rather general reps by using starter / pauser / stoppers

e.g. open / closed BPS state counting and their wall crossings



conifold : ∞-chamber



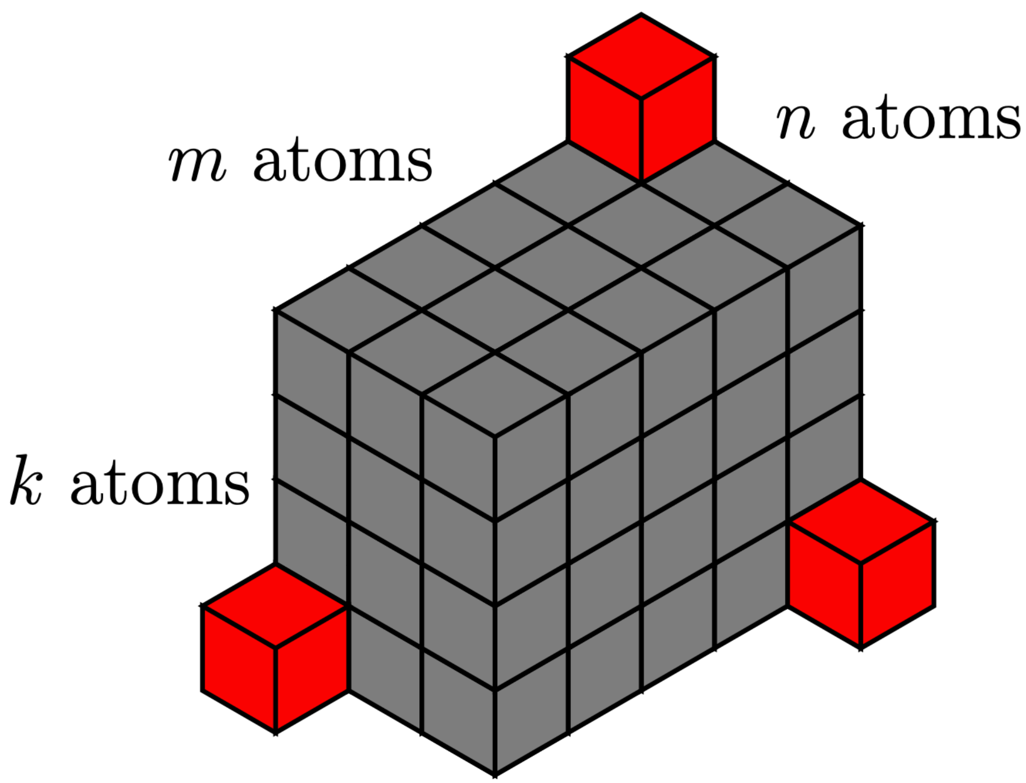
conifold : finite chamber

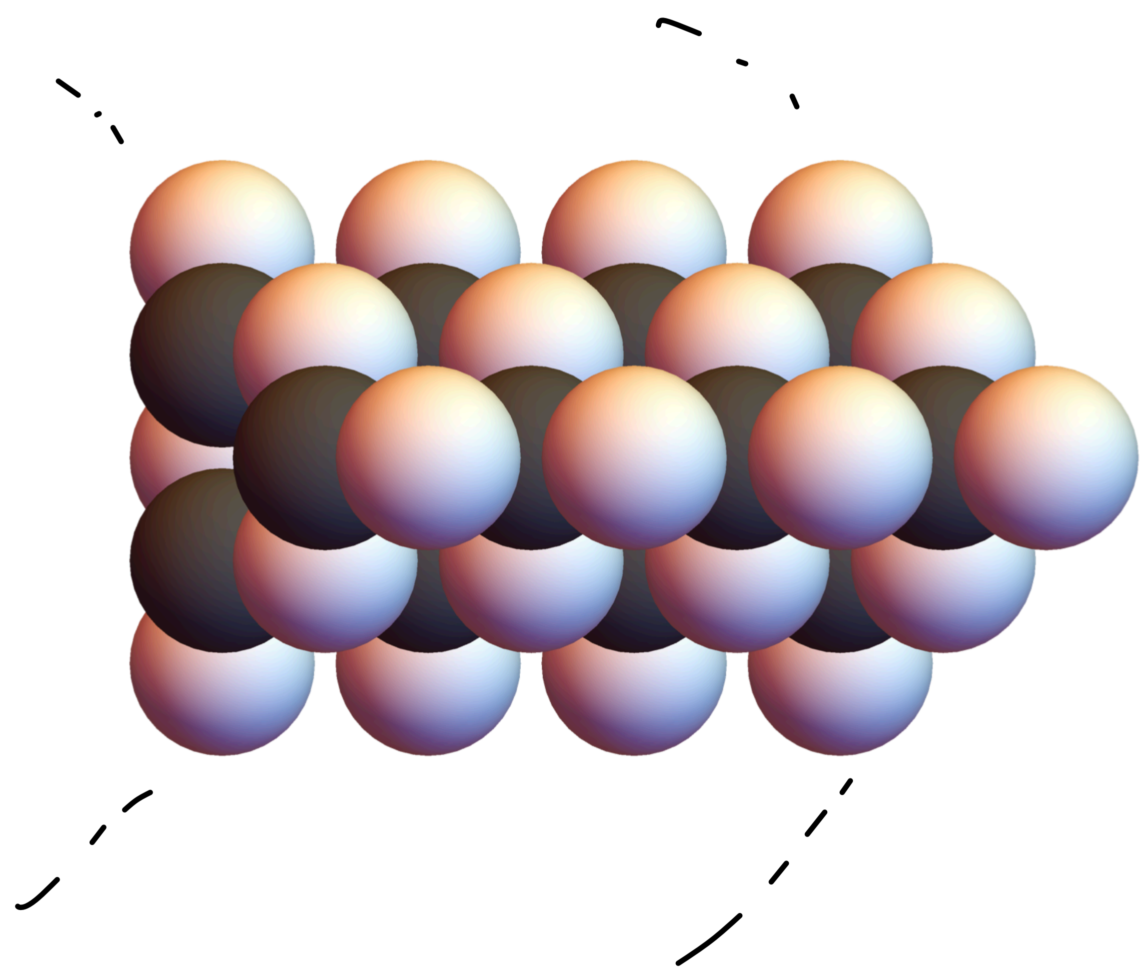
[Nagao-Nakajima; Jafferis-Moore; Chuang-Jafferis,⋯ '08]

Some representations have no known CY_3/geometry counterparts

$Y(\hat{\mathfrak{gl}}_1)$ $\mathbb{C}^3$-like

$Y(\hat{\mathfrak{gl}}_{1|1})$ conifold-like





semi-infinite

[Galakhov-Li-MY '21]

We can derive quiver Yangian representation by equivariant localization in SUSY QM

[Galakhov-MY '20]

$(Q, W) \rightsquigarrow$ SUSY QM

We can <u>derive</u> quiver Yangian representation by equivariant localization in SUSY QM

[Galakhov-MY '20]

(Q, W) $\rightsquigarrow$ SUSY QM $\rightsquigarrow$ $\mathcal{H}_{\mathrm{VAC}}$

Q supercharge $(\sim d)$

We can derive quiver Yangian representation by equivariant localization in SUSY QM

[Galakhov-MY '20]

$(Q, W) \rightsquigarrow$ SUSY QM $\rightsquigarrow$ $\mathcal{M}_{\text{VAC}}$

Q supercharge $(\sim d)$

$\Downarrow$

Ω-deformation by equiv. param. $\{h_I\}$

fixed pts : crystal Λ

We can derive quiver Yangian representation by equivariant localization in SUSY QM

[Galakhov-MY '20]

$(Q, W) \rightsquigarrow$ SUSY QM $\rightsquigarrow$ $\mathcal{M}_{\text{VAC}}$

Q supercharge $(\sim d)$

$\Downarrow$

Ω-deformation by equiv. param. $\{h_I\}$

fixed pts : crystal Λ

effective wavefunction

$\Psi_\Lambda \sim \mathrm{Eul}_\Lambda$

We can derive quiver Yangian representation by equivariant localization in SUSY QM

[Galakhov-MY '20]

(Q, W) ⇝ SUSY QM ⇝ $\mathcal{M}_{\text{VAC}}$

Q supercharge $(\sim d)$

Ω- deformation by equiv. param. $\{h_I\}$

fixed pts : crystal Λ

effective wavefunction

$\Psi_\Lambda \sim \mathrm{Eul}_\Lambda$

e/f generators: "Hecke modification"

$\langle \Psi_{\Lambda+\square} | e | \Psi_\Lambda \rangle$

FMT in $\mathcal{M}_\Lambda \times \mathcal{M}_{\Lambda+\square}$

We obtain algebras / repr. by equiv. localization of $SQM_{(Q,W)}$

In all cases reproduce $Y_{(Q,W)}$ but no general proof

[Galakhov-MY '20; Galakhov-Li-MY '20]

Highly non-trivial cancellations!

[Galakhov-MY '20]

For example, for one of the Serre relations of $Y(\widehat{\mathfrak{gl}}_{3|1})$

$$\mathrm{Sym}_{z_1,z_2}\left[e^{(2)}(z_1),\left[e^{(3)}(w_1),\left[e^{(2)}(z_2),e^{(1)}(w_2)\right\}\right\}\right\}$$

$$A_2 := \operatorname*{Res}_{z_1,z_2,w_1,w_2}\langle\Lambda|A_1|\Lambda_0\rangle =$$

$$= [1,2,4,3]+[1,3,4,2]-[2,1,3,4]+[2,1,4,3]-[2,3,1,4]+[2,4,1,3]+$$
$$+[2,4,3,1]-[3,1,2,4]+[3,1,4,2]-[3,2,1,4]+[3,4,1,2]+[3,4,2,1]-$$
$$-[4,1,2,3]-[4,1,3,2]-[4,2,1,3]-[4,3,1,2] = 0\,!$$

$$[2,4,1,3]=-\frac{1}{48}\,,\quad [4,2,1,3]=-\frac{1}{96}\,,\quad [2,1,4,3]=-\frac{1}{48}\,,\quad [1,2,4,3]=\frac{1}{32}\,,$$
$$[4,1,2,3]=\frac{1}{64}\,,\quad [1,4,2,3]=\frac{1}{64}\,,\quad [4,1,3,2]=-\frac{1}{64}\,,\quad [1,4,3,2]=-\frac{1}{64}\,,$$
$$[2,4,3,1]=\frac{2\hbar_1+\hbar_2}{24\,(4\hbar_1+\hbar_2)}\,,\quad [4,2,3,1]=\frac{2\hbar_1+\hbar_2}{48\,(4\hbar_1+\hbar_2)}\,,$$
$$[2,3,4,1]=\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,\quad [3,2,4,1]=-\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[4,3,2,1]=-\frac{2\hbar_1+\hbar_2}{48\,(4\hbar_1+\hbar_2)}\,,\quad [3,4,2,1]=-\frac{(2\hbar_1+\hbar_2)^2}{24\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[2,1,3,4]=-\frac{2\hbar_1+\hbar_2}{24\,(4\hbar_1+3\hbar_2)}\,,\quad [1,2,3,4]=\frac{2\hbar_1+\hbar_2}{16\,(4\hbar_1+3\hbar_2)}\,,$$
$$[2,3,1,4]=\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,\quad [3,2,1,4]=-\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[1,3,2,4]=-\frac{2\hbar_1+\hbar_2}{16\,(4\hbar_1+3\hbar_2)}\,,\quad [3,1,2,4]=\frac{(2\hbar_1+\hbar_2)^2}{8\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[4,3,1,2]=\frac{2\hbar_1+\hbar_2}{32\,(4\hbar_1+\hbar_2)}\,,\quad [3,4,1,2]=\frac{(2\hbar_1+\hbar_2)^2}{16\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[1,3,4,2]=-\frac{2\hbar_1+\hbar_2}{32\,(4\hbar_1+3\hbar_2)}\,,\quad [3,1,4,2]=\frac{(2\hbar_1+\hbar_2)^2}{16\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}.$$

Summary

String theory

toric CY3

Quiver Yangian

Y(Q,W)

new algebras

SUSY QM (Q,W)

repr. in crystal melting

new repr.